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Relatively hyperbolic extensions of groups and Cannon–Thurston maps

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Abstract. Let $1 \to (K, K_1) \to (G, N_G(K_1)) \to (Q, Q_1) \to 1$ be a short exact sequence of pairs of finitely generated groups with K_1 a proper non-trivial subgroup of K and K strongly hyperbolic relative to K_1. Assuming that, for all $g \in G$, there exists $k_g \in K$ such that $gK_1g^{-1} = k_gK_1k_g^{-1}$, we will prove that there exists a quasi-isometric section $s\colon Q \to G$. Further, we will prove that if G is strongly hyperbolic relative to the normalizer subgroup $N_G(K_1)$ and weakly hyperbolic relative to K_1, then there exists a Cannon–Thurston map for the inclusion $i\colon \Gamma_K \to \Gamma_G$.

Keywords. Cannon–Thurston maps; relatively hyperbolic groups.

1. Introduction

Let us consider the short exact sequence of finitely generated groups

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$$

such that K is non-elementary word hyperbolic. In [12], Mosher proved that if G is hyperbolic, then Q is hyperbolic. To prove that Q is hyperbolic, Mosher (in [12]) constructed a quasi-isometric section from Q to G, that is, a map $s\colon Q \to G$ satisfying

$$\frac{1}{k} d_Q(q, q') - \epsilon \le d_G(s(q), s(q')) \le k d_Q(q, q') + \epsilon,$$

for all $q, q' \in Q$, where d_G and d_Q are word metrics and $k \ge 1, \epsilon \ge 0$ are constants. In [6], the existence of a Cannon–Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$ was proved, where Γ_K and Γ_G are respectively the Cayley graphs of K and G. In this paper, we will generalize these results to the case where the kernel is strongly hyperbolic relative to a cusp subgroup.

One of our main theorems states:

Theorem 2.10. *Suppose we have a short exact sequence of finitely generated groups*

$$1 \to K \to G \xrightarrow{p} Q \to 1,$$

with K strongly hyperbolic relative to a non-trivial proper subgroup K_1 and suppose that, for all $g \in G$, there exists $k_g \in K$ such that $gK_1g^{-1} = k_gK_1k_g^{-1}$. Then there exists a (k, ϵ)-quasi-isometric section $s\colon Q \to G$ for some constants $k \ge 1, \epsilon \ge 0$.

As a corollary, under the hypotheses as in the above theorem, we can take the image of the quasi-isometric section to lie in $N_G(K_1)$.

Let S be a once-punctured torus. Then its fundamental group $\pi_1(S) = \mathbb{F}(a, b)$ is strongly hyperbolic relative to the peripheral subgroup $H = \langle aba^{-1}b^{-1}\rangle$. Let M be a 3-manifold fibering over the unit circle with fiber S such that the fundamental group $\pi_1(M)$ is strongly hyperbolic relative to the subgroup $H \oplus \mathbb{Z}$. Then we have a short exact sequence of pairs of finitely generated groups:

$$1 \to (\pi_1(S), H) \to (\pi_1(M), H \oplus \mathbb{Z}) \to (\mathbb{Z}, \mathbb{Z}) \to 1.$$

Let $K = \pi_1(S)$, $G = \pi_1(M)$; and let Γ_K and Γ_G be the Cayley graphs of K and G respectively. Bowditch [3] and Mj [9], proved the existence of a Cannon–Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$.

Motivated by this example, we will prove the following theorem:

Theorem 3.11. *Suppose we have a short exact sequence of pairs of finitely generated groups*

$$1 \to (K, K_1) \xrightarrow{i} (G, N_G(K_1)) \xrightarrow{p} (Q, Q_1) \to 1$$

with K_1 a proper non-trivial subgroup of K and K strongly hyperbolic relative to K_1. Assume further that, for all $g \in G$, there exists $k_g \in K$ such that $gK_1g^{-1} = k_gK_1k_g^{-1}$. If G is strongly hyperbolic relative to $N_G(K_1)$ and weakly hyperbolic relative to the subgroup K_1, then there exists a Cannon–Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$, where Γ_K and Γ_G are Cayley graphs of K and G respectively.

2. Relative hyperbolicity and quasi-isometric section

For basic notions and properties about hyperbolic metric spaces, refer to [1]. Let $K \geq 1$ and $\epsilon \geq 0$. A (K, ϵ)-*quasigeodesic* in a metric space X is a (K, ϵ)-quasi-isometric embedding $\gamma : J \to X$, where J is an interval (bounded or unbounded) of the real line. A (K, K)-quasigeodesic in X will be called as K-*quasigeodesic*.

We recall Gromov's definition of a strongly relatively hyperbolic group.

DEFINITION 2.1 [5]

Let G be a finitely generated group acting freely and properly discontinously by isometries on a proper and δ-hyperbolic metric space X, such that the quotient space X/G is quasi-isometric to $[0, \infty)$. Let H denote the stabilizer subgroup of the endpoint on ∂X of a lift of this ray to X. Then G is said to be strongly hyperbolic relative to H. The subgroup H is said to be a parabolic or cusp subgroup. The end points on ∂X of lifts of $[0, \infty)$ will be called as parabolic end points.

For a group G strongly hyperbolic relative to a subgroup H, the stabilizer subgroup of a parabolic end point is aHa^{-1} for some $a \in G$.

DEFINITION 2.2 [4]

Let G be a finitely generated group, and let H be a finitely generated subgroup of G. Let Γ_G be the Cayley graph of G. Let $\hat{\Gamma}_G$ be a new graph obtained from Γ_G as follows:

For each left coset gH of H in G, we add a new vertex $v(gH)$ to Γ_G, and add an edge $e(gh)$ of length $1/2$ from each element gh of gH to the vertex $v(gH)$. We call this new graph the coned-off Cayley graph of G with respect to H, and denote it by $\hat{\Gamma}_G = \hat{\Gamma}_G(H)$.

