Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 1, February 2010, pp. 83–96.

John disks, the Apollonian metric, and min-max properties

M HUANG[1], S PONNUSAMY[2] and X WANG[1,*]

[1]Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China
[2]Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
E-mail: mzhuang79@yahoo.com.cn; samy@iitm.ac.in; xtwang@hunnu.edu.cn
*Corresponding author.

Dedicated to Matti Vuorinen on the occasion of his 60th birthday

MS received 18 December 2008; revised 26 October 2009

Abstract. The main results of this paper are characterizations of John disks–the simply connected proper subdomains of the complex plane that satisfy a twisted double cone connectivity property. One of the characterizations of John disks is an analog of a result due to Gehring and Hag who found such a characterization for quasidisks. In both situations the geometric condition is an estimate for the domain's hyperbolic metric in terms of its Apollonian metric. The other characterization is in terms of an arc min-max property.

Keywords. John disk; Apollonian metric; quasihyperbolic metric; internal distance; min-max property.

1. Introduction and main results

The purpose of this paper is to present some properties of John disks. We speak of the center of a carrot domain (§2 of [14]). According to this, a domain $D \subset \bar{\mathbb{C}}$ is called a *c-carrot domain* if there exists a point $z_0 \in \bar{D}$, called a center of D, and a positive constant c such that any point $z_1 \in D\backslash\{z_0\}$ may be joined to z_0 by an arc α such that for every $z \in \alpha\backslash\{\infty\}$ we have $\ell(\alpha[z, z_1]) \leq c\,\mathrm{dist}(z, \partial D)$, and α is called a *c-carrot arc*. Here $\ell(\alpha[z, z_1])$ denotes the Euclidean arclength of the part of α with endpoints z and z_1, and $\mathrm{dist}(z, \partial D)$ denotes the Euclidean distance from z to the boundary ∂D of D. If D is unbounded, then $z_0 = \infty$ (Remark 2.3.2 of [14]).

We now introduce the cigar domains. A domain $D \subset \bar{\mathbb{C}}$ is called a *c-cigar domain* if for every pair of points $z_1, z_2 \in D$, there is an arc $\alpha \subset D$ joining them with

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c\,\mathrm{dist}(z, \partial D) \quad \text{for all } z \in \alpha\backslash\{\infty\},$$

where c is a positive constant, and α is called a *c-cigar arc*.

It follows from Theorem 2.14 of [14] that any c-cigar domain is a c'-carrot domain, and by §2.26 of [14], when D is bounded, D is a c-cigar domain if and only if it is a c'-carrot domain, where c and c' depend only on each other.

If D is a c-cigar domain, then D is called a c-John domain. A simply connected c-John domain $D \subset \mathbb{C}$ with at least two boundary points is called a *c-John disk* (cf. [14]).

We call a domain D a *John domain* (resp. *John disk*) if it is a c-John domain (resp. c-John disk) for some positive constant c (see [11, 14]).

It is known that a Jordan domain $D \subset \mathbb{C}$ is a quasidisk if and only if both D and $D^* = \bar{\mathbb{C}} \backslash \bar{D}$ are John disks (cf. [12]), and every quasidisk is a John disk (see [5]). Hence John disks can be thought of as 'one-sided quasidisks'. This is also because of the fact that John disks enjoy similar geometric properties as quasidisks (see [12–14]).

Let D be a simply connected proper subdomain of $\bar{\mathbb{C}}$. D is called a *c-crosscut domain* if for every straight crosscut α of D dividing D into two subdomains D_1 and D_2, we have

$$\min_{j=1,2} \operatorname{diam}(D_j) \le c \operatorname{diam}(\alpha),$$

where c is a constant and $\operatorname{diam}(\alpha)$ means the diameter of α. Here a *straight crosscut* is a line segment in D with endpoints in $\partial D \backslash \{\infty\}$.

Let D be a simply connected domain of $\mathbb{C}$ with more than one boundary point and g a conformal mapping of D onto the unit disk $\mathbb{B} = \{z \in \mathbb{C}\colon |z| < 1\}$. Then the hyperbolic (or Poincaré) density of D is defined by

$$\rho_D(z) = \rho_{\mathbb{B}}(g(z))|g'(z)|,$$

where $\rho_{\mathbb{B}}(z) = 2/(1 - |z|^2)$. Here ρ_D is well-defined, for it does not depend on the particular choice of the conformal mapping g. It is well-known that ρ_D is a conformal invariant, (cf. §4.6 of [15]).

