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A heat kernel version of Cowling–Price theorem for the Laguerre hypergroup

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Abstract. In this paper, we prove a heat kernel version of Cowling–Price theorem for the Laguerre hypergroup.

Keywords. Laguerre hypergroup; uncertainty principle; Cowling–Price theorem.

1. Introduction

The uncertainty principle says that a function and its Fourier transform cannot simultaneously decay very rapidly at infinity. A classical version of uncertainty principle, known as Hardy's theorem, was first proved by Hardy [6] on $\mathbb{R}$. Several generalizations of the Hardy's theorem have appeared since, one of these is the Cowling–Price theorem, which was first proved by Cowling and Price in [4]. The following version of Cowling–Price theorem on $\mathbb{R}^n$ can be found in [2] or [12].

Theorem 1. *Suppose $f \in L^2(\mathbb{R}^n)$ and satisfies*

$$\left(\int_{\mathbb{R}^n} \left(\frac{|f(x)|\mathrm{e}^{ax^2}}{(1+|x|)^N}\right)^p \mathrm{d}x\right)^{1/p} < \infty \quad \text{and}$$

$$\left(\int_{\mathbb{R}^n} \left(\frac{|\hat{f}(y)|\mathrm{e}^{by^2}}{(1+|y|)^N}\right)^q \mathrm{d}y\right)^{1/q} < \infty,$$

where $a, b > 0$, $N \geq 0$ and $1 \leq p, q \leq \infty$. Then $f = 0$ whenever $ab > \frac{1}{4}$ and when $ab = \frac{1}{4}$, $f(x) = P(x)\,\mathrm{e}^{-a|x|^2}$ where P is a polynomial of degree $\leq N - n/p$.

The Cowling–Price theorem has been extended to various settings. More results can be found in the book [15] by Thangavelu and the references therein.

We note that the heat kernel h_s on $\mathbb{R}^n$ is given by

$$h_s(x) = (4\pi s)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|x|^2}{4s}}, \quad \hat{h}_s(y) = \mathrm{e}^{-s|y|^2}.$$

Thus Cowling–Price theorem is well explained in terms of the heat kernel. In view of this point, Parui and Thangavelu [11] proved a heat kernel version of Cowling–Price theorem for the nilpotent Lie group. For the Heisenberg group, their result is remarkable because the heat kernel with respect to the sub-Laplacian on the Heisenberg group decays as $\mathrm{e}^{-a|t|}$ much slower than $\mathrm{e}^{-a|t|^2}$ where the central variable t is concerned. In this paper, we will prove a heat kernel version of Cowling–Price theorem for the Laguerre hypergroup.

Throughout the paper, we will use C to denote the positive constant, which is not necessarily same at each occurrence.

2. Preliminaries

In this section, we give some notations and collect some basic results about the Laguerre hypergroup. For more about the Laguerre hypergroup we refer the reader to [7], [10], [13] and [16]. We also give Hardy's theorem for the Hankel transform, which we will use in the sequel.

Given $\alpha \geq 0$, let $\mathbb{K} = [0, \infty) \times \mathbb{R}$ equipped with the measure

$$\mathrm{d}m_\alpha(x, t) = \frac{1}{\pi\,\Gamma(\alpha+1)}\, x^{2\alpha+1}\mathrm{d}x\mathrm{d}t.$$

We simply write $L^p(\mathbb{K})$ instead of $L^p(\mathbb{K}, \mathrm{d}m_\alpha)$.