We say that G is *weakly hyperbolic* relative to the subgroup H if the coned-off Cayley Graph $\hat{\Gamma}_G$ is hyperbolic.

Geodesics in the coned-off space $\hat{\Gamma}_G$ will be called as electric geodesics. For a path $\gamma \subset \Gamma_G$, there is an induced path $\hat{\gamma}$ in $\widehat{\Gamma_G}$ obtained by replacing the portion of γ inside a left coset by an edge path of length 1 passing through the cone point corresponding to that left coset. If $\hat{\gamma}$ is an electric geodesic (resp. P-quasigeodesic), γ is called a *relative geodesic* (resp. *relative P-quasigeodesic*). If $\hat{\gamma}$ passes through some cone point $v(gH)$, we say that $\hat{\gamma}$ *penetrates* the coset gH.

DEFINITION 2.3 [4]

$\hat{\gamma}$ is said to be an electric (K, ϵ)-quasigeodesic in (the electric space) $\widehat{\Gamma_G}$ *without backtracking* if $\hat{\gamma}$ is an electric (K, ϵ)-quasigeodesic in $\widehat{\Gamma_G}$ and $\hat{\gamma}$ does not return to any left coset after leaving it.

DEFINITION 2.4 [4]

Bounded coset penetration: The pair (G, H) is said to satisfy the bounded coset penetration property if, for every $P \geq 1$, there is a constant $D = D(P)$ such that, whenever α and β are two electric P-quasigeodesics without backtracking starting and ending at same points, the following conditions hold:

1. If α penetrates a coset gH but β does not penetrate gH, then α travels a Γ_G-distance of at most D in gH.
2. If both α and β penetrate a coset gH, then the vertices in Γ_G, at which α and β first enter gH, lie at a Γ_G-distance of at most D from each other; similarly for the last exit vertices.

We state below Farb's definition of a strongly relatively hyperbolic group.

DEFINITION 2.5 [4]

G is said to be strongly hyperbolic relative to H if G is weakly hyperbolic relative to H and the pair (G, H) satisfies the bounded coset penetration property.

Theorem 2.6 [2, 13]. *G is strongly hyperbolic relative to the cusp subgroup H in the sense of Farb if and only if G is strongly hyperbolic relative to H in the sense of Gromov.*

Let G be a group strongly hyperbolic relative to a subgroup G_1. Let $X = \Gamma_G$ and H_{gG_1} be the closed set in Γ_G corresponding to the left coset gG_1 of G_1 in G. Let $\mathcal{H}_G = \{H_{gG_1}: g \in G\}$. Let X_h be the space obtained from X by gluing $H_{gG_1} \times [0, \infty)$ to H_{gG_1} for all $H_{gG_1} \in \mathcal{H}_G$, where $H_{gG_1} \times [0, \infty)$ is equipped with the path metric d_h induced from the following two pieces of data:

(a) $d_{h,t}((x, t), (y, t)) = 2^{-t} d_H(x, y)$, where $d_{h,t}$ is the path metric on $H_t = H \times \{t\}$.
(b) $d_h((x, t), (x, s)) = |t - s|$ for all $x \in H$ and for all $t, s \in [0, \infty)$.

Then it is shown by Bowditch (in [2]) that X_h is hyperbolic.

$H_{gG_1} \in \mathcal{H}_G$ were referred to as *horosphere-like sets* by Mj and Reeves in [11] and $H_{gG_1} \times [0, \infty)$ was referred to as *hyperbolic cones or horoball-like sets* in [10].

DEFINITION 2.7 [2]

Relative hyperbolic boundary: For the relatively hyperbolic group (G, G_1), the boundary ∂X_h of X_h will be called as relative hyperbolic boundary of (G, G_1) and will be denoted by $\partial\Gamma(G, G_1)$.

Bowditch in [2] showed that if G acts properly discontinuously by isometries on a proper hyperbolic space Z and the action of G on ∂Z is geometrically finite (i.e. every point of ∂Z is either a conical limit point or a bounded parabolic point) and minimal (i.e. if the limit set $\Lambda G = \partial Z$) then ∂Z is homeomorphic to $\partial\Gamma(G, H)$.

DEFINITION 2.8

Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of finitely generated groups with K strongly hyperbolic relative to K_1. We say that G *preserves cusps*, if for all $g \in G$, there exists $a_g \in K$ such that $gK_1g^{-1} = a_gK_1a_g^{-1}$.

DEFINITION 2.9 [12]

Quasi-isometric section: Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of finitely generated groups. A map $s\colon Q \to G$ is said to be a (R, ϵ) quasi-isometric section if

$$\frac{1}{R}d_Q(q, q') - \epsilon \le d_G(s(q), s(q')) \le Rd_Q(q, q') + \epsilon,$$

for all $q, q' \in Q$, where d_G and d_Q are word metrics and $R \ge 1, \epsilon \ge 0$ are constants.

Let K be a strongly hyperbolic group relative to a cusp subgroup K_1. For each parabolic point $\alpha \in \partial\Gamma(K, K_1)$, there is a unique subgroup of the form aK_1a^{-1}. Now the Hausdorff distance between two sets aK_1 and aK_1a^{-1} is uniformly bounded by the length of the word a. Hence α corresponds to a left coset aK_1 of K_1 in K.

Let $1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be a short exact sequence of finitely generated groups with K non-elementary and strongly hyperbolic relative to a subgroup K_1.