For any pair of points z_1, z_2 in D and a rectifiable path $\alpha \subset D$ connecting z_1 and z_2, we define the *hyperbolic length* $h_D(\alpha)$ of α and the *hyperbolic distance* $h_D(z_1, z_2)$ between z_1 and z_2 as follows:

$$h_D(\alpha) = \int_\alpha \rho_D(z)\, |\mathrm{d}z|$$

and

$$h_D(z_1, z_2) = \inf_\alpha h_D(\alpha) = \inf_\alpha \int_\alpha \rho_D(z)\, |\mathrm{d}z|,$$

where the infimum in the above is taken over all rectifiable paths α in D from z_1 to z_2. Minimal hyperbolic length paths are called *hyperbolic geodesics*. In the following, sometimes, a hyperbolic geodesic also means a hyperbolic ray (this means that one of its endpoints is in the boundary of D) or a hyperbolic line (both of its endpoints are in the boundary of D). It will be clear from the context which of the classes the path belongs to. It follows from [2, 7, 14] that the cigar arcs in the definition of c-John disks can be replaced by hyperbolic geodesics.

As usual, the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1, z_2 in D is the infimum of the quasihyperbolic lengths

$$k_D(\alpha) = \int_\alpha \frac{1}{\operatorname{dist}(z, \partial D)}\, |\mathrm{d}z|$$

of all rectifiable paths α which join z_1, z_2 in D. This infimum is always achieved and the minimal quasihyperbolic length paths are called *quasihyperbolic geodesics*, (cf. Lemma 1 of [8]).

As in [1], the *Apollonian distance* $a_D(z_1, z_2)$ between z_1, z_2 in D is defined by

$$a_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial D} \log(|z_1, z_2, w_1, w_2|),$$

where

$$|z_1, z_2, w_1, w_2| = \frac{|z_1 - w_1|\,|z_2 - w_2|}{|z_1 - w_2|\,|z_2 - w_1|}.$$

At this point, it may be important to recall the fact that $a_D \le j_D$ (see §3 for the definition of j_D). See [1] for more details about the metric a_D.

We can now formulate the known characterizations for John disks in a convenient form (cf. [12, 14]).

For a simply connected proper subdomain D of $\mathbb{C}$, the following are equivalent:

(a) D is a John disk.
(b) D is a cigar domain.
(c) D is a crosscut domain.
(d) The hyperbolic distance is bounded by inner-j distance; i.e., $h_D \le c j'_D$ (see §3 for the definition of j'_D) for some $c > 0$.

When D is bounded, we also get (1), (2), (3) of Lemmas 3.2 and 4.1. When D is Jordan, we further get the equivalence in Theorem A.

Similarly, one can describe quasidisks.

For a simply connected proper subdomain D of $\mathbb{C}$, the following are equivalent:

(I) D is a quasidisk.
(II) The hyperbolic distance is bounded by the Apollonian metric; i.e., $h_D \le c a_D$ for some $c > 0$.
(III) The hyperbolic geodesic min-max condition holds, i.e., there exists a constant $b_1 \ge 1$ such that for each hyperbolic segment γ joining z_1 and $z_2 \in D$,

$$\frac{1}{b_1} \min_{j=1,2} |z_j - w| \le |z - w| \le b_1 \max_{j=1,2} |z_j - w| \tag{1.1}$$

for any $z \in \gamma$ and $w \in \mathbb{C} \backslash D$.

We refer to Theorem 26 of [4] and Corollary 2.9 of [5] for the equivalence of (I) and (III) above, and Theorem 3.1 of [6] for the equivalence of (I) and (II). Our primary result is the following characterization of John disks in terms of the hyperbolic and Apollonian metrics.

Theorem 1.1. *A simply connected proper subdomain D of $\mathbb{C}$ is a c-John disk with center z_0 if and only if there are two positive constants C and H such that each point z in D can be joined to z_0 via an arc γ in D such that for all points z_1, z_2 on γ, we have*

$$h_D(\gamma[z_1, z_2]) \le C\, a_D(z_1, z_2) + H. \tag{1.2}$$

(*When D is unbounded we take $z_0 = \infty$.*) *The constants C and H depend only on c, and conversely, c depends only on C, H and* $\operatorname{diam}(D)/\operatorname{dist}(z_0, \partial D)$ *when D is bounded, and only on C and H when D is unbounded.*

Notice that this is the John disk analog of the equivalence of (I) and (II) above (see Theorem 3.1 of [6]). We conjecture that the constant H in Theorem 1.1 is not required. The proof of Theorem 1.1 will be presented in §3.