For $(x, t) \in \mathbb{K}$, the generalized translation operators $T^{(\alpha)}_{(x,t)}$ are defined by (cf. [10] or [13])

$$T^{(\alpha)}_{(x,t)} f(y, s) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta}, s+t+xy\sin\theta)\mathrm{d}\theta, \\ \quad \text{if } \alpha = 0, \\ \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f(\sqrt{x^2+y^2+2xyr\cos\theta}, s+t+xyr\sin\theta) \\ \quad r(1-r^2)^{\alpha-1}\,\mathrm{d}r\mathrm{d}\theta, \quad \text{if } \alpha > 0. \end{cases}$$

Let $M_b(\mathbb{K})$ denote the space of bounded Radon measures on $\mathbb{K}$. The convolution on $M_b(\mathbb{K})$ is defined by

$$(\mu * \nu)(f) = \int_{\mathbb{K}\times\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y, s)\,\mathrm{d}\mu(x, t)\mathrm{d}\nu(y, s).$$

It is easy to see that $\mu * \nu = \nu * \mu$. If $f, g \in L^1(\mathbb{K})$ and $\mu = f m_\alpha$, $\nu = g m_\alpha$, then $\mu * \nu = (f * g)m_\alpha$, where $f * g$ is the convolution of functions f and g defined by

$$(f * g)(x, t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y, s) g(y, -s)\,\mathrm{d}m_\alpha(y, s).$$

$(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett (cf. [9], [1]), where i denotes the involution defined by $i(x, t) = (x, -t)$. If $\alpha = n - 1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathbb{H}^n$.

The dilations on $\mathbb{K}$ are defined by

$$\delta_r(x, t) = (rx, r^2 t), \qquad r > 0.$$

It is clear that the dilations are consistent with the structure of hypergroup. Let

$$(\delta_r f)(x, t) = r^{-(2\alpha+4)} f\left(\frac{x}{r}, \frac{t}{r^2}\right).$$

Then we have

$$\|(\delta_r f)\|_{L^1(\mathbb{K})} = \|f\|_{L^1(\mathbb{K})}.$$

Let us consider the partial differential operator

$$\mathcal{L} = -\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}\right).$$

$\mathcal{L}$ is positive and symmetric in $L^2(\mathbb{K})$, and is homogeneous of degree 2 with respect to the dilations defined above. When $\alpha = n-1$, $\mathcal{L}$ is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$. We call $\mathcal{L}$ the generalized sublaplacian.

Let $L_m^{(\alpha)}$ be the Laguerre polynomial of degree m and order α defined in terms of the generating function by

$$\sum_{m=0}^{\infty} s^m L_m^{(\alpha)}(x) = \frac{1}{(1-s)^{\alpha+1}} \exp\left(-\frac{xs}{1-s}\right).$$

Set

$$\varphi_m^{(\alpha)}(x) = \mathrm{e}^{-\frac{x^2}{2}} L_m^{(\alpha)}(x^2). \tag{1}$$

The proof of the following Lemma can be found in [14].

Lemma 1. *For any* $\lambda \neq 0$, *the system*

$$\left\{\left(\frac{2|\lambda|^{\alpha+1} m!}{\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} \varphi_m^{(\alpha)}(\sqrt{|\lambda|}x) : m \in \mathbb{N}\right\}$$

forms an orthonormal basis of the space $L^2([0,\infty),\ x^{2\alpha+1}\mathrm{d}x)$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, we put

$$\psi_{(\lambda,m)}(x, t) = \frac{m!\,\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} \mathrm{e}^{i\lambda t}\, \varphi_m^{(\alpha)}\big(\sqrt{|\lambda|}x\big).$$

Lemma 2. *The functions* $\psi_{(\lambda,m)}$ *satisfy that*

(a) $\|\psi_{(\lambda,m)}\|_{L^\infty} = \psi_{(\lambda,m)}(0,0) = 1$,
(b) $T_{(x,t)}^{(\alpha)} \psi_{(\lambda,m)}(y, s) = \psi_{(\lambda,m)}(x, t)\, \psi_{(\lambda,m)}(y, s)$,
(c) $\mathcal{L}\psi_{(\lambda,m)} = 2|\lambda|(2m+\alpha+1)\psi_{(\lambda,m)}$.