We use the following notations for our further purpose:

- Let Π be the set of all parabolic end points for the relatively hyperbolic group K with cusp subgroup as K_1.
- Let $\Pi^2 = \{(\alpha_1, \alpha_2)\colon \alpha_1 \text{ and } \alpha_2 \text{ are distinct elements in } \Pi\}$.
- For $a \in K$, let $i_a\colon K \to K$ denote the inner automorphism and $L_a\colon K \to K$ the left translation.
- For $g \in G$, let $I_g\colon K \to K$ be the outer automorphism, that is, $I_g(k) = gkg^{-1}$ and $L_g\colon G \to G$ be the left translation.

G preserves cusps, so for each $g \in G$, there exists $a_g \in K$ such that $a_g^{-1}g \in N_G(K_1)$. If $b \in K$, then it can be easily proved that $d_K(a_gK_1, gbg^{-1}a_gK_1) \le d_K(K_1, bK_1) + 2l_K(a_g^{-1}g)$. Since $I_g(bK_1) = g(bK_1)g^{-1} = gbg^{-1}a_gK_1a_g^{-1}$ and the Hausdorff distance between $gbg^{-1}a_gK_1$ and $gbg^{-1}a_gK_1a_g^{-1}$ is bounded, I_g will induce a map $\tilde{I}_g\colon \Pi \to \Pi$

and $\tilde{I}_g$ is a bijection. Therefore, $\tilde{I}_g$ will induce a bijective map $\tilde{I}_g^2\colon \Pi^2 \to \Pi^2$. For the sake of convenience of notation we will use I_g for $\tilde{I}_g$ and $\tilde{I}_g^2$. Similarly, for $a \in K$, i_a and L_a will induce bijective maps (with same notation) from Π to Π and Π^2 to Π^2.

In the following theorem we generalize Mosher's construction of quasi-isometric section to the relatively hyperbolic case.

Theorem 2.10. *Suppose we have a short exact sequence of finitely generated groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{p} Q \to 1,$$

such that K is strongly hyperbolic relative to a non-trivial proper subgroup K_1 and G preserves cusps. Then there exists an (R, ϵ) quasi-isometric section $s\colon Q \to G$ for some $R \geq 1, \epsilon \geq 0$.

Proof. Let $\alpha = (\alpha_1, \alpha_2) \in \Pi^2$, then the stabilizer subgroup of each α_i is $a_i K_1 a_i^{-1}$ for some $a_i \in K$, $i = 1, 2$. Let λ, μ be two relative geodesics in Γ_K starting from a point of $a_1 K_1$ and ending at some point of $a_2 K_1$ and further assume that $\hat{\lambda}$, $\hat{\mu}$ passes through the cone points $v(a_i K_1)$, $i = 1, 2$. Let x, y be the exit points of λ, μ from the left coset $a_1 K_1$ respectively. Then due to bounded coset penetration property 2, $d_K(x, y) \leq D$, where D is the constant as in Definition 2.4. Let B_α be the set of all exit points from $a_1 K_1$ of relative geodesics λ starting at some point of $a_1 K_1$ and ending at some point of $a_2 K_1$ such that $\hat{\lambda}$ passes through $v(a_i K_1)$'s, $i = 1, 2$. Then B_α is a bounded set with diameter less than or equal to D.

Let $C = \{\alpha \in \Pi^2\colon e_K \in B_\alpha\}$, where e_K is the identity element in K. We fix an element $\eta = (\eta_1, \eta_2) \in \Pi^2$. Let $\Sigma = \{g \in G\colon \eta \in I_g(C)\}$. A subset of Σ will be proved to be the image of a quasi-isometric section.

Step 1. We first prove that, for any $g \in G$, $\cup_{a \in K} I_{ga}(C) = \Pi^2$. Let $\alpha = (\alpha_1, \alpha_2) \in \Pi^2$. Each α_i corresponds to a left coset $a_i K_1$, $i = 1, 2$. Let λ be a relative geodesic in Γ_K starting at some point of $a_1 K_1$ and ending at some point of $a_2 K_1$ such that $\hat{\lambda}$ passes through cone points $v(a_i K_1)$, $i = 1, 2$. Let x_α be the exit point of λ from $a_1 K_1$. Then $x_\alpha \in B_\alpha$. Now there exists $k \in K$ such that $L_k(x_\alpha) = e_K$. Since L_k is an isometry, $L_k(\lambda)$ will be a relative geodesic joining point of $k a_1 K_1$ and $k a_2 K_1$ with $\widehat{L_k(\lambda)}$ containing the cone points $v(k a_1 K_1)$, $v(k a_2 K_1)$ and e_K the exit point of $L_k(\lambda)$ from $k a_1 K_1$. Now there exists $\beta_i \in \Pi$ such that β_i corresponds to the left coset $k a_i K_1$, $i = 1, 2$. Therefore $\beta = (\beta_1, \beta_2) \in \Pi^2$, $e_K \in B_\beta$ and $L_k(\alpha) = \beta \in C$. Since L_k and i_k are same on the relative hyperbolic boundary, we have $i_k(\alpha) \in C$. Thus $\cup_{a \in K}(i_a(C)) = \Pi^2$. Consequently, for any $g \in G$, we have $\cup_{a \in K} I_{ga}(C) = \cup_{a \in K} I_g i_a(C) = I_g(\cup_{a \in K}(i_a(C))) = I_g(\Pi^2) = \Pi^2$.

Step 2. Now we prove that $p(\Sigma) = Q$. Let $q \in Q$. Then there exists $g \in G$ such that $p(g) = q$. Now $\cup_{a \in K} I_{ga}(C) = \Pi^2$ for any $g \in G$. Therefore, for $\eta \in \Pi^2$, there exists $a \in K$ such that $\eta \in I_{ga}(C)$. Hence $ga \in \Sigma$ and $p(ga) = p(g) = q$.