Let $D \subset \mathbb{C}$ be a domain. The *internal distance* between z_1 and z_2 in D is defined by

$$\lambda_D(z_1, z_2) = \inf\{\ell(\alpha) \colon \alpha \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2\}.$$

We call λ_D the *internal metric* on D. A point w in the boundary ∂D of D is said to be *rectifiably accessible* if there is a half open rectifiable path α in D ending at w, that is $\alpha \colon [0, 1) \to D$ with $w = \lim_{t \nearrow 1} \alpha(t)$. Let $\partial_r D$ denote the subset of ∂D that consists of all the rectifiably accessible points, that is

$$\partial_r D = \{w \in \partial D \colon w \text{ is rectifiably accessible}\}.$$

It is known that $\partial_r D$ is dense in ∂D (cf. [2]). In particular, if D is a John disk, then $\partial_r D = \partial D$ (cf. [14]). For convenience, we denote $D_r = D \cup \partial_r D$. Let d denote a distance function such that for any $z_1, z_2 \in D_r$,

$$|z_1 - z_2| \le d(z_1, z_2) \le \lambda_D(z_1, z_2).$$

Then we have

Theorem 1.2. *A simply connected proper subdomain D of $\mathbb{C}$ is a c-John disk with center z_0 if and only if there is a positive constant b such that each point x in D can be joined to z_0 via an arc γ in D such that for all successive points z_1, z, z_2 along γ, we have*

$$\frac{1}{b} \min_{j=1,2} d(z_j, w) \le d(z, w) \le b \max_{j=1,2} d(z_j, w) \tag{1.3}$$

for every $w \in \partial_r D$ (when D is unbounded we take $z_0 = \infty$), where b depends only on c and z_0 when D is bounded, and depends only on c when D is unbounded, and conversely, c depends only b.

Theorems 1.2 is a generalization of Theorems A and B stated in §2. This result may be regarded as a variation of results due to Broch which in turn are 'one sided' versions of a result for quasidisks by Gehring and Hag.

The proof of Theorems 1.2 is presented in §4.

2. A comparison

For the purpose of comparison of our main results with earlier known results, we need to recall a few of them that will be partly used in the discussion.

Theorem A (Theorem 3.4 of [3] or Theorem 4.3.8 of [2]). *Suppose that D is a Jordan domain of $\mathbb{C}$. Then the following conditions are equivalent.*

(1) *D is a c-John disk.*
(2) *There is a positive constant b such that for any $z_1, z_2 \in D_r$,*

$$\min_{j=1,2} |z_j - w| \le b\, |z - w|$$

for every $w \in \mathbb{C} \backslash D$ and every $z \in \gamma[z_1, z_2]$.

(3) *There is a positive constant a such that for any* $z_1, z_2 \in D_r$,

$$\min_{j=1,2} \lambda_D(z_j, w) \leq a\, \lambda_D(z, w)$$

for every $w \in \partial_r D$ *and every* $z \in \gamma[z_1, z_2]$.

The constants a, b *and* c *depend only on each other, and* $\gamma[z_1, z_2]$ *denotes the hyperbolic geodesic joining* z_1 *to* z_2.

By using the center of a John domain and the hyperbolic geodesics from the center to the boundary of D, Broch also obtained

Theorem B (Remark 4.3.12 of [2]). *Suppose that* $D \subset \mathbb{C}$ *is a bounded c-John disk with center* $z_0 \in D$. *If* z_1, z, z_2 *is an ordered triple of points on a hyperbolic geodesic* γ *from* z_0 *to* ∂D, *then* (1.1) *holds for every* $w \in \mathbb{C} \backslash D$, *where* $b_1 = b_1(c, z_0)$.

3. Proof of Theorem 1.1

In order to prove the theorem we need some preparations. It follows from §3.2 of [4] and Inequalities (1.2) of [8] that for each simply connected proper subdomain D of $\mathbb{C}$ and all points $z, z_1, z_2 \in D$,

$$\frac{1}{2} \frac{1}{\operatorname{dist}(z, \partial D)} \leq \rho_D(z) \leq \frac{2}{\operatorname{dist}(z, \partial D)}, \tag{3.1}$$

$$\frac{1}{2} h_D(z_1, z_2) \leq k_D(z_1, z_2) \leq 2 h_D(z_1, z_2) \tag{3.2}$$

and

$$k_D(z_1, z_2) \geq \left| \log \frac{\operatorname{dist}(z_1, \partial D)}{\operatorname{dist}(z_2, \partial D)} \right|. \tag{3.3}$$

Also, we remark that the inequalities between the hyperbolic and quasihyperbolic metrics are consequences of Schwarz lemma and Koebe's one quarter theorem. The lower estimate for the quasihyperbolic distance was first noticed by Gehring and Palka [9].

