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On the Iwasawa algebra associated to a normal element of $\mathbb{C}_p$

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Abstract. Given a prime number p and the Galois orbit $O(x)$ of a normal element x of $\mathbb{C}_p$, the topological completion of the algebraic closure of the field of p-adic numbers, we study the Iwasawa algebra of $O(x)$ with scalars drawn from $\mathbb{Q}_p$ and relate it with $\mathbb{Q}_p$-distributions and functionals.

Keywords. Iwasawa algebra; local fields; distributions.

1. Introduction

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\bar{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\bar{\mathbb{Q}}_p$ with respect to the p-adic valuation. Let $O(x)$ be the orbit of a normal element $x \in \mathbb{C}_p$, with respect to the Galois group $G_{(p)} = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$. In this paper we are going to study the Iwasawa algebra of $O(x)$ with scalars drawn from $\mathbb{Q}_p$, and relate this with distributions defined on $O(x)$ with values in $\mathbb{Q}_p$ and $\mathbb{Q}_p$-functionals defined on the closure of the polynomial ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$.

The paper consists of three sections. The first one contains notations and some basic results of Iwasawa algebra, and relations with distributions. Moreover, the measures correspond to the sequences of Iwasawa algebra that are uniformly bounded. The second section is concerned with Iwasawa algebra of $O(x)$ with scalars drawn from $\mathbb{Q}_p$, in the case of a normal element x. In this case the Iwasawa algebra coincides, via an ismorphism of $\mathbb{Q}_p$-vector spaces, with the space of $\mathbb{Q}_p$-linear morphisms defined on the algebraic part of the closure of the polynomial ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$ (see Theorem 2). Also, we give a new interpretation of a theorem of Serre (see Remark 7). In the last section, our goal is to obtain an isomorphism of Banach spaces between the measures on $O(x)$ with values in the algebraic closure of $\mathbb{Q}_p$, which are equivariant with respect to the absolute Galois group, on the one hand, and the Banach algebra of formal series with bounded coefficients in $\mathbb{Q}_p$ on the other (see Theorem 3).

2. Notations and basic results

Let $\{G_n, \varphi_n\}_{n\geq 1}$ be a projective system, that is a sequence of pairs where G_n is a finite group and $\varphi_n\colon G_{n+1} \to G_n$ is a surjective homomorphism of groups, for all $n \geq 1$. Denote

$G = \varprojlim G_n$ the profinite group defined by this projective system. Let A be a closed subring of $\mathbb{C}_p$. The canonical homomorphism φ_n induces a homomorphism of the corresponding group rings $A[G_{n+1}] \to A[G_n]$. We obtain the completed group ring $A[[G]]$ in such a way, that is an A-algebra, defined as the projective limit

$$A[[G]] = \varprojlim A[G_n],$$

usually called the Iwasawa algebra of G with scalars drawn from A.

Now let $f_n\colon G \to G_n$ be the natural projection. By definition, the elements of G are sequences $(g_1, g_2, \ldots, g_n, \ldots)$ where $g_n \in G_n$ and $\varphi_n(g_{n+1}) = g_n$ for all $n \geq 1$. By a *ball* in G we mean a subset of the form $f_n^{-1}(g)$, where $g \in G_n$. The balls in G define a topology, the topology of projective limit. Then G is a compact space. Moreover, the topology on G may be defined by a distance (metric). Namely, let $\{\varepsilon_n\}_{n\geq 1}$ be a strictly decreasing sequence of positive real numbers with zero limit. If $g = (g_n)_{n\geq 1}$, $h = (h_n)_{n\geq 1}$ are two points of G we put $d(g, h) = \varepsilon_m$, where m is the smallest natural number n such that $g_n \neq h_n$. The metric just defined on G is an ultrametric, i.e. for all $g, h, u \in G$ one has

$$d(g, h) \leq \max\{(d(g, u), d(h, u)\}.$$

According to [MS] we have the following

DEFINITION 1

By a *distribution* on $\{G_n, \varphi_n\}_{n\geq 1}$ (or on G) with values in A we mean a set $\mu = \{\mu_n\}_{n\geq 1}$ of mappings: $\mu_n\colon G_n \to A$ such that the following compatibility relations

$$\mu_n(g) = \sum_{h\in\varphi_n^{-1}(g)} \mu_{n+1}(h) \tag{1}$$

hold for all $n \geq 1$ and all $g \in G_n$.

Denote by $\mathcal{D}(G, A)$ the set of all distributions $\mu = \{\mu_n\}_{n\geq 1}$ as above. For any $\mu \in \mathcal{D}(G, A)$ denote

$$\|\mu\| = \sup_{n\geq 1}\{|\mu_n(g)|\colon g \in G_n\}, \tag{2}$$

the norm of μ. If $\|\mu\| < \infty$ we say that μ is a measure and, consider $\mathcal{M}(G, A)$ the set of all measures on G with values in A.