Step 3. Now we prove that there exist constants $R \geq 1, \epsilon \geq 0$ such that, for all $g, g' \in \Sigma$, the following holds:

$$\frac{1}{R} d_Q(p(g), p(g')) - \epsilon \leq d_G(g, g') \leq R d_Q(p(g), p(g')) + \epsilon.$$

We can choose a finite symmetric generating set S of G such that $p(S)$ is also a generating set for Q. Obviously, $d_Q(p(g), p(g')) \leq d_G(g, g')$ for all $g, g' \in G$. To prove $d_G(g, g') \leq Rd_Q(p(g), p(g')) + \epsilon$ for all $g, g' \in \Sigma$, it suffices to prove that there exists $R \geq 1$ such that $d_G(g, g') \leq R$ whenever $d_Q(p(g), p(g')) \leq 1$ for some $g, g' \in \Sigma$.

Let $d_Q(p(g), p(g')) \leq 1$ for some $g, g' \in \Sigma$. Then $g^{-1}g' = ka$ for some $k \in K$ and a is either the identity element of G or a generator of G. Since $g, g' \in \Sigma$, $I_g(C) \cap I_{g'}(C) \neq \Phi$. This implies $I_{ka}(C) \cap C = I_{g^{-1}g'}(C) \cap C \neq \Phi$. Now $I_{ka} = i_k(I_a)$. Therefore $i_k(I_a(C)) \cap C \neq \Phi$.

For each $\alpha \in \Pi^2$, we choose an element $a_\alpha \in B_\alpha$. Define a map $F\colon \Pi^2 \to \Gamma_K$ by $F(\alpha) = a_\alpha$.

Since L_k is an isometry, for $k \in K$, $ka_\alpha \in B_{k\alpha}$ and hence

$$d_K(a_{k\alpha}, ka_\alpha) = d_K(F(k\alpha), kF(\alpha)) \leq D, \tag{1}$$

where $k\alpha$ denotes the image of α under the map $L_k\colon \Pi^2 \to \Pi^2$.

Let $B_D(e_K)$ be the closed D-neighborhood of e_K. Now $F(C)$ is contained in the union of the B_α's containing identity e_K. Therefore $F(C)$ is contained in $B_D(e_K)$. Since G preserves cusps, there exists $s \in K$ such that $F(I_a(C))$ is contained in the union of the B_α's containing s and hence $F(I_a(C)) \subset B_D(s)$, where $B_D(s)$ is a closed D-neighborhood of s. From (1), the Hausdorff distance between two sets $F(kI_a(C))$ and $kF(I_a(C))$ is bounded by D. For a set $A \subset \Gamma_K$, let $N_D(A)$ denote the closed D-neighborhood of A. Thus

$$F(kI_a(C)) \subset N_D(kF(I_a(C))) = kN_D(F(I_a(C))) \subset kB_{2D}(s).$$

Now K acts properly discontinuously on Γ_K. Therefore

$$B_D(e_K) \cap kB_{2D}(s) \neq \Phi$$

for finitely many k's in K. This implies $F(C) \cap F(kI_a(C)) \neq \Phi$ for finitely many k's in K. And hence $C \cap L_k(I_a(C)) = C \cap kI_a(C) \neq \Phi$ for finitely many k's in K. $L_k = i_k$ on the relative hyperbolic boundary. Hence $C \cap (I_{ka}(C)) \neq \Phi$ for finitely many k's in K. Thus $g^{-1}g' = ka$ for finitely many k's. Since the number of generators of G is finite, there exists a constant $R \geq 1$ such that $d_G(g, g') \leq R$.

Now we define $s\colon Q \to G$ as follows:

Let $q \in Q$ and let there exist $g, g' \in \Sigma$ such that $p(g) = p(g') = q$. Then by the above inequality $d_G(g, g') \leq R$. We choose one element $g \in p^{-1}(q) \cap \Sigma$ for each $q \in Q$ and define $s(q) = g$. Then s defines a single valued map satisfying:

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq Rd_Q(q, q') + \epsilon$$

for some constants $R \geq 1$, $\epsilon \geq 0$ and for all $q, q' \in Q$. □

COROLLARY 2.11

Suppose we have a short exact sequence of pairs of finitely generated groups

$$1 \to (K, K_1) \to (G, N_G(K_1)) \xrightarrow{p} (Q, Q_1) \to 1$$

with K strongly relatively hyperbolic with respect to the cusp subgroup K_1. If G preserves cusps, then $Q_1 = Q$ and there is a quasi-isometric section $s\colon Q \to N_G(K_1)$ satisfying

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_{N_G(K_1)}(s(q), s(q')) \leq Rd_Q(q, q') + \epsilon$$

where $q, q' \in Q$ and $R \geq 1, \epsilon \geq 0$ are constants. Further, if G is weakly hyperbolic relative to K_1, then Q is hyperbolic.

Proof. Let $q \in Q$. Then there exists $g \in G$ such that $p(g) = q$. Since G preserves cusps, $gK_1g^{-1} = aK_1a^{-1}$ for some $a \in K$. Therefore, $a^{-1}g \in N_G(K_1)$ and $q = p(a^{-1}g) \in Q_1$. Thus $Q_1 = Q$.

