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A note on the tangent bundle of G/P

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Abstract. Let P be a parabolic subgroup of a complex simple linear algebraic group G. We prove that the tangent bundle $T(G/P)$ is stable.

Keywords. Homogeneous space; tangent bundle; stability.

1. Introduction

Let G be a simple linear algebraic group defined over $\mathbb{C}$. Fix a proper parabolic subgroup G/P. In [1], a Kähler–Einstein metric on G/P was constructed. This implies that the tangent bundle $T(G/P)$ is polystable (the definition is recalled below). We show that $T(G/P)$ is stable. Since $T(G/P)$ is polystable, it suffices to show that $T(G/P)$ is simple.

2. Stability of the tangent bundle of G/P

Let d be the complex dimension of G/P, where G and P are as above. The anti-canonical line bundle $\bigwedge^d T(G/P) \longrightarrow G/P$ is ample. Set

$$\xi := c_1(T(G/P)) \in H^2(G/P, \mathbb{Z}).$$

The degree of a torsionfree coherent sheaf F on G/P is defined to be

$$\text{degree}(F) := (c_1(F) \cup \xi^{d-1}) \cap [G/P] \in \mathbb{Z}.$$

The *slope* of F is $\mu(F) := \text{degree}(F)/\text{rank}(F)$. A vector bundle $E \longrightarrow G/P$ is called *stable* if $\mu(F) < \mu(E)$ for all coherent subsheaves $F \subset E$ with $\text{rank}(F) < \text{rank}(E)$. A vector bundle is called *polystable* if it is a direct sum of stable vector bundles of same slope (see [4]).

Theorem 2.1. *The tangent bundle $T(G/P)$ is stable.*

Proof. There is a Kähler–Einstein metric on G/P (see p. 487, §5 of [1]). Hence $T(G/P)$ is polystable (see [3]). To prove the theorem it suffices to show that

$$H^0(G/P,\ \text{End}(T(G/P))) = \mathbb{C} \cdot \text{Id}_{T(G/P)}\,, \tag{1}$$

where $\text{End}(T(G/P)) = T(G/P) \otimes T(G/P)^*$.

Let ∇ denote the Kähler–Einstein connection on $T(G/P)$ defined by the Kähler–Einstein metric constructed in [1]. The connection on $\mathrm{End}(T(G/P))$ induced by ∇ will be denoted by $\hat{\nabla}$. So $\hat{\nabla}$ is a Hermitian–Einstein connection. Note that $\mathrm{End}(T(G/P)) = \mathrm{End}(T(G/P))^*$, in particular,

$$c_1(\mathrm{End}(T(G/P))) = 0.$$

Therefore, any holomorphic section

$$\varphi \in H^0(G/P, \mathrm{End}(T(G/P))) \tag{2}$$

is flat with respect to the Hermitian–Einstein connection $\hat{\nabla}$ (p. 49, Proposition 3 of [2]).

Let $\mathfrak{p}$ be the Lie algebra of P. Let $\mathfrak{n} \subset \mathfrak{p}$ be the nilpotent radical. The vector bundle $\mathrm{End}(T(G/P))$ is identified with the one associated to the principal P-bundle $G \longrightarrow G/P$ for the P-module $\mathrm{End}_{\mathbb{C}}(\mathfrak{n}) = \mathfrak{n} \otimes \mathfrak{n}^*$ (the vector bundle TG/P corresponds to the P-module $\mathfrak{n}^*$). In particular, we have an injective homomorphism

$$\mathrm{End}_{\mathbb{C}}(\mathfrak{n})^P \longrightarrow H^0(G/P, \mathrm{End}(T(G/P))). \tag{3}$$

The left translation action of G on G/P gives an action of G on $T(G/P)$. This action of G on $T(G/P)$ induces an action of G on $H^0(G/P, \mathrm{End}(T(G/P)))$. The action of a maximal compact subgroup $K \subset G$ (see p. 482 of [1]) preserves the connection ∇. Since φ in (2) is flat with respect to $\hat{\nabla}$, it follows that the action of K on $H^0(G/P, \mathrm{End}(T(G/P)))$ fixes φ. This implies that

$$\varphi \in H^0(G/P, \mathrm{End}(T(G/P)))^G.$$

Hence the homomorphism in (3) is surjective. Therefore,

$$H^0(G/P, \mathrm{End}(T(G/P))) = \mathrm{End}_{\mathbb{C}}(\mathfrak{n})^P. \tag{4}$$

We will show that $\mathrm{End}_{\mathbb{C}}(\mathfrak{n})^P = \mathbb{C} \cdot \mathrm{Id}_{\mathfrak{n}}$.

Let

$$\mathfrak{n} = \bigoplus_{\alpha \in I} L_\alpha \quad \text{and} \quad \mathfrak{p} = \bigoplus_{\beta \in \hat{I}} L_\beta \tag{5}$$

be the root space decompositions under a maximal torus $T \subset P$. We note that any element of $\mathrm{End}_{\mathbb{C}}(\mathfrak{n})^T$ must be of the form $\bigoplus_{\alpha \in I} c_\alpha \cdot L_\alpha \otimes L_\alpha^*$, where $c_\alpha \in \mathbb{C}$. Take any

$$\eta = \bigoplus_{\alpha \in I} c_\alpha \cdot L_\alpha \otimes L_\alpha^* \in \mathrm{End}_{\mathbb{C}}(\mathfrak{n})^P. \tag{6}$$

Since η is fixed by P,

$$\eta(\mathrm{ad}(Y)(X)) = \mathrm{ad}(Y)(\eta(X)) \tag{7}$$

for all $X \in \mathfrak{n}$ and $Y \in \mathfrak{p}$. Take $X \in L_\alpha$ and $Y \in L_\beta$. From (6) we have $\eta([Y, X]) = c_{\alpha+\beta}[Y, X]$ and $[Y, \eta(X)] = [Y, c_\alpha \cdot X] = c_\alpha[Y, X]$. Hence from (7),

$$c_{\alpha+\beta} = c_\alpha.$$

Given any $\alpha \in I$, there is an integer n and $\beta_1, \ldots, \beta_n \in \hat{I}$ (see (5)) such that for $Y_i \in L_{\beta_i}$, the endomorphism $\mathrm{ad}(Y_1) \circ \cdots \circ \mathrm{ad}(Y_n)$ takes L_α to the unique highest root space in $\mathfrak{n}$. Consequently, $\eta = c \cdot \mathrm{Id}_{\mathfrak{n}}$ for some $c \in \mathbb{C}$. Now the proof is completed using (4). □

Theorem 2.1 is known under the assumption that $\mathfrak{n}$ is an irreducible P-module (see [5, 6]).

It would be interesting to decide whether Theorem 2.1 remains valid in positive characteristics.

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