As in [9] (see also [2]), we define the metric $j_D(z_1, z_2)$ for $z_1, z_2 \in D$ by

$$j_D(z_1, z_2) = \log \left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} \right) \left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} \right).$$

Replacing the Euclidean distance by internal distance we obtain

$$j'_D(z_1, z_2) = \log \left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)} \right) \left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)} \right).$$

Lemma 3.1 (*Theorem* 4.1 *of* [12]). *A simply connected proper subdomain* D *of* $\mathbb{C}$ *is a c-John disk if and only if there exists a constant* $b \geq 1$ *such that*

$$h_D(z_1, z_2) \leq b\, j'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. *Here the constants b and c depend only on each other.*

In [10], Herron proved the following:

Lemma 3.2 (*Corollary of* [10]). *Let D be a bounded domain of $\mathbb{C}$. Then D is a c-John domain if and only if each point z in D can be joined to z_0 (the center of D) via an arc α in D with any one (all) of the following conditions holding for each point u in α:*

(1) $k_D(\alpha[z,u]) \le a \left|\log \dfrac{\operatorname{dist}(u,\partial D)}{\operatorname{dist}(z,\partial D)}\right| + b;$

(2) $k_D(\alpha[z,u]) \le a \log\left(1 + \dfrac{|z-u|}{\operatorname{dist}(z,\partial D)}\right) + b;$

(3) $k_D(\alpha[z,u]) \le a \log\left(1 + \dfrac{|z-u|}{\min\{\operatorname{dist}(z,\partial D), \operatorname{dist}(u,\partial D)\}}\right) + b.$

The various constants a depend only on c, $b = c(3+\log 2c)$, and conversely, c depends only on a, b and $\operatorname{diam}(D)/\operatorname{dist}(z_0,\partial D)$.

Elementary arguments reveal that the constants b in items (2), (3) of Lemma 3.2 are not necessary, but the one in (1) is not removable.

Lemma 3.3. *Suppose D is an unbounded simply connected proper subdomain of $\mathbb{C}$. Let $z_1 \in D$ and suppose that there is an arc γ joining z_1 to ∞ in D such that for any pair of points v_1 and $v_2 \in \gamma$,*

$$h_D(\gamma[v_1,v_2]) \le C\, a_D(v_1,v_2) + H.$$

Then, for each $z \in \gamma$, we have

$$|z_1 - z| \le a \operatorname{dist}(z,\partial D),$$

where the positive constant a depends only on C and H.

Proof. Let $z \in \gamma$. Clearly, we may assume $z_1 \neq z$. Now we divide the proof into two cases.

Case I. There exists a point $x_0 \in \gamma[z,\infty)$ such that

$$\operatorname{dist}(x_0,\partial D) = \sup_{p\in\gamma[z_1,x_0]} \operatorname{dist}(p,\partial D).$$

Then there is some integer $m \ge 0$ such that

$$2^m \operatorname{dist}(z_1,\partial D) \le \operatorname{dist}(x_0,\partial D) < 2^{m+1} \operatorname{dist}(z_1,\partial D).$$

Let y_0 be the first point in $\gamma[z_1,x_0]$ along the direction from z_1 to x_0 with

$$\operatorname{dist}(y_0,\partial D) = 2^m \operatorname{dist}(z_1,\partial D),$$

where if for any $x \in \gamma[z_1,x_0]\backslash\{z_1,x_0\}$,

$$\operatorname{dist}(x,\partial D) < \operatorname{dist}(x_0,\partial D) = \operatorname{dist}(z_1,\partial D)$$

or $\operatorname{dist}(x,\partial D) = \operatorname{dist}(x_0,\partial D) = \operatorname{dist}(z_1,\partial D)$, then we let $y_0 = z_1$.

Let $y_1 = z_1$ and choose points $y_2, \dots, y_{m+1} \in \gamma[z_1, x_0]$ so that y_i is the first point in $\gamma[z_1, x_0]$ such that

$$\operatorname{dist}(y_i, \partial D) = 2^{i-1} \operatorname{dist}(y_1, \partial D), \tag{3.4}$$

where, when $y_0 = z_1$, we let $y_1 = x_0$.

Then $y_{m+1} = y_0$. We let $y_{m+2} = x_0$ (may be $y_{m+2} = y_{m+1}$). Fix $i \in \{1, \dots, m+1\}$ and let

$$s_i = \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}.$$

If $v \in \gamma[y_i, y_{i+1}]$, then

$$\operatorname{dist}(v, \partial D) \le 2 \operatorname{dist}(y_i, \partial D).$$

We see from (3.1) that

$$\begin{aligned}
s_i &\le \int_{\gamma[y_i, y_{i+1}]} \frac{|dz|}{\operatorname{dist}(y_i, \partial D)} \\
&\le 4\, h_D(\gamma[y_i, y_{i+1}]) \\
&\le 4C\, a_D(y_i, y_{i+1}) + 4H \\
&\le 4C\, \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right)\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_{i+1}, \partial D)}\right) + 4H \\
&\le 8C\, \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right) + 4H.
\end{aligned}$$