Let $\Pi^2_{K_1} = \{(\alpha_1, \alpha_2) \in \Pi^2 \colon \alpha_1$ corresponds to subgroup $K_1\}$ and $C = \{\alpha \in \Pi^2_{K_1} \colon e_K \in B_\alpha\}$, where B_α is defined as in the above theorem. We fix an element $\eta \in \Pi^2_{K_1}$ and set $\Sigma = \{g \in N_G(K_1) \colon \eta \in I_g(C)\}$. We can choose a finite symmetric generating set S of G such that $p(S)$ is a generating set of Q and S contains the generators of $N_G(K_1)$. Using the same argument as in the above theorem, by replacing G with $N_G(K_1)$, we get a quasi-isometric section $s \colon Q \to N_G(K_1)$ satisfying:

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_{N_G(K_1)}(s(q), s(q')) \leq R d_Q(q, q') + \epsilon.$$

for some constants $R \geq 1, \epsilon \geq 0$ and for all $q, q' \in Q$.

Since $d_Q(q, q') \leq d_G(s(q), s(q')) \leq d_{N_G(K_1)}(s(q), s(q'))$, we can take the quasi-isometric section $s \colon Q \to N_G(K_1)$ such that

$$\frac{1}{R} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq R d_Q(q, q') + \epsilon.$$

Now, let Γ_G^{pel} denote the space obtained from Γ_G by coning left cosets of K_1 in G. Since G is weakly hyperbolic with respect to K_1, Γ_G^{pel} is hyperbolic. We will prove that Q is hyperbolic.

The quasi-isometric section $s \colon Q \to N_G(K_1)(\subset G)$ will induce a map $\hat{s} \colon Q \to \Gamma_G^{\mathrm{pel}}$. Now, for all $q, q' \in Q$, $d_{G^{\mathrm{pel}}}(\hat{s}(q), \hat{s}(q')) \leq d_G(s(q), s(q')) \leq R\, d_Q(q, q') + \epsilon$, where $d_{G^{\mathrm{pel}}}$ is the metric on Γ_G^{pel}. Obviously, $d_Q(q, q') \leq d_{G^{\mathrm{pel}}}(\hat{s}(q), \hat{s}(q'))$. Hence $\hat{s}$ is a quasi-isometric section from Q to Γ_G^{pel}. Therefore, $s(Q)$ is quasiconvex in Γ_G^{pel}. Since Γ_G^{pel} is hyperbolic, Q is hyperbolic. □

3. Existence of Cannon–Thurston maps

Consider the inclusion between pairs of relatively hyperbolic groups $(H, H_1) \overset{i}{\hookrightarrow} (G, G_1)$. i will induce a proper embedding $i \colon \Gamma_H \to \Gamma_G$. Let $X = \Gamma_G$ and $Y = \Gamma_H$. Recall that X_h is the space obtained from X by gluing the hyperbolic cones. Inclusion of a horosphere-like set in its hyperbolic cone is uniformly proper. Therefore, the inclusion of X in X_h is uniformly proper, i.e., for all $M > 0$ and $x, y \in X$, there exists $N > 0$ such that $d_G(x, y) \leq N$ whenever $d_{X_h}(x, y) \leq M$, where d_G is the word metric corresponding to G.

Since G preserves cusps, i will induce a proper embedding $i_h \colon Y_h \to X_h$.

DEFINITION 3.1

A Cannon–Thurston map for $i \colon (\Gamma_H, \mathcal{H}_{\mathcal{H}}) \to (\Gamma_G, \mathcal{H}_{\mathcal{G}})$ is said to exist if there exists a continuous extension $\tilde{i_h} \colon Y_h \cup \partial Y_h \to X_h \cup \partial X_h$ of $i_h \colon Y_h \to X_h$.

To prove the existence of a Cannon–Thurston map for the inclusion $i \colon (K, K_1) \to (G, N_G(K_1))$, we need the notion of partial electrocution.

DEFINITION 3.2 [11] (Partial electrocution)

Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple such that the following holds:

1. X is a proper geodesic metric space, $\mathcal{H}$ is a collection of subsets H_α of X, and X is hyperbolic relative to $\mathcal{H}$, i.e., the space X_h, obtained from X by gluing $H_\alpha \times [0, \infty)$ to each H_α with the metric d_h satisfying properties 1 and 2 above, is hyperbolic.
2. There exists $\delta > 0$ such that $\mathcal{L}$ is a collection of δ-hyperbolic metric spaces L_α and $\mathcal{G}$ is a collection of (uniformly) coarse Lipschitz maps $g_\alpha\colon H_\alpha \to L_\alpha$. Note that the indexing set for $H_\alpha, L_\alpha, g_\alpha$ is common.

The *partially electrocuted space* or *partially coned off space* corresponding to $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is obtained from X by gluing in the (metric) mapping cylinders for the maps $g_\alpha\colon H_\alpha \to L_\alpha$.

Lemma 3.3 [11]. *Given $K, \epsilon \geq 0$, there exists $C > 0$ such that the following holds: Let γ_{pel} and γ denote respectively a (K, ϵ) partially electrocuted quasigeodesic in (X, d_{pel}) and a hyperbolic (K, ϵ)-quasigeodesic in (X_h, d_h) joining a, b. Then $\gamma \cap X$ lies in a (hyperbolic) C-neighborhood of (any representative of) γ_{pel}. Further, outside of a C-neighborhood of the horoballs that γ meets, γ and γ_{pel} track each other.*

Let G be a group strongly hyperbolic relative to the subgroup G_1. Let $X = \Gamma_G$ be the Cayley graph of G and X_h be the complete hyperbolic space obtained from X. We describe a special type of quasigeodesic in X_h which will be essential for our purpose:

DEFINITION 3.4 [8]

We start with an electric quasi-geodesic $\hat{\lambda}$ in the electric space $\hat{X}$ without backtracking. For any horosphere-like set H penetrated by $\hat{\lambda}$, let x_H and y_H be the respective entry and exit points to H. We join x_H and y_H by a hyperbolic geodesic segment in $H \times [0, \infty)$. This results a path, say λ, in X_h. The path λ will be called an *electro-ambient path.*

Lemma 3.5 [8]. *The Hausdorff distance between any subsegment λ_{st} of the electroambient path λ and the geodesic in X_h joining the end points of λ_{st} is bounded by some constant L, where L depends only upon the hyperbolicity constant of X_h.*

For the rest of the paper, we will work with the following pair of short exact sequence of finitely generated groups:

$$1 \to (K, K_1) \to (G, N_G(K_1)) \xrightarrow{p} (Q, Q_1) \to 1.$$

Since all the above groups are finitely generated, we can choose a finite symmetric generating set S of G such that

- S contains generators of $K, K_1, N_G(K_1)$ and
- $p(S)$ is also a finite generating set of Q.