Let $f \in L^1(\mathbb{K})$, the generalized Fourier transform of f is defined by

$$\hat{f}(\lambda, m) = \int_{\mathbf{K}} f(x, t)\, \psi_{(-\lambda,m)}(x, t)\, \mathrm{d}m_\alpha(x, t).$$

We note that

$$\hat{f}(\lambda, m) = \frac{m!}{\pi\,\Gamma(m+\alpha+1)} \int_0^\infty f^\lambda(x)\,\varphi_m^{(\alpha)}\big(\sqrt{|\lambda|}\,x\big)\,x^{2\alpha+1}\mathrm{d}x \tag{2}$$

where

$$f^\lambda(x) = \int_{-\infty}^{\infty} f(x,t)\,\mathrm{e}^{-i\lambda t}\,\mathrm{d}t$$

is the Fourier transform of $f(x, t)$ in the t-variable.

Let $\mathrm{d}\gamma_\alpha$ be the positive measure defined on $\mathbb{R} \times \mathbb{N}$ by

$$\int_{\mathbb{R}\times\mathbb{N}} g(\lambda, m)\,\mathrm{d}\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)} \int_{\mathbb{R}} g(\lambda, m)|\lambda|^{\alpha+1}\,\mathrm{d}\lambda.$$

Write $L^p(\hat{\mathbb{K}})$ instead of $L^p(\mathbb{R}\times\mathbb{N}, \mathrm{d}\gamma_\alpha)$. We have the following Plancherel formula.

$$\|f\|_{L^2(\mathbb{K})} = \|\hat{f}\|_{L^2(\hat{\mathbb{K}})}, \qquad f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K}).$$

We also have the inverse formula of the generalized Fourier transform.

$$f(x,t) = \int_{\mathbb{R}\times\mathbb{N}} \hat{f}(\lambda, m)\,\psi_{(\lambda,m)}(x,t)\,\mathrm{d}\gamma_\alpha(\lambda, m)$$

provided $\hat{f} \in L^1(\hat{\mathbb{K}})$.

Let $\{H^s : s > 0\} = \{\mathrm{e}^{-s\mathcal{L}} : s > 0\}$ be the heat semigroup generated by $\mathcal{L}$. There is an unique smooth function $h((x,t),s) = h_s(x,t)$ on $\mathbb{K}\times(0,+\infty)$ such that

$$H^s f(x,t) = f * h_s(x,t).$$

h_s is called the heat kernel associated to $\mathcal{L}$.

By the definition of the generalized Fourier transform and Lemma 2, it is easy to show that

$$\widehat{\delta_r f}(\lambda, m) = \hat{f}(r^2\lambda, m),$$

$$\widehat{\mathcal{L}f}(\lambda, m) = 2|\lambda|(2m+\alpha+1)\hat{f}(\lambda, m),$$

$$(f * g)^\wedge(\lambda, m) = \hat{f}(\lambda, m)\,\hat{g}(\lambda, m).$$

Therefore,

$$\hat{h}_s(\lambda, m) = \mathrm{e}^{-2|\lambda|(2m+\alpha+1)s},$$

$$h_{s_1} * h_{s_2} = h_{s_1+s_2},$$

$$h_s(x,t) = s^{-(\alpha+2)} h_1\left(\frac{x}{\sqrt{s}}, \frac{t}{s}\right).$$

Although the heat kernel $h_s(x,t)$ is not explicitly known, we do have the explicit expression of $h_s^\lambda(x)$, from which the estimate for $h_s(x,t)$ is obtained (cf. [7]).

Lemma 3.

$$h_s^\lambda(x) = 2\pi \left(\frac{\lambda}{2\sinh(2\lambda s)}\right)^{\alpha+1} \mathrm{e}^{-\frac{1}{2}\lambda\coth(2\lambda s)x^2}.$$

Lemma 4. *There exists* $A > 0$ *such that*

$$0 < h_s(x,t) \le C\, s^{-(\alpha+2)} \mathrm{e}^{-\frac{A}{s}(|x|^2+|t|)}.$$

Now we turn to the Hankel transform. For $z \in \mathbb{C}$, the Bessel function of first kind and order α is defined by