If $s_i \le 1$, then

$$s_i \le 4\, h_D(\gamma[y_i, y_{i+1}]) \le 8C\, \log 2 + 4H.$$

If $s_i > 1$, then

$$\begin{aligned}
s_i &\le 4\, h_D(\gamma[y_i, y_{i+1}]) \\
&\le 8C\, \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right) + 4H \\
&\le 8C\, \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right) + 4H\, \log\left(1 + \frac{2|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right) \\
&\le 8C\, \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right) + 8H\, \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_i, \partial D)}\right) \\
&\le 16(C + H)\, (1 + s_i)^{1/2} \\
&\le c_1 (c_1^2 + 2)^{1/2} \quad (\text{since } (1/2) \log(1 + x) < (1 + x)^{1/2}),
\end{aligned}$$

where $c_1 = 16(C + H)$. This yields

$$|y_i - y_{i+1}| \le 2c_0 \operatorname{dist}(y_i, \partial D) \tag{3.5}$$

so that

$$s_i \leq 4\, h_D(\gamma[y_i, y_{i+1}]) \leq 2c_0,$$

where $c_0 = \frac{1}{2} c_1 (c_1^2 + 2)^{1/2}$.

Then for $u \in \gamma[y_i, y_{i+1}]$ $(i = 1, \ldots, m+1)$, (3.2) and (3.3) show that

$$\log \frac{\mathrm{dist}(y_{i+1}, \partial D)}{\mathrm{dist}(u, \partial D)} \leq 2\, h_D(\gamma[u, y_{i+1}]) \leq 2\, h_D(\gamma[y_i, y_{i+1}]) \leq c_0,$$

and thus

$$\mathrm{dist}(y_{i+1}, \partial D) \leq \exp(c_0)\, \mathrm{dist}(u, \partial D). \tag{3.6}$$

We may fix some $i_0 \in \{1, \ldots, m+1\}$ such that $z \in \gamma[y_{i_0}, y_{i_0+1}]$. Since

$$\begin{aligned}\frac{|y_{i_0} - z|}{\mathrm{dist}(y_{i_0}, \partial D)} &\leq \int_{\gamma[y_{i_0}, z]} \frac{|dz|}{\mathrm{dist}(y_{i_0}, \partial D)} \\ &\leq 4\, h_D(\gamma[y_{i_0}, z]) \leq 4\, h_D(\gamma[y_{i_0}, y_{i_0+1}]) \leq 2c_0,\end{aligned}$$

we see that

$$|y_{i_0} - z| \leq 2c_0\, \mathrm{dist}(y_{i_0}, \partial D). \tag{3.7}$$

Hence (3.4), (3.5) and (3.7) imply that

$$\begin{aligned}|z_1 - z| &\leq \sum_{i=1}^{i_0-1} |y_i - y_{i+1}| + |y_{i_0} - z| \\ &\leq 2c_0 \sum_{i=1}^{i_0} \mathrm{dist}(y_i, \partial D) \\ &\leq 2c_0\, 2^{i_0} \mathrm{dist}(y_1, \partial D).\end{aligned}$$

If $i_0 \in \{1, \ldots, m\}$, then (3.4) and (3.6) imply that

$$\begin{aligned}|z_1 - z| &\leq 2c_0\, \mathrm{dist}(y_{i_0+1}, \partial D) \\ &\leq 2c_0\, \exp(c_0) \mathrm{dist}(z, \partial D).\end{aligned}$$

If $i_0 = m + 1$, then (3.6) implies that

$$\begin{aligned}|z_1 - z| &\leq 4c_0\, 2^m \mathrm{dist}(y_1, \partial D) \\ &\leq 4c_0\, \mathrm{dist}(y_{m+2}, \partial D) \\ &\leq 4c_0\, \exp(c_0) \mathrm{dist}(z, \partial D).\end{aligned}$$

The above shows that for any $z \in \gamma$, we have

$$|z_1 - z| \leq 4c_0\, \exp(c_0) \mathrm{dist}(z, \partial D).$$

The proof is finished by taking $a = 4c_0 \exp(c_0)$.