We will assume the hypotheses of Corollary 2.11. As a consequence, $Q_1 = Q$ and there exists an (R, ϵ) quasi-isometric section $s\colon Q \to N_G(K_1)$ such that $\frac{1}{R} d_Q(q, q') - \epsilon \leq d_G(s(q), s(q')) \leq R d_Q(q, q') + \epsilon$ for all $q, q' \in Q$. Further, we assume that G is strongly hyperbolic relative to the subgroup $N_G(K_1)$. Also, using a left translation L_k by an element $k \in K_1$, we can assume that $s(Q)$ contains the identity element e_K of

K and $s(Q) \subset N_G(K_1)$. We have assumed that G is weakly hyperbolic relative to the subgroup K_1 and hence the coned-off space Γ_G^{pel} obtained by coning left cosets gK_1 of K_1 to a point $v(gK_1)$ is hyperbolic. As Γ_Q is quasi-isometrically embedded in Γ_G^{pel}, Q is hyperbolic. We have also assumed that G is strongly hyperbolic relative to $N_G(K_1)$. Thus Γ_G^{pel} becomes a partially electrocuted space obtained from Γ_G by partially electrocuting the closed sets (horosphere-like sets) $H_{gN_G(K_1)}$ in Γ_G corresponding to the left cosets $gN_G(K_1)$ to the hyperbolic space $g(s(Q))$, where $g(s(Q))$ denotes the image of $s(Q)$ under the left translation L_g for $g \in G$.

Let $\lambda^b = \hat{\lambda} \backslash \mathcal{H}_{\mathcal{K}}$ denote the portions of $\hat{\lambda}$ that do not penetrate horosphere-like sets in $\mathcal{H}_{\mathcal{K}}$. The following lemma gives a sufficient condition for the existence of a Cannon–Thurston map for the inclusion $i\colon (\Gamma_K, \mathcal{H}_{\mathcal{K}}) \to (\Gamma_G, \mathcal{H}_{\mathcal{G}})$.

Lemma 3.6 [10]. *A Cannon–Thurston map for* $i\colon (\Gamma_K, \mathcal{H}_{\mathcal{K}}) \to (\Gamma_G, \mathcal{H}_{\mathcal{G}})$ *exists if there exists a non-negative function* $M(N)$ *with* $M(N) \to \infty$ *as* $N \to \infty$ *such that the following holds*:

Given $y_0 \in \Gamma_K$, *and an electric quasigeodesic segment* $\hat{\lambda}$ *in* $\hat{\Gamma}_K$, *if* $\lambda^b = \hat{\lambda} \backslash \mathcal{H}_{\mathcal{K}}$ *lies outside an* N*-ball around* $y_0 \in \Gamma_K$ *then, for any partially electrocuted quasigeodesic* β_{pel} *in* Γ_G^{pel} *joining end points of* $\hat{\lambda}$, $\beta^b = \beta_{\mathrm{pel}} \backslash \mathcal{H}_{\mathcal{G}}$ *lies outside an* $M(N)$*-ball around* $i(y_0)$ *in* Γ_G.

3.1 *Construction of quasiconvex sets and retraction map*

Recall that for $g \in G$, $L_g\colon G \to G$ denotes the left translation by g and $I_g\colon K \to K$ denotes the automorphism $I_g(k) = gkg^{-1}$. Let $\phi_g = I_{g^{-1}}$ then $\phi_g(a) = g^{-1}ag$. Since L_g is an isometry, L_g preserves distance between left cosets of G_1 in G. Hence L_g induces an isometry. The embedding $i\colon \Gamma_K \to \Gamma_G$ will induce an embedding $\hat{i}\colon \hat{\Gamma}_K \to \Gamma_G^{\mathrm{pel}}$. $\hat{L}_g\colon \Gamma_G^{\mathrm{pel}} \to \Gamma_G^{\mathrm{pel}}$. The embedding $i\colon \Gamma_K \to \Gamma_G$ will induce an embedding $\hat{i}\colon \hat{\Gamma}_K \to \Gamma_G^{\mathrm{pel}}$.

Let $\hat{\lambda}$ be an electric geodesic segment in $\hat{\Gamma}_K$ with end points a and b in Γ_K. Let $\hat{\lambda}_g$ be an electric geodesic in $\hat{\Gamma}_K$ joining $\phi_g(a)$ and $\phi_g(b)$.

Define

$$B_{\hat{\lambda}} = \bigcup_{g \in s(Q)} \hat{L}_g.\hat{i}(\hat{\lambda}_g).$$

On $\hat{\Gamma}_K$, define a map $\pi_{\hat{\lambda}_g}\colon \hat{\Gamma}_K \to \hat{\lambda}_g$ taking $k \in \hat{\Gamma}_K$ to one of the points on $\hat{\lambda}_g$ closest to k in the metric $d_{\hat{K}}$.