The space $\mathcal{D}(G, A)$ can be endowed with the convolution product of distributions. More precisely, if $\mu, \nu \in \mathcal{D}(G, A)$ then $\mu \star \nu$ is defined by the sequence $(\mu \star \nu)_n$, where by definition

$$(\mu \star \nu)_n(g) = \sum_{h\in G_n} \mu_n(h)\nu_n(h^{-1}g). \tag{3}$$

One has one-to-one correspondence $\theta_n\colon \mathcal{D}(G_n, A) \to A[G_n]$, $\theta_n(\mu) = \sum_{\sigma\in G_n} \mu(\sigma)\sigma$, which induces commutative squares

$$\begin{array}{ccc}
\mathcal{D}(G_{n+1}, A) & \xrightarrow{\theta_{n+1}} & A[G_{n+1}] \\
{\scriptstyle N}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}} \\
\mathcal{D}(G_n, A) & \xrightarrow[\theta_n]{} & A[G_n]
\end{array}$$

where N is 'the norm', $N(\mu_{n+1}) = \mu_n$, as in formula (1).

Here and henceforth $A[[G]]$ will be denoted by $\Lambda_G(A)$. By passing to the projective limit we obtain an isomorphism of A-algebras $\theta\colon \mathcal{D}(G, A) \to \Lambda_G(A)$ defined by

$$\theta(\mu) := (\theta_n(\mu_n))_{n\geq 1}.$$

We have

$$\begin{aligned}(\theta(\mu) \star \theta(\nu))_n &= \theta_n(\mu_n) \cdot \theta_n(\nu_n) \\ &= \left(\sum_{\sigma\in G_n} \mu_n(\sigma)\sigma\right) \cdot \left(\sum_{\tau\in G_n} \nu_n(\tau)\tau\right) \\ &= \sum_{\epsilon\in G_n} \left(\sum_{\sigma\in G_n} \mu_n(\sigma)\nu_n(\sigma^{-1}\epsilon)\right)\epsilon. \qquad (4)\end{aligned}$$

On the other hand,

$$\theta_n((\mu \star \nu)_n) = \sum_{\epsilon\in G_n} ((\mu \star \nu)_n(\epsilon))\epsilon. \qquad (5)$$

By (4) and (5) we obtain $\theta(\mu \star \nu) = \theta(\mu) \cdot \theta(\nu)$ for any $\mu, \nu \in \mathcal{D}(G, A)$. The measures $\mathcal{M}(G, A) \subset \mathcal{D}(G, A)$ correspond to $\Lambda_G^{(b)}(A) := \{u \in \Lambda_G(A)\colon u = (u_n)_{n\geq 1}$ such that there exists $M > 0$ with $|u_n| \leq M$, for any $n \geq 1\}$, that means sequences of $\Lambda_G(A)$ are uniformly bounded. One has the following well-known result: There exists an isomorphism of A-algebras between $\mathcal{D}(G, A)$ and the Iwasawa algebra $\Lambda_G(A)$ and, moreover, the measures correspond to the sequences $\Lambda_G^{(b)}(A)$ of $\Lambda_G(A)$ that are uniformly bounded.

Remark 1. In fact $\|\mu\| := \sup_{n\geq 1} \sup_{\sigma\in G_n} |\mu_n(\sigma)|$ and if $u \in \Lambda_G(A)$, $u = (u_n)_{n\geq 1}$, $u_n = \sum_{\sigma\in G_n} a_\sigma^{(n)}\sigma$, $\|u\| := \sup_{n\geq 1} \sup_{\sigma\in G_n} |a_\sigma^{(n)}|$ we have $\|\mu\| = \|\theta(\mu)\|$, so $\mathcal{M}(G, A)$ is isomorphic, via θ, with $\Lambda_G^{(b)}(A) = \{u \in \Lambda_G(A)\colon \|u\| < \infty\}$.

Remark 2.

(1) In the case $G = \mathbb{Z}_p$ the ring of p-adic integers, $\Lambda_{\mathbb{Z}_p}(A)$ is isomorphic to $A[[X]]$, the ring of formal power series in one variable with coefficients in A (see [La], [Pa], [VZ], [MS], [Iw]).
(2) One has the natural mappings $G \to A[G] \to A[[G]]$. The image of $g \in G$ is the Dirac measure supported at g. For more details, see [MS].

3. The case of a normal element of $\mathbb{C}_p$

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\bar{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\bar{\mathbb{Q}}_p$ (see [Ar], [APZ1], [APZ2]). Denote by $G_{(p)}$ the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. One knows that $G_{(p)}$ is canonically isomorphic to $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$, the group of all continuous automorphisms of $\mathbb{C}_p$. In what follows we shall identify these two groups.

For any subset E of $\mathbb{C}_p$ we denote by $\tilde{E}$ the topological closure of E in $\mathbb{C}_p$. In case E is a field, $\tilde{E}$ is a closed subfield of $\mathbb{C}_p$.