Case II. There does not exist a point $y_0 \in \gamma[z, \infty)$ such that

$$\operatorname{dist}(y_0, \partial D) = \sup_{w \in \gamma[z_1, y_0]} \operatorname{dist}(w, \partial D).$$

Obviously, there exists a point $x_0 \in \gamma[z_1, z]$ which has maximal distance to ∂D, that is,

$$\operatorname{dist}(x_0, \partial D) = \sup_{p \in \gamma[z_1, z]} \operatorname{dist}(p, \partial D).$$

Then, we can find a point $z_2 \in \gamma[z, \infty)$ such that $|z_1 - z| = |z_2 - z|$ and

$$\operatorname{dist}(x_0, \partial D) = \sup_{w \in \gamma[z_2, x_0]} \operatorname{dist}(w, \partial D).$$

Similar reasoning as in the proof of Case I implies that

$$|z_1 - z| = |z_2 - z| \leq a \operatorname{dist}(z, \partial D),$$

where $a = 4c_0 \exp(c_0)$. □

Proof of Theorem 1.1. First we prove the necessary part. Assume D is a c-John disk with center z_0. Let $z \in D$, γ be a c'-carrot arc joining z_0 to z in D, and let $z_1, z_2 \in \gamma$. We may assume that

$$\min_{j=1,2} \operatorname{dist}(z_j, \partial D) = \operatorname{dist}(z_1, \partial D).$$

We present the proof into two cases.

Case I. $|z_1 - z_2| \geq 3\operatorname{dist}(z_1, \partial D)$.

Pick $w \in \partial D$ with $\operatorname{dist}(z_1, \partial D) = |z_1 - w|$. Then

$$|z_2 - w| \geq |z_1 - z_2| - |z_1 - w| \geq \frac{1}{3}|z_1 - z_2| + |z_1 - w|$$

which yields

$$\begin{aligned} a_D(z_1, z_2) &\geq \log \frac{|z_2 - w|}{|z_1 - w|} \\ &\geq \log\left(1 + \frac{|z_1 - z_2|}{3\operatorname{dist}(z_1, \partial D)}\right) \\ &\geq \frac{1}{3} \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)}\right). \end{aligned}$$

We have

$$\begin{aligned} \lambda_D(z_1, z_2) &\leq \ell(\gamma[z_1, z_2]) \\ &\leq c' \operatorname{dist}(z_2, \partial D) \\ &\leq \log c' \left(|z_1 - z_2| + \operatorname{dist}(z_1, \partial D)\right) \\ &\leq \frac{4}{3} c' |z_1 - z_2| \end{aligned}$$

and for any $z \in \gamma[z_1, z_2]$,

$$\mathrm{dist}(z, \partial D) \geq \frac{\mathrm{dist}(z_1, \partial D) + \ell(\gamma[z_1, z])}{4c'},$$

which by Lemma 3.1 implies that

$$\begin{aligned} h_D(\gamma[z_1, z_2]) &\leq 4c' \log\left(1 + \frac{\ell(\gamma[z_1, z_2])}{\mathrm{dist}(z_1, \partial D)}\right) \\ &\leq \frac{16(c')^2}{3} \log\left(1 + \frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right) \\ &\leq \frac{16(c')^2}{3} h_D(z_1, z_2) \\ &\leq \frac{16b(c')^2}{3}\, j'_D(z_1, z_2) \\ &\leq \frac{32b(c')^2}{3} \log\left(1 + \frac{4c'|z_1 - z_2|}{3\mathrm{dist}(z_1, \partial D)}\right) \\ &\leq \frac{128b(c')^2(c'+1)}{9} \log\left(1 + \frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right) \\ &\leq c_2\, a_D(z_1, z_2), \end{aligned}$$

where $c' = c'(c)$ and $c_2 = \frac{128b(c')^2(c'+1)}{3}$.

Case II. $|z_1 - z_2| < 3\mathrm{dist}(z_1, \partial D)$.

It follows from Lemma 3.1 and

$$\begin{aligned} \ell(\gamma[z_1, z_2]) &\leq c'\mathrm{dist}(z_2, \partial D) \\ &\leq 4c'\mathrm{dist}(z_1, \partial D) \end{aligned}$$

that

$$\begin{aligned} h_D(\gamma[z_1, z_2]) &\leq 4c' \log(1 + \frac{\ell(\gamma[z_1, z_2])}{\mathrm{dist}(z_1, \partial D)}) \\ &\leq h_D(z_1, z_2) + 4c' \log(1 + 4c') \\ &\leq 2b \log\left(1 + \frac{4c'|z_1 - z_2|}{3\mathrm{dist}(z_1, \partial D)}\right) \\ &\leq 2b \log(1 + 4c') + 4c' \log(1 + 4c') \\ &\leq a_D(z_1, z_2) + (2b + 4c') \log(1 + 4c'). \end{aligned}$$

For the proof of the sufficiency part, we need to deal with two cases one for which D is bounded and the other for which D is unbounded.