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-\alpha-2k} z^{\alpha+2k}}{\Gamma(k+1)\Gamma(k+\alpha+1)} = \frac{2^{-\alpha} z^\alpha}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} \mathrm{e}^{izs}(1-s^2)^{\alpha-\frac{1}{2}} \mathrm{d}s.$$

In this paper we only concern about the case $\alpha \ge 0$ although $J_\alpha(z)$ are well defined for all $\alpha \in \mathbb{C}$. We refer the reader to Watson's book [17] for the reference about the Bessel function. Here we point out that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{-\alpha} J_\alpha(z)) = -z^{-\alpha} J_{\alpha+1}(z), \tag{3}$$

$$|J_\alpha(it)| \le C\, t^{-\frac{1}{2}} \mathrm{e}^t, \quad t > 0. \tag{4}$$

Then the Hankel transform of order α of $f \in L^1([0,\infty), x^{2\alpha+1}\mathrm{d}x)$ is defined by

$$(\mathcal{H}_\alpha f)(y) = \int_0^\infty f(x) \frac{J_\alpha(xy)}{(xy)^\alpha}\, x^{2\alpha+1} \mathrm{d}x.$$

The functions $\varphi_m(x)$ defined by (1) are the eigenfunctions of the Hankel transform, i.e.,

$$(\mathcal{H}_\alpha \varphi_m)(x) = (-1)^m \varphi_m(x) \tag{5}$$

(cf. p. 42 of [5]). It follows that $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$. Also we have

$$(\mathcal{H}_\alpha f(r\,\cdot))(y) = r^{-2(\alpha+1)} (\mathcal{H}_\alpha f)\left(\frac{y}{r}\right), \quad r > 0. \tag{6}$$

We state Cowling–Price theorem for the Hankel transform as follows.

PROPOSITION 1

Suppose $f \in L^2([0,\infty),\ x^{2\alpha+1}\mathrm{d}x)$ *and satisfies*

$$\|f(x)\mathrm{e}^{ax^2}\|_p < \infty \quad \textit{and} \quad \|(\mathcal{H}_\alpha f)(y)\mathrm{e}^{by^2}\|_q < \infty,$$

where a, b *are positive constants and* $1 \le p, q \le \infty$. *Then* $f(x) = 0$ *whenever* $ab > \frac{1}{4}$.

In [3], the authors proved Beurling's theorem for the Chébli–Trimèche transform. We can get Cowling–Price theorem for the Chébli–Trimèche transform by Beurling's theorem, the proof of this is standard (cf. [15]). Proposition 1 is a special case of Cowling–Price theorem for the Chébli–Trimèche transform.

3. Cowling–Price theorem in terms of heat kernel on $\mathbb{K}$

In this section, we can prove the main result of this paper.

Theorem 2. *Suppose $f \in L^2(\mathbb{K})$ and satisfies*

$$\left(\int_{\mathbb{K}} |f(x,t)h_a^{-1}(x,t)|^p \mathrm{d}m_\alpha(x,t)\right)^{1/p} < \infty, \tag{7}$$

$$\left(\int_{\mathbb{R}\times\mathbb{N}} |\hat{f}(\lambda,m)\mathrm{e}^{2|\lambda|b(2m+a+1)}|^q \mathrm{d}\gamma_\alpha(\lambda,m)\right)^{1/q} < \infty, \tag{8}$$

where a, b are positive constants and $1 \le p, q \le \infty$.

Then, if $a < b$, we have $f(x,t) = 0$, a.e. $(x,t) \in \mathbb{K}$.

Proof. When $p = q = \infty$, Theorem 2 reduces to Hardy's theorem which has been proved in [8]. In the following, we assume $\min\{p,q\} < \infty$.