For any closed subgroup H of $G_{(p)}$ denote $\mathrm{Fix}(H) = \{x \in \mathbb{C}_p: \sigma(x) = x \text{ for all } \sigma \in H\}$. Then $\mathrm{Fix}(H)$ is a closed subfield of $\mathbb{C}_p$. If $x \in \mathbb{C}_p$, denote $H(x) = \{\sigma \in G_{(p)}: \sigma(x) = x\}$. Then $H(x)$ is a subgroup of $G_{(p)}$, and $\mathrm{Fix}(H(x)) = \widetilde{\mathbb{Q}_p[x]}$ is the closure of the polynomial ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$. We say that x is a *topological generic element* of $\widetilde{\mathbb{Q}_p[x]}$. Moreover, by [APZ1] one knows that any closed subfield K of $\mathbb{C}_p$ has a topological generic element, i.e. there exists $x \in K$ such that $K = \widetilde{\mathbb{Q}_p[x]}$. It is proved in [APZ3] that for any element y of $\mathbb{C}_p$, the ring $\mathbb{Q}_p[y]$ and the field $\mathbb{Q}_p(y)$ have the same topological closure in $\mathbb{C}_p$, that is, $\widetilde{\mathbb{Q}_p[y]} = \widetilde{\mathbb{Q}_p(y)}$. As a consequence, the topological closure in $\mathbb{C}_p$ of a ring of the form $\mathbb{Q}_p[y]$ is always a field.

Let $x \in \mathbb{C}_p$. Denote $O(x) = \{\sigma(x): \sigma \in G_{(p)}\}$, the orbit of x. The map $\sigma \rightsquigarrow \sigma(x)$ from $G_{(p)}$ to $O(x)$ is continuous, and it defines a homeomorphism from $G_{(p)}/H(x)$ (endowed with the quotient topology) to $O(x)$ (endowed with the induced topology from $\mathbb{C}_p$) (see [APZ1]). In this way, $O(x)$ is a closed compact and totally disconnected subspace of $\mathbb{C}_p$, and the group $G_{(p)}$ acts continuously on $O(x)$: if $\sigma \in G_{(p)}$, $\tau(x) \in O(x)$ then $\sigma \star \tau(x) = (\sigma\tau)(x)$.

Let us now recall a few results from [APZ] and [APZ1].

Theorem 1 [APZ]. *Let $\mathbb{Q}_p \subseteq K \subseteq L \subseteq \bar{\mathbb{Q}}_p$, with L/K a normal extension. Then there exists an element u in the topological closure $\tilde{L}$ of L in $\mathbb{C}_p$ such that the K-vector space generated by all the conjugates of u over K is dense in $\tilde{L}$.*

PROPOSITION 1 [APZ1]

(1) *The subfield $\widetilde{\mathbb{Q}_p[x]}$ is canonically isomorphic to the set of all equivariant continuous functions $f: O(x) \to \mathbb{C}_p$, i.e. the continuous functions which verify the condition: $f(\sigma \star y) = \sigma(f(y))$ for all $\sigma \in G_{(p)}$ and $y \in O(x)$.*

(2) *There exists a family $\{M_n(x)\}_{n\geq 0}$ of polynomials in $\mathbb{Q}_p[x]$ such that*

   (i) $\deg M_n(x) = n$ *for all* $n \geq 0$,

   (ii) $\frac{1}{p} < |M_n(x)| \leq 1$,

   (iii) *Any element $f \in \widetilde{\mathbb{Q}_p[x]}$ can be written uniquely in the form: $f = \sum_{n\geq 0} a_n M_n(x)$ where $\{a_n\}_n$ is a sequence of elements in $\mathbb{Q}_p$ such that $\lim_n a_n = 0$. Moreover one has $\|f\| = \sup_{n\geq 0} |a_n M_n(x)|$.*

(3) *If $K_x = \widetilde{\mathbb{Q}_p[x]} \cap \bar{\mathbb{Q}}_p$, then $\tilde{K}_x = \widetilde{\mathbb{Q}_p[x]}$ and $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K_x)$ is canonically isomorphic to $H(x)$.*

DEFINITION 2

An element $x \in \mathbb{C}_p$ is called normal if the extension $\widetilde{\mathbb{Q}_p(x)}/\mathbb{Q}_p$ is normal, i.e. $\mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\widetilde{\mathbb{Q}_p(x)}) = H(x)$ is a normal subgroup of $G_{(p)} = \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$.

This means that the algebraic extension $\widetilde{\mathbb{Q}_p(x)} \cap \bar{\mathbb{Q}}_p/\mathbb{Q}_p$ is normal. Then $G' = \mathrm{Gal}_{\mathrm{cont}}(\widetilde{\mathbb{Q}_p(x)}/\mathbb{Q}_p)$ is canonically isomorphic to $G_{(p)}/H(x) \simeq O(x)$, the orbit of x.