First assume that D is bounded. In this case, take $z_1 \in D$ and let γ be an arc joining z_0 to z_1 in D satisfying (1.2). For any $z \in \gamma$, we have

$$\begin{aligned}
k_D(\gamma[z_1, z]) &\leq 2\, h_D(\gamma[z_1, z]) \\
&\leq 2C\, a_D(z_1, z) + 2H \\
&\leq 2C\, \log\left(1 + \frac{|z_1 - z|}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{|z_1 - z|}{\operatorname{dist}(z, \partial D)}\right) + 2H \\
&\leq 4C\, \log\left(1 + \frac{|z_1 - z|}{\min\{\operatorname{dist}(z_1, \partial D), \operatorname{dist}(z, \partial D)\}}\right) + 2H.
\end{aligned}$$

By Lemma 3.2, we see that D is a John disk.

Next we consider the case where D is unbounded. Let $\alpha = [\zeta_1, \zeta_2]$ be a straight crosscut of D with $\zeta_1, \zeta_2 \in \partial D$, where D_1, D_2 are two components of $D \backslash \alpha$. We may assume that $\min_{j=1,2} \operatorname{diam}(D_j) = \operatorname{diam}(D_1)$. Take $w \in \partial D_1 \backslash \alpha$ and $z_1 \in D_1$ with $|w - z_1| < |\zeta_1 - \zeta_2|$. Let γ be an arc joining z_1 to ∞ in D satisfying (1.2) and $z \in \gamma \cap \alpha$.

Lemma 3.3 implies that

$$\begin{aligned}
|\zeta_1 - w| &\leq |w - z_1| + |\zeta_1 - z_1| \\
&\leq |\zeta_1 - \zeta_2| + |z_1 - z| + |\zeta_1 - z| \\
&\leq |\zeta_1 - \zeta_2| + a\, \operatorname{dist}(z, \partial D) + |\zeta_1 - z| \\
&\leq |\zeta_1 - \zeta_2| + \frac{a}{2}\, |\zeta_1 - \zeta_2| + |\zeta_1 - \zeta_2| \\
&\leq (2 + (a/2))\, \operatorname{diam}(\alpha).
\end{aligned}$$

This yields $\operatorname{diam}(D_1) \leq (4 + a)\operatorname{diam}(\alpha)$, which shows that D is a crosscut domain. By Theorem 4.5 of [14], D is a John disk. □

4. Proof of Theorem 1.2

The following lemma is crucial for the proof of Theorem 1.2.

Lemma 4.1. *Suppose that D is a bounded simply connected domain in $\mathbb{C}$. Then D is a c-John disk with center z_0 if and only if there exists a positive constant c_1 such that each point z_1 in D can be joined to z_0 via an arc γ in D such that for any $z \in \gamma$,*

$$\operatorname{dist}(z, \partial D) \geq \frac{1}{c_1}\, |z - z_1|. \tag{4.1}$$

The constants c_1 and c depend on each other only.

The proof of Lemma 4.1 easily follows from Theorem 2.16 of [14].

Proof of Theorem 1.2. First we prove the necessity. Assume that D is a c-John disk with center z_0. Take $x \in D$.

If D is bounded, then let γ be a hyperbolic geodesic joining z_0 to x in D, and z_1, z, z_2 be any successive points in D. For any $w \in \partial D$, it follows from the proof of Theorem B (cf. [2]) that

$$\begin{aligned}\min_{j=1,2} d(z_j, w) &\leq \min_{j=1,2} d(z_j, z) + d(z, w)\\ &\leq c' \operatorname{dist}(z, \partial D) + d(z, w)\\ &\leq (c' + 1)\, d(z, w).\end{aligned}$$

We also get

$$\begin{aligned}d(z, w) &\leq \ell(\gamma[z_1, z_2]) + \max_{j=1,2} d(z_j, w)\\ &\leq c' \max_{j=1,2} \operatorname{dist}(z_j, \partial D) + \max_{j=1,2} d(z_j, w)\\ &\leq (c' + 1) \max_{j=1,2} d(z_j, w).\end{aligned}$$

If D is unbounded, then $z_0 = \infty$. Let γ be a c'-carrot arc joining z_0 to x in γ and z_1, z, z_2 be any successive points in γ. For any $w \in \partial D$, we have

$$\begin{aligned}\min_{j=1,2} d(z_j, w) &\leq \min_{j=1,2} \ell(\gamma[z_j, z]) + d(z, w)\\ &\leq c' \operatorname{dist}(z, \partial D) + d(z, w)\\ &\leq (c' + 1)\, d(z, w).\end{aligned}$$

We easily get

$$\begin{aligned}d(z, w) &\leq \frac{1}{2}(d(z_1, w) + d(z_1, z) + d(z_2, w) + d(z_2, z))\\ &\leq \frac{1}{2}(d(z_1, w) + d(z_2, w) + \ell(\gamma[z_1, z_2]))\\ &\leq \frac{1}{2}(d(z_1, w) + d(z_2, w) + c' \max_{j=1,2} \operatorname{dist}(z_j, \partial D))\\ &\leq \frac{1}{2}(d(z_1, w) + d(z_2, w) + c' \max_{j=1,2} d(z_j, w))\\ &\leq \frac{1}{2}(c' + 2) \max_{j=1,2} d(z_j, w).\end{aligned}$$

Now we proceed to prove the sufficiency. We divide the proof into two cases.