Lemma 3.7 [7]. *For* $\pi_{\hat{\lambda}_g}$ *defined above*,

$$d_{\hat{K}}(\pi_{\hat{\lambda}_g}(k), \pi_{\hat{\lambda}_g}(k')) \leq C d_{\hat{K}}(k, k') + C$$

for all $k, k' \in \hat{\Gamma}_K$, *where* C *depends only on the hyperbolic constant of* $\hat{\Gamma}_K$.

DEFINITION 3.8 (Retraction map)

Define $\Pi_{\hat{\lambda}}\colon \Gamma_G^{\mathrm{pel}} \to B_{\hat{\lambda}}$ as follows:

Let $x \in \Gamma_G^{\mathrm{pel}}$. Then there exists a unique $g \in s(Q)$ such that $\widehat{L_g}(\hat{i}(k)) = x$ for some unique $k \in K$, define $\Pi_{\hat{\lambda}}(x) = \widehat{L_g}(\hat{i}(\hat{\pi}_{\hat{\lambda}_g}(k)))$. $\Pi_{\hat{\lambda}}$ will be called a retraction map.

Theorem 3.9 [6, 7]. *There exists* $C_0 > 0$ *such that*

$$d_{\hat{G}}(\Pi_{\hat{\lambda}}(g), \Pi_{\hat{\lambda}}(g')) \leq C_0 d_{\hat{G}}(g, g') + C_0$$

for all $g, g' \in \Gamma_G^{\mathrm{pel}}$. *In particular, if* Γ_G^{pel} *is hyperbolic then* $B_{\hat{\lambda}}$ *is uniformly* (*independent of* $\hat{\lambda}$) *quasiconvex.*

3.2 *Proof of Theorem* 3.11

Since $i\colon \Gamma_K \to \Gamma_G$ is an embedding, we identify $k \in K$ with its image $i(k)$. Let

- $\hat{\mu}_g = \hat{L}_g(\hat{\lambda}_g)$, where $g \in s(Q)$.
- $\mu_g^b = \hat{\mu}_g \backslash \mathcal{H}_G$.
- $B_{\lambda^b} = \bigcup_{g \in s(Q)} \mu_g^b$.
- $Y = \Gamma_K$ and $X = \Gamma_G$.

Lemma 3.10. *There exists* $A > 0$ *such that for all* $x \in \mu_g^b \subset B_{\lambda^b} \subset B_{\hat{\lambda}}$, *if* λ^b *lies outside* $B_N(p)$ *for a fixed reference point* $p \in \Gamma_K$ *then* x *lies outside an* $\frac{f(N)}{A+1}$ *ball about* p *in* Γ_G, *where* $f(N) \to \infty$ *as* $N \to \infty$.

Proof. Let $x \in \mu_g^b$ for some $g \in s(Q)$. Let γ be a geodesic path in Γ_Q joining the identity element e_Q of Γ_Q and $p(x) \in \Gamma_Q$. Order the vertices on γ so that we have a finite sequence $e_Q = q_0, q_1, \ldots, q_n = p(x) = p(g)$ such that $d_Q(q_i, q_{i+1}) = 1$ and $d_Q(e_Q, p(x)) = n$. Since s is a quasi-isometric section, this gives a sequence $s(q_i) = g_i$ such that $d_G(g_i, g_{i+1}) \leq R + \epsilon = R_1$ (say). Observe that $g_n = g$ and $g_0 = e_G$. Let $B_{R_1}(e_G)$ be a closed ball around e_G of radius R_1. Then $B_{R_1}(e_G)$ is finite. Now for each $g \in G$, the outer automorphism ϕ_g is a quasi-isometry. Thus there exists $K \geq 1$ and $\epsilon \geq 0$ such that for all $g \in B_{R_1}(e_G)$, ϕ_g is a (K, ϵ) quasi-isometry and K, ϵ are independent of elements of G. Let $s_i = g_{i+1}^{-1} g_i$, then $s_i \in B_{R_1}(e_G)$, where $i = 0, \ldots, n-1$. Hence ϕ_{s_i} is a (K, ϵ) quasi-isometry. Therefore, ϕ_{s_i} will induce a $(\hat{K}, \hat{\epsilon})$ quasi-isometry $\hat{\phi}_{s_i}$ from $\hat{\Gamma}_K$ to $\hat{\Gamma}_K$, where $\hat{K}, \hat{\epsilon}$ depends only on K and ϵ.

Now $x \in \mu_{g_n}^b$ and L_g preserves distance between left cosets for all $g \in G$. Hence there exists $x_1 \in \lambda_{g_n}^b$ such that $x = L_{g_n}(x_1)$.

Let $[p, q]_{g_n} \subset \lambda_{g_n}^b$ be the connected portion of $\lambda_{g_n}^b$ on which x_1 lies. Since $\hat{\phi}_{s_{n-1}}$ is a quasi-isometry, $\hat{\phi}_{s_{n-1}}([p, q]_{g_n})$ will be an electric quasigeodesic lying outside horosphere-like sets and hence it is a quasigeodesic in Y_h lying at a uniformly bounded distance $\leq C_1$ from $\lambda_{g_{n-1}}^h$ in Y_h (and hence in X_h), where $\lambda_{g_{n-1}}^h$ is the electroambient representative of $\hat{\lambda}_{g_{n-1}}$ and Y_h, X_h are respectively the complete hyperbolic metric spaces corresponding to Y, X. Thus there exists $x_2 \in \lambda_{g_{n-1}}^h$ such that $d_{X_h}(\phi_{s_{n-1}}(x_1), x_2) \leq C_1$. But x_2 may lie inside horoball-like set penetrated by $\hat{\lambda}_{g_{n-1}}$. Due to bounded coset (horosphere) penetration properties, there exists $y \in \lambda_{g_{n-1}}^b$ such that $d_{X_h}(x_2, y) \leq D$.