Now, let $K/\mathbb{Q}_p$ be a normal extension and $\mathbb{Q}_p \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset K = \cup_{n=1}^{\infty} K_n$ a tower such that $K_n/\mathbb{Q}_p$ is normal for each $n \geq 1$. Denote $G_n = \mathrm{Gal}(K_n/\mathbb{Q}_p)$,

$G = \mathrm{Gal}(K/\mathbb{Q}_p)$, $H_n = \mathrm{Gal}(K/K_n)$. We have $K_n = \mathrm{Fix}(H_n)$, $G_n \simeq G/H_n$ and $G \simeq \varprojlim G_n$.

From Theorem 1 there exists $x \in \tilde{K}$ such that $B_x = \{\sigma(x)\colon \sigma \in G\}$ is $\mathbb{Q}_p$-linear independent and $\overline{\mathbb{Q}_p\langle B_x\rangle} = \tilde{K}$. Let μ be an equivariant measure on $O(x) \simeq G'$, which means $\mu(\sigma B) = \sigma\mu(B)$ for any $\sigma \in G_{(p)}$ and any B open ball in $O(x)$. (Instead of μ, we may consider a distribution on $O(x)$ like Haar distribution with x a Lipschitz element as in [APZ2].)

Let us consider

$$e^{(n)} = \int_{H_n} \sigma(x)\mathrm{d}\mu(\sigma) = \int_{H_n(x)} t\mathrm{d}\mu_x(t). \tag{6}$$

One has that $B_n = \{ge^{(n)}\colon \hat{g} \in G/H_n\}$ is a normal basis of $K_n/\mathbb{Q}_p$ (see [APZ]). It is easy to see that $e^{(n)} = \sum_{\hat{\tau}\in H_n/H_{n+1}} \tau e^{(n+1)}$ and moreover $ge^{(n)} = \sum_{\hat{\tau}\in H_n/H_{n+1}} g\tau e^{(n+1)}$, when $\hat{g} \in G/H_n$.

Remark 3. From (6) one has $\lim_{n\to\infty} \frac{e^{(n)}}{\mu(H_n)} = x$.

For any $n \geq 1$, each $\varphi \in \mathrm{Hom}(K_n, \mathbb{Q}_p)$ can be identified with $\tilde{\varphi} \in \mathbb{Q}_p[G_n]$ given by

$$\tilde{\varphi} = \sum_{\sigma\in G_n} \varphi(\sigma e^{(n)})\sigma.$$

In such a way to any $f = \sum_{\sigma\in G_n} a_\sigma\sigma \in \mathbb{Q}_p[G_n]$ it corresponds to a unique $\varphi \in \mathrm{Hom}(K_n, \mathbb{Q}_p)$ such that $\varphi(\sigma e^{(n)}) = a_\sigma$. So $\mathrm{Hom}(K_n, \mathbb{Q}_p) \simeq \mathbb{Q}_p[G_n]$ is an isomorphism of $\mathbb{Q}_p$-vector spaces. The restriction map from $\mathrm{Hom}(K_{n+1}, \mathbb{Q}_p)$ to $\mathrm{Hom}(K_n, \mathbb{Q}_p)$, via the above isomorphism, gives us the natural map $\phi_n^{n+1}\colon \mathbb{Q}_p[G_{n+1}] \to \mathbb{Q}_p[G_n]$, which is a morphism of $\mathbb{Q}_p$-algebras. The augumentation ideal

$$I = \left\{ \sum_{g\in G_{n+1}} a_g g\colon \sum_{\bar{h}\in H_n/H_{n+1}} a_{hg} = 0 \right\},$$

is if fact Ker ϕ_n^{n+1}, which could be described as the ideal $\big\{ \sum_{g\in G_{n+1}} a_g g$: If $A \subset G_{n+1}$ is a maximal set with the property that any two elements of A are inequivalent mod H_n/H_{n+1}, then $\sum_{g\in A} a_g = 0\big\}$. It is clear that if an element $y \in \mathbb{Q}_p[G_{n+1}]$, $y = \sum_{g\in G_{n+1}} a_g g$ has the above property, then for any $\sigma \in G_n$, σy and $y\sigma$ have the same property. Moreover, I is a $\mathbb{Q}_p$-vector space. One has $\Lambda_{O(x)}(\mathbb{Q}_p) \simeq \mathrm{Hom}(K, \mathbb{Q}_p)$ and this isomorphism g can be given as follows: If $u = (u_n)_n \in \mathbb{Q}_p[[G]] = \Lambda_{O(x)}(\mathbb{Q}_p)$ and $u_n = \sum_{\bar{\sigma}\in G_n} a_{\bar{\sigma}}\bar{\sigma} \in \mathbb{Q}_p[G_n]$ we can define $g(u)\colon K \to \mathbb{Q}_p$ by $g(u)(\bar{\sigma}e^{(n)}) = a_{\bar{\sigma}}$, for any $\bar{\sigma} \in G_n$.