By (7), we have

$$\left(\int_{\mathbb{R}} |f(x,t)h_a^{-1}(x,t)|^p \mathrm{d}t\right)^{1/p} < \infty, \quad \text{a.e.} \quad x \in [0,\infty).$$

Therefore, by Lemma 4, for $\lambda \in \left\{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| < \frac{1}{4a}\right\}$ and almost all $x \in [0,\infty)$, we get

$$\begin{aligned}\int_{\mathbb{R}} |f(x,t)|\mathrm{e}^{|\mathrm{Im}\,\lambda||t|}\mathrm{d}t &= \int_{\mathbb{R}} |f(x,t)h_a^{-1}(x,t)|h_a(x,t)\mathrm{e}^{|\mathrm{Im}\,\lambda||t|}\mathrm{d}t \\ &\le \left(\int_{\mathbb{R}} |f(x,t)h_a^{-1}(x,t)|^p \mathrm{d}t\right)^{1/p} \left(\int_{\mathbb{R}} \left(h_a(x,t)\mathrm{e}^{|\mathrm{Im}\,\lambda||t|}\right)^{p'} \mathrm{d}t\right)^{1/p'} \\ &< \infty.\end{aligned}$$

Therefore, for almost all $x \in [0,\infty)$, $f^\lambda(x)$ can be extended to a holomorphic function of λ in the strip $\left\{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| < \frac{1}{4a}\right\}$.

Choose $\delta > 0$, such that $b < \frac{1}{2\delta}$. Then, we will prove that $f^\lambda(x) = 0$, for $0 < \lambda < \delta$. This means that $f^\lambda(x) = 0$ for all $\lambda \in \mathbb{R}$, therefore $f(x,t) = 0$, a.e. $(x,t) \in \mathbb{K}$.

By Lemma 4 and (7), we have

$$\begin{aligned}&\left(\int_0^\infty |f^\lambda(x)\mathrm{e}^{\frac{x^2}{4a}}|^p x^{2\alpha+1}\mathrm{d}x\right)^{1/p} \\ &\le \left(\int_0^\infty \left|\left(\int_{\mathbb{R}} |f(x,t)|\mathrm{d}t\right)\mathrm{e}^{\frac{x^2}{4a}}\right|^p x^{2\alpha+1}\mathrm{d}x\right)^{1/p} \\ &\le \int_{\mathbb{R}} \left(\int_0^\infty \left|f(x,t)\mathrm{e}^{\frac{x^2}{4a}}\right|^p x^{2\alpha+1}\mathrm{d}x\right)^{1/p} \mathrm{d}t \\ &= \int_{\mathbb{R}} \left(\int_0^\infty |f(x,t)h_a^{-1}(x,t)h_a(x,t)\mathrm{e}^{\frac{x^2}{4a}}|^p x^{2\alpha+1}\mathrm{d}x\right)^{1/p} \mathrm{d}t\end{aligned}$$

$$\leq C\int_{\mathbb{R}}\left(\int_0^\infty |f(x,t)h_a^{-1}(x,t)\mathrm{e}^{-\frac{|t|}{4a}}|^p x^{2\alpha+1}\mathrm{d}x\right)^{1/p}\mathrm{d}t$$

$$= C\int_{\mathbb{R}}\left(\int_0^\infty |f(x,t)h_a^{-1}(x,t)|^p x^{2\alpha+1}\mathrm{d}x\right)^{1/p}\mathrm{e}^{-\frac{|t|}{4a}}\mathrm{d}t$$

$$\leq C\int_{\mathbb{R}}\left(\int_0^\infty |f(x,t)h_a^{-1}(x,t)|^p x^{2\alpha+1}\mathrm{d}x\mathrm{d}t\right)^{1/p}\left(\int_{\mathbb{R}}\mathrm{e}^{-\frac{p'|t|}{4a}}\mathrm{d}t\right)^{1/p'}$$

$$< \infty. \tag{9}$$

By (8) and Hölder inequality, for any $0 < \epsilon < 1$, we have

$$\int_{\mathbb{R}\times\mathbb{N}} |\hat{f}(\lambda,m)\mathrm{e}^{2\epsilon b|\lambda|(2m+\alpha+1)}|\mathrm{d}\gamma_\alpha(\lambda,m) < \infty,$$

i.e.,

$$\sum_{m=0}^{\infty}\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}\int_{\mathbb{R}}|\hat{f}(\lambda,m)\mathrm{e}^{2\epsilon b|\lambda|(2m+\alpha+1)}||\lambda|^{\alpha+1}\mathrm{d}\lambda < \infty. \tag{10}$$