Thus $d_{X_h}(\phi_{s_{n-1}}(x_1), y) \leq C_1 + D$. Since $X = \Gamma_G$ is properly embedded in X_h, there exists $M > 0$ depending only upon C_1, D such that $d_G(\phi_{s_{n-1}}(x_1), y) \leq M$. Hence $d_G(L_{g_{n-1}}(\phi_{s_{n-1}}(x_1)), L_{g_{n-1}}(y)) = d_G(\phi_{s_{n-1}}(x_1), y) \leq M$ and $L_{g_{n-1}}(y) \in \mu_{g_{n-1}}^b$.

Let $z = L_{g_{n-1}}(y)$. Then

$$\begin{aligned} d_G(x,z) &\le d_G(x, L_{g_{n-1}}(\phi_{s_{n-1}}(x_1))) + d_G(L_{g_{n-1}}(\phi_{s_{n-1}}(x_1)), L_{g_{n-1}}(y)) \\ &\le d_G(x, xs_{n-1}) + M \\ &\le R_1 + M = A \text{ (say)}. \end{aligned}$$

Thus, we have shown that, for $x \in \mu^b_{g_n}$ there exists $z \in \mu^b_{g_{n-1}}$ such that $d_G(x,z) \le A$. Proceeding in this way, for each $y \in \mu^b_{g_i}$ there exists $y' \in \mu^b_{g_{i-1}}$ such that $d_G(y,y') \le A$.

Hence there exists $x' \in \lambda^b$ such that $d_G(x,x') \le An$.

Since Γ_K is properly embedded in Γ_G, there exists $f(N)$ such that λ^b lies outside an $f(N)$-ball about p in Γ_G and $f(N) \to \infty$ as $N \to \infty$.

Therefore, $d_G(x', p) \ge f(N)$.

Thus

$$d_G(x,p) \ge f(N) - d_G(x,x') \ge f(N) - An.$$

Also $d_G(x,p) \ge n$.

Therefore, $d_G(x,p) \ge \frac{f(N)}{A+1}$, that is, x lies outside the $\frac{f(N)}{A+1}$-ball about p in Γ_G. □

Theorem 3.11. *Consider a short exact sequence of pairs of finitely generated groups*

$$1 \to (K, K_1) \to (G, N_G(K_1)) \xrightarrow{p} (Q, Q_1) \to 1$$

with K strongly hyperbolic relative to the non-trivial proper subgroup K_1. If G preserves cusps, strongly hyperbolic relative to $N_G(K_1)$ and weakly hyperbolic relative to the subgroup K_1, then there exists a Cannon–Thurston map for the embedding $i\colon \Gamma_K \to \Gamma_G$, where Γ_K and Γ_G are Cayley graphs of K and G respectively.

Proof. It suffices to prove the condition of Lemma 3.6.

So, for a fixed reference point $p \in \Gamma_K$, we assume that $\hat{\lambda}$ is an electric geodesic segment in $\hat{\Gamma}_K$ such that $\lambda^b (\subset \Gamma_K)$ lies outside an N-ball $B_N(p)$ around p. Let β_{pel} be a quasigeodesic in the partially electrocuted space Γ_G^{pel} joining the end points of $\hat{\lambda}$. Let $Pr_{\hat{\lambda}}$ be a nearest point projection from Γ_G^{pel} onto the quasiconvex set $B_{\hat{\lambda}}$. Let $\beta'_{\text{pel}} = Pr_{\hat{\lambda}}(\beta_{\text{pel}})$, then β'_{pel} is a quasigeodesic in Γ_G^{pel} lying on $B_{\hat{\lambda}}$. So β'_{pel} lies in a P-neighborhood of β_{pel} in Γ_G^{pel}. But β'_{pel} might backtrack. β'_{pel} can be modified to form a quasigeodesic γ_{pel} in Γ_G^{pel} of the same type (i.e., lying in a P-neighborhood of β_{pel}) without backtracking with end points remaining the same. By Lemma 3.3, β_{pel} and γ_{pel} satisfy bounded coset (horosphere) penetration properties with the closed sets (horosphere-like sets) in Γ_G corresponding to the left cosets of $N_G(K_1)$ in G. Thus if γ_{pel} penetrates a horosphere-like set $C_{gN_G(K_1)}$ corresponding to the left coset $gN_G(K_1)$ of $N_G(K_1)$ in G and β_{pel} does not, then the length of the geodesic traversed by γ^h, where γ^h is the electroambient path representative of γ_{pel}, inside $C_{gN_G(K_1)} \times [0,\infty)$ is uniformly bounded. Let $\mathcal{C} = \{C_{gN_G(K_1)}\colon g \in G\}$. Thus there exists $C_1 \ge 0$ such that if $x \in \beta^b_{\text{pel}} = \beta_{\text{pel}} \backslash \mathcal{C}$, then there exists $y \in \gamma^b_{\text{pel}} = \gamma_{\text{pel}} \backslash \mathcal{C}$ such that $d_G(x,y) \le C_1$.

Since $y \in \gamma^b_{\text{pel}} \subset B_{\lambda^b}$, by Lemma 3.10, $d_G(y,p) \ge \frac{f(N)}{A+1}$.

Therefore, $d_G(x,p) \ge \frac{f(N)}{A+1} - C_1 (= M(N)$, say) and $M(N) \to \infty$ as $N \to \infty$. By Lemma 3.6, a Cannon–Thurston map for $i\colon \Gamma_K \to \Gamma_G$ exists. □

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