We can define another isomorphism $g_K\colon \Lambda_{O(x)}(\mathbb{Q}_p) \to \mathrm{Hom}(K, \mathbb{Q}_p)$, which in some sense is closer to K than g defined above. More precisely, let $u = (u_n)_n \in \Lambda_{O(x)}(\mathbb{Q}_p)$, $u_n = \sum_{\bar{\sigma}\in G_n} a_{\bar{\sigma}}\bar{\sigma} \in \mathbb{Q}_p[G_n]$ and $\alpha_n := \sum_{\bar{\sigma}\in G_n} a_{\bar{\sigma}}\bar{\sigma}e^{(n)} \in K_n$. We have $\mathrm{Tr}_{K_n/K_{n-1}}(\alpha_n) = \alpha_{n-1}$, for any $n \geq 2$. Let us define $g_K(u_n)(z) := \mathrm{Tr}_{K_n/\mathbb{Q}_p}(\alpha_n z)$, for any $z \in K_n$. We see that for any $z \in K_{n-1}$ one has $\mathrm{Tr}_{K_n/\mathbb{Q}_p}(\alpha_n z) = \mathrm{Tr}_{K_{n-1}/\mathbb{Q}_p}(\alpha_{n-1}z)$, so $g_K(u_n)|_{K_{n-1}} = g_K(u_{n-1})$. By definition $g_K(u)\colon K \to \mathbb{Q}_p$ is $g_K(u_n)$ on K_n, for any $n \geq 1$. It is easy to see that g_K is well-defined and is an isomorphism of $\mathbb{Q}_p$-vector spaces.

Let $\Lambda_{O(x)}^{(*)}(\mathbb{Q}_p)$ be the subspace of $\Lambda_{O(x)}(\mathbb{Q}_p)$ which consists of the sequences $u = (u_n)_n \in \Lambda_{O(x)}(\mathbb{Q}_p)$, $u_n = \sum_{\bar{\sigma}\in G_n} a_{\bar{\sigma}}^{(n)}\bar{\sigma} \in \mathbb{Q}_p[G_n]$, such that there exists a positive real number M that satisfies

$$\frac{|\mathcal{D}_{K_n/\mathbb{Q}_p}| \cdot \left| \sum_{\bar{\sigma} \in G_n} a_{\bar{\sigma}}^{(n)} \bar{\sigma} e^{(n)} \right|}{[K_n : \mathbb{Q}_p]} < M, \text{ for any } n \geq 1,$$

where $\mathcal{D}_{K_n/\mathbb{Q}_p}$ is different from $K_n/\mathbb{Q}_p$. It is clear that by using Proposition 2 of [APVZ], $g_K(\Lambda_{O(x)}^{(*)}(\mathbb{Q}_p)) = \mathrm{Hom}_{\mathrm{cont}}(K, \mathbb{Q}_p)$. Let $\mathcal{D}^*(G, \mathbb{Q}_p)$ be the subspace of $\mathcal{D}(G, \mathbb{Q}_p)$ that corresponds to $\Lambda_{O(x)}^{(*)}(\mathbb{Q}_p)$, via θ, which is the isomorphism between the $\mathbb{Q}_p$-algebras $\mathcal{D}(G, \mathbb{Q}_p)$ and $\Lambda_{O(x)}(\mathbb{Q}_p)$ described in §1.

Finally, let $\varphi \colon K \to \mathbb{Q}_p$ be a $\mathbb{Q}_p$-linear morphism that is bounded on the unit ball by an absolute constant M. Let $a \in \tilde{K}$, $a = \lim_{n\to\infty} a_n$, $a_n \in K$. We have $|\varphi(a_{n+1}) - \varphi(a_n)| = |\varphi(a_{n+1} - a_n)| \leq M|a_{n+1} - a_n|$ for n large enough so the sequence $\{\varphi(a_n)\}_{n\geq 1}$ is convergent. We can define $\tilde{\varphi}(a) = \lim_{n\to\infty} \varphi(a_n)$ and $|\tilde{\varphi}(a)| = |\varphi(a_n)| \leq M|a_n|$ for n large enough, which means $|\tilde{\varphi}(a)| \leq M|a|$, and $\tilde{\varphi}$ is continuous. To sum up, one has the following.

Theorem 2. *Let K be a Galois extension of $\mathbb{Q}_p$ with $G = \mathrm{Gal}(K/\mathbb{Q}_p)$. Let $\tilde{K}$ be the closure of K in $\mathbb{C}_p$, and let x be an element of $\tilde{K}$ (which always exists by Theorem 1) such that $B_x := \{\sigma(x) \colon \sigma \in G\}$ is $\mathbb{Q}_p$-linear independent and $\mathbb{Q}_p\langle B_x\rangle$ is dense in $\tilde{K}$. Then for $O(x)$ the orbit of x under the Galois group of K, and $\mathcal{D}^*(G, \mathbb{Q}_p)$ (respectively $\Lambda_{O(x)}^{(*)}(\mathbb{Q}_p)$) the subspace of $\mathcal{D}(G, \mathbb{Q}_p)$ (respectively of $\Lambda_{O(x)}(\mathbb{Q}_p)$) defined before, we have the following commutative diagram*