Since $f \in L^2(\mathbb{K})$, we get $f^\lambda(x) \in L^2([0,\infty), x^{2\alpha+1}\mathrm{d}x)$. Therefore, by Lemma 1, we can get

$$f^\lambda(x) = 2\pi|\lambda|^{\alpha+1}\sum_{m=0}^{\infty}\hat{f}(\lambda,m)\varphi_m^{(\alpha)}(|\lambda|^{1/2}x).$$

Taking Hankel transform on the both sides of the above equality, we obtain from (6),

$$(\mathcal{H}_\alpha f^\lambda)(y) = 2\pi\sum_{m=0}^{\infty}\hat{f}(\lambda,m)(-1)^m\varphi_m^{(\alpha)}(|\lambda|^{-1/2}y).$$

Therefore

$$\left(\int_0^\infty |(\mathcal{H}_\alpha f^\lambda)(y)\mathrm{e}^{by^2}|^q y^{2\alpha+1}\mathrm{d}y\right)^{1/q}$$

$$\leq C\sum_{m=0}^{\infty}|\hat{f}(\lambda,m)|\left(\int_0^\infty |\varphi_m^{(\alpha)}(|\lambda|^{-1/2}y)\mathrm{e}^{by^2}|^q y^{2\alpha+1}\mathrm{d}y\right)^{1/q}$$

$$= C|\lambda|^{(\alpha+1)/q}\sum_{m=0}^{\infty}|\hat{f}(\lambda,m)|\left(\int_0^\infty |\varphi_m^{(\alpha)}(y)\mathrm{e}^{b|\lambda|y^2}|^q y^{2\alpha+1}\mathrm{d}y\right)^{1/q}.$$

By $0 < \lambda < \delta$ and $b < \frac{1}{2\delta}$, we know $b\lambda < b\delta < \frac{1}{2}$. Then, we have

$$\left(\int_0^\infty |(\mathcal{H}_\alpha f^\lambda)(y)\mathrm{e}^{by^2}|^q y^{2\alpha+1}\mathrm{d}y\right)^{1/q}$$

$$\leq C\,\lambda^{(\alpha+1)/q}\sum_{m=0}^{\infty}|\hat{f}(\lambda,m)|\left(\int_0^\infty |\mathrm{e}^{-\left(\frac{1}{2}-b\lambda\right)y^2}L_m^{(\alpha)}(y^2)|^q y^{2\alpha+1}\mathrm{d}y\right)^{1/q}$$

$$\leq C\,\delta^{(\alpha+1)/q}\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}|\hat{f}(\lambda,m)|\mathrm{e}^{2\epsilon b|\lambda|(2m+\alpha+1)}\right).$$

By (10) and the fact that $\hat{f}(\lambda, m)$ is continuous with λ, for $0 \le \lambda \le \delta$, we have

$$\sum_{m=0}^{\infty} \left(\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)} |\hat{f}(\lambda, m)| \mathrm{e}^{2\epsilon b|\lambda|(2m+\alpha+1)} \right) < \infty.$$

Therefore

$$\left(\int_0^{\infty} |(\mathcal{H}_\alpha f^\lambda)(y) \mathrm{e}^{by^2}|^q y^{2\alpha+1} \mathrm{d}y \right)^{1/q} < \infty. \tag{11}$$

By (9), (11) and Proposition 1, we get $f^\lambda(x) = 0$, a.e. $x \in [0, \infty)$ for $0 < \lambda < \delta$ and $a < b$.

This completes the proof of Theorem 2. □

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