Case I. D is bounded.

Let $z_1 \in D$ and γ be the arc joining z_0 to z_1 in D satisfying (1.3). Take $z \in \gamma[z_0, z_1]$, $w \in \partial D$ with $\operatorname{dist}(z, \partial D) = |z - w|$. Since $w \in \partial_r D$, by (1.3), we have

$$\begin{aligned}\min_{j=0,1} d(z_j, z) &\leq \min_{j=0,1} d(z_j, w) + d(z, w)\\ &\leq b\, d(z, w) + d(z, w)\\ &= (b + 1) \operatorname{dist}(z, \partial D).\end{aligned}$$

Subcase 1. $\min_{j=0,1} d(z_j, z) = d(z_1, z)$.

Then it follows that

$$\mathrm{dist}(z, \partial D) \geq \frac{d(z_1, z)}{b+1} \geq \frac{1}{b+1}|z_1 - z|.$$

Subcase 2. $\min_{j=0,1} d(z_j, z) = d(z_0, z)$.

It follows from the inequality

$$\mathrm{dist}(z, \partial D) \geq \mathrm{dist}(z_0, \partial D) - d(z_0, z)$$

that

$$\mathrm{dist}(z, \partial D) \geq \frac{\mathrm{dist}(z_0, \partial D)}{b+2} \geq \frac{1}{c_2}|z_1 - z|,$$

where $c_2 = (b+2)\mathrm{diam}(D)/\mathrm{dist}(z_0, \partial D)$.

Consequently, Subcases 1 and 2 show that for any $z \in \gamma$,

$$\mathrm{dist}(z, \partial D) \geq \frac{1}{c_2}|z_1 - z|.$$

Thus Lemma 4.1 shows that D is a c-John disk ($c = c(c_2)$).

Case II. D is unbounded.

Let $\alpha = [\zeta_1, \zeta_2]$ be a straight crosscut of D with $\zeta_1, \zeta_2 \in \partial D$, and D_1, D_2 be the two components of $D \backslash \alpha$. Then $\zeta_1,\ \zeta_2 \in \partial_r D$. We may assume that $\min_{j=1,2} \mathrm{diam}(D_j) = \mathrm{diam}(D_1)$. For any $z_1 \in (\partial_r D_1 \backslash \alpha) \cup \{\zeta_1,\ \zeta_2\}$, choose $z_2 \in D_1$ with $d(z_2, z_1) < \frac{|\zeta_1 - \zeta_2|}{2}$. Let γ be an arc joining z_2 to ∞ in D satisfying (1.3), and let $z' \in \gamma \bigcap \alpha$.

By (1.3), we then have

$$\begin{aligned} d(z_1, \zeta_1) &\leq d(z_2, z_1) + d(z_2, \zeta_1) \\ &\leq d(z_2, z_1) + b\, d(z', \zeta_1) \\ &\leq d(z_2, z_1) + b\, |\zeta_1 - \zeta_2| \\ &\leq \left(b + \frac{1}{2}\right) |\zeta_1 - \zeta_2|. \end{aligned}$$

It follows that for any positive number ϵ, there exist two points $z_{1\epsilon}$ and $z_{2\epsilon} \in (\partial_r D_1 \backslash \alpha) \cup \{\zeta_1,\ \zeta_2\}$ such that

$$\begin{aligned} \mathrm{diam}(D_1) &\leq d(\zeta_1, z_{1\epsilon}) + d(\zeta_1, z_{2\epsilon}) + \epsilon \\ &\leq (2b+1)\, |\zeta_1 - \zeta_2| + \epsilon \\ &= (2b+1)\mathrm{diam}(\alpha) + \epsilon. \end{aligned}$$

The arbitrariness of ϵ shows that

$$\mathrm{diam}(D_1) \leq (2b+1)\mathrm{diam}(\alpha),$$

which implies that D is a crosscut domain. Hence D is a c-John disk ($c = c(b)$). □

Acknowledgements

This research was partly supported by NSF of China (No. 10771059). The first author thanks 'INSA-JRD Tata Fellowship' and the second and the third authors thank the National Board for High Mathematics, India, and the Commission on Development and Exchanges (IMU) for their support. The authors thank the referee for valuable comments for the improvement of the presentation of the article.

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