$$\begin{array}{ccccc}
\mathcal{D}(G, \mathbb{Q}_p) & \xrightarrow{\theta} & \Lambda_{O(x)}(\mathbb{Q}_p) & \xrightarrow{g, g_K} & \mathrm{Hom}(K, \mathbb{Q}_p) \\
{\scriptstyle i}\uparrow & & {\scriptstyle i}\uparrow & & {\scriptstyle i}\uparrow \\
\mathcal{D}^*(G, \mathbb{Q}_p) & \xrightarrow{\theta} & \Lambda_{O(x)}^{(*)}(\mathbb{Q}_p) & \xrightarrow{g_K} & \mathrm{Hom}_{\mathrm{cont}}(\tilde{K}, \mathbb{Q}_p)
\end{array}$$

where the isomorphisms θ, g, g_K are isomorphisms of $\mathbb{Q}_p$-vector spaces. The vertical rows are the natural inclusion morphisms.

Remark 4.

(1) Using Theorem 1 above we may replace in Theorem 2, $\mathbb{Q}_p$ with K, K with L and $O(x)$ with $\mathrm{Gal}(L/K)$.
(2) The canonical projections $\mathbb{Q}_p[[G]] \to \mathbb{Q}_p[G_n]$ correspond to the restriction of functionals.

Remark 5. We have a functoriality of the above diagram: Let $\bar{G}$ be a factor group of G, $\bar{K}$ the subfield of K that corresponds to $\bar{G}$ such that $\mathrm{Gal}(\bar{K}/\mathbb{Q}_p) = \bar{G}$, $\bar{x}$ a generic element of $\tilde{\bar{K}}$ such that $\overline{\langle O(\bar{x})\rangle_{\mathbb{Q}_p}} = \tilde{\bar{K}}$. In a natural way, via Theorem 2 in [APZ], using the trace map, one has a continuous surjection on $O(x)$ with values in $O(\bar{x})$. This gives us a surjective map from $\Lambda_{O(x)}(\mathbb{Q}_p)$ to $\Lambda_{O(\bar{x})}(\mathbb{Q}_p)$.

Next, let us consider an algebraic extension K of $\mathbb{Q}_p$ as above that is normal and, moreover, we assume that the trace Tr: $K \to \mathbb{Q}_p$ is continuous. Such an example of K on which Tr is continuous is $K = \cup_{n=1}^{\infty} \mathbb{Q}_p(\varepsilon_{p^n})$, where ε_{p^n} is a p^n-th root of unity.

We can find an element $x \in \tilde{K}$ (see [APZ]) such that:

(1) The element x has a trace, $\mathrm{Tr}(x) = \int_{O(x)} t \mathrm{d}\pi_x(t)$, where π_x is the Haar measure on $O(x)$.
(2) The $\mathbb{Q}_p$-vector space generated by $\{\sigma(x)\colon \sigma \in G_K = \mathrm{Gal}_{\mathrm{cont}}(\tilde{K}/\mathbb{Q}_p) \simeq O(x)\}$ is dense in $\tilde{K} = \widetilde{\mathbb{Q}_p[x]}$ and, moreover, the elements of $\{\sigma(x)\colon \sigma \in G_K\}$ are $\mathbb{Q}_p$-linear independent.

In what follows, our goal is to relate the functionals Φ defined on $\mathcal{C}_{G_K}(O(x), \mathbb{C}_p) \simeq \tilde{K}$ with values in $\mathbb{Q}_p$ on the one hand, and with $\mathcal{C}(O(x), \mathbb{Q}_p)$ and the functional Tr on the other hand.

As we can see in [APZ], if we have $\mathbb{Q}_p \subset K_n \subset K$, a tower of fields such that $[K_n\colon \mathbb{Q}_p] = d_n < \infty$, $H_n = \mathrm{Gal}(K/K_n) \leq G_K$ and $e_i^{(n)} = \mathrm{Tr}(\sigma_i(x)) = \mathrm{Tr}_{K_n}(\sigma_i(x)) = \mathrm{Tr}_{\sigma_i}(x)$, $\hat{\sigma}_i \in G/H_n$, i.e. $e_i^{(n)} = \int_{H_n(x)} \sigma_i(t) \mathrm{d}\pi_x(t)$, then the set $\{e_1^{(n)}, \ldots, e_{d_n}^{(n)}\}$ gives a basis of $K_n/\mathbb{Q}_p$. One has the following result.

PROPOSITION 2

Let $f\colon O(x) \to \mathbb{Q}_p$ be a locally constant function. Then there exists a unique functional $\Phi_f\colon \tilde{K} \to \mathbb{Q}_p$ that extends f. Moreover, there exists a field K_n, $\mathbb{Q}_p \subset K_n \subset \bar{\mathbb{Q}}_p$, $[K_n\colon \mathbb{Q}_p] < \infty$ and $\alpha_n \in K_n$ such that $\Phi_f(z) = \mathrm{Tr}(\alpha_n z)$, for any $z \in \tilde{K}$.

Proof. Because f is locally constant, there is $H_n \leq G_K$ of finite index such that f is constant on $\{\sigma_i H_n(x)\colon \hat{\sigma}_i \in G/H_n\}$. Let K_n be the subfield of K fixed by H_n, $[K_n\colon \mathbb{Q}_p] = d_n < \infty$. Let $\{e_1^{(n)}, \ldots, e_{d_n}^{(n)}\}$ be the basis of $K_n/\mathbb{Q}_p$ defined above. There exists a unique $\alpha_n \in K_n$ such that $\mathrm{Tr}_{K_n/\mathbb{Q}_p}(\alpha_n e_i^{(n)}) = f(\sigma_i x)$, for any $1 \leq i \leq d_n$. The functional $z \rightsquigarrow \mathrm{Tr}(\alpha_n z)$ on $\tilde{K}$ extends f. Indeed, for any $\sigma(x) \in O(x)$ we have $\mathrm{Tr}_{K_n}(\alpha_n \sigma(x)) = \alpha_n \mathrm{Tr}_{K_n}(\sigma(x)) = \alpha_n \mathrm{Tr}_{K_n}(\sigma_i(x)) = \alpha_n e_i^{(n)}$, if $\hat{\sigma} = \hat{\sigma}_i \pmod{H_n}$. It is easy to see that $\mathrm{Tr}(z) = \mathrm{Tr}_{K/\mathbb{Q}_p}(\mathrm{Tr}_K(z))$ so $\mathrm{Tr}(\alpha_n \sigma(x)) = \mathrm{Tr}_{K_n/\mathbb{Q}_p}(\alpha_n e_i^{(n)}) = f(\sigma_i(x)) = f(\sigma(x))$, which completes the proof. ■

Let us remark that for any $\alpha \in K = \cup_{n \geq 1} K_n$, the functional $z \rightsquigarrow \mathrm{Tr}(\alpha z)$ defined on $\tilde{K}$ with values in $\mathbb{Q}_p$ is locally constant on $O(x)$. Because $\alpha \in K$ there exists $n \geq 1$ such that $\alpha \in K_n$. Let $O(x) = \cup_{\hat{\sigma} \in G/H_n} \sigma H_n(x)$ be a decompositon of the orbit of x. We show that $z \rightsquigarrow \mathrm{Tr}(\alpha z)$ is a constant on $\sigma H_n(x)$. For the sake of simplicity let us suppose $\sigma = \{e\}$, the neutral element, and let $z \in H_n(x)$, say $z = \tau(x)$, $\tau \in H_n$. Then $\mathrm{Tr}(\alpha z) = \mathrm{Tr}_{K_n/\mathbb{Q}_p}(\mathrm{Tr}_{K_n}(\alpha \tau(x))) = \mathrm{Tr}_{K_n/\mathbb{Q}_p}(\alpha \mathrm{Tr}_{K_n} \tau(x)) = \mathrm{Tr}_{K_n/\mathbb{Q}_p}(\alpha \mathrm{Tr}_{K_n}(x)) = \mathrm{Tr}_{K_n/\mathbb{Q}_p}(\mathrm{Tr}_{K_n}(\alpha x)) = \mathrm{Tr}(\alpha x)$, so the claim is true. Now, we consider $\varphi_\alpha\colon \tilde{K} \to \mathbb{Q}_p$, $\varphi_\alpha(z) = \mathrm{Tr}(\alpha z)$. Then $\alpha \rightsquigarrow \varphi_\alpha$ is a bijective map from K to the set of locally constant functions on $O(x)$ with values in $\mathbb{Q}_p$. This map is continuous, but its inverse is not. Indeed, if this were true, for any continuous function $f\colon O(x) \to \mathbb{Q}_p$ we obtain a functional $\varphi\colon \tilde{K} \to \mathbb{Q}_p$ that extends f. But, if $\varphi\colon \tilde{K} \to \mathbb{Q}_p$ is a functional then the restriction f of φ to $O(x)$ must be Lipschitz, because for any $y, z \in O(x)$ we have $|f(y) - f(z)| = |\varphi(y - z)| \leq A_\varphi |y - z|$ for a constant $A_\varphi > 0$ that depends only on φ, so one has a contradiction.

Thus, a natural question arises: Is it true that any Lipschitz function on $O(x)$ with values in $\mathbb{Q}_p$ can be extended to a functional on $\tilde{K}$ with values in $\mathbb{Q}_p$?

Example. We consider $K = \cup_{n=1}^{\infty} \mathbb{Q}_p(\varepsilon_{p^n})$, where ε_{p^n} is a p^n-th root of unity. The extension $K/\mathbb{Q}_p$ is Galois, totally ramified and $\mathrm{Gal}(K/\mathbb{Q}_p) = \varprojlim U(\mathbb{Z}_{p^n}) =$

$\varprojlim U(\mathbb{Z}/p^n\mathbb{Z})$. We have $\tilde{K} \subset \mathbb{C}_p$, and apply the considerations from this section. We have $\varprojlim U(\mathbb{Z}/p^n\mathbb{Z}) \simeq U(O_p)$, the group of units of the ring of p-adic integers. We obtain the Iwasawa algebra from 'cyclotomic fields I' [La]: Let $T \in \tilde{K}$ be such that $O_T = \{\sigma(T)\colon \sigma \in G\}$ gives us a topological normal basis in $\tilde{K}$. Then the Iwasawa algebra associated to the orbit of T coincides with the classical Iwasawa algebra. We remark here that the Haar measure π_T is continuous on O_T because the trace Tr is continuous on K.

Remark 6. Let $K/\mathbb{Q}_p$ be such that $\tilde{K}/\mathbb{Q}_p$ is a normal extension. We know that there exists $x \in \tilde{K}$ which is a generic element for $\tilde{K}$. With the above notations, any $f \in \mathbb{Q}_p[G_n]$, $f = \sum_{g\in G_n} a_g g,\ a_g \in \mathbb{Q}_p$, is a continuous function on $O(x)$ with values in $\mathbb{Q}_p$, constant on the elements of the form $g\sigma(x)$, $\sigma \in H_n$. Indeed, $(\sum_{g\in G_n} a_g g)(g'\sigma(x)) := a_{g'}$, for any $\sigma \in H_n$, so f is locally constant on $O(x)$. In such a way, we can identify $\mathbb{Q}_p[G_n]$ with those elements of $\mathbb{Q}_p[G]$ that are constant modulo H_n and also, with the continuous functions defined on $O(x)$ with values in $\mathbb{Q}_p$ that are constant on the subsets of the form $\{g\sigma(x)\colon \sigma \in H_n\}$. In fact, these functions coincide with the restriction to $O(x)$ of linear maps of $\mathrm{Hom}_{\mathrm{cont}}(\tilde{K}/\mathbb{Q}_p)$ that are of the following form: $\mathrm{Tr}(\alpha x)$ with $\alpha \in K_n = \mathrm{Fix}(H_n)$.

Now, let $\mu \in \mathcal{M}(O(x), \mathbb{Q}_p)$ be a measure and Φ_μ the functional that corresponds to μ, defined on $\mathcal{C}_{G_{(p)}}(O(x), \mathbb{C}_p) \simeq \tilde{K}$ with values in $\mathbb{Q}_p$, $\Phi_\mu(f) := \int_{O(x)} f(t)\mathrm{d}\mu(t)$. This functional, as a function in $\mathrm{Hom}_{\mathrm{cont}}(\tilde{K}/\mathbb{Q}_p)$, is well determined by its restriction on $O(x)$ because $\langle O(x)\rangle$ is dense in $\tilde{K}$. Its restriction on $O(x)$ is a continuous function of the form $g_\mu(\sigma(x)) = \int_{O(x)} \sigma(t)\mathrm{d}\mu(t)$ and, we can view it as an element of the Iwasawa algebra $\Lambda^{(b)}_{O(x)}(\mathbb{Q}_p)$. In what follows, we suppose that $|x| \le 1$. If μ takes values in $\mathbb{Z}_p$, we define $f(\sigma(x)) = \int_{O(x)} \sigma(t)\mathrm{d}\mu(t)$. Because $\overline{\langle O(x)\rangle} = \tilde{K}$, this f gives us a functional on $\tilde{K}$ with values in $\mathbb{Z}_p$.

Remark 7. Let us suppose that K is a finite extension of $\mathbb{Q}_p$ and let L be a Galois extension of K such that $\mathrm{Gal}(L/K) \simeq U_1 \simeq \mathbb{Z}_p$. For example, in the case $K = \mathbb{Q}_p$ we can find such a $L \subseteq \cup_{s=1}^{\infty}\mathbb{Q}_p(\varepsilon_{p^s})$. The trace map is continuous on $\cup_{s=1}^{\infty}\mathbb{Q}_p(\varepsilon_{p^s})$ (see [APZ2]), so the Haar distribution is a measure and we can apply Remark 6. As in the case $K = \mathbb{Q}_p$, we can find a generic element $x \in \tilde{L}$ such that $\overline{\langle O(x)\rangle_K} = \tilde{L}$. Then $O(x)$ is homeomorphic to U_1 and $G \simeq U_1 \simeq \mathbb{Z}_p$ acts in a natural way on $O(x)$: $(\sigma, g(x)) \rightsquigarrow (\sigma g)(x)$. Of course, this action is equivalent to the action of $\mathbb{Z}_p \simeq U_1$ on U_1 given by: $(s, u) \rightsquigarrow u^s$.

Let us consider $f \in \mathcal{C}(G, O_K)$. Then $f \in \varprojlim O_K[G_n] \simeq O_K[[T]]$ if and only if there exists a measure μ_f on U_1 with values in O_K (i.e. $\|\mu_f\| \le 1$) such that $f(\sigma) = \int_{u\in U_1} \sigma(u)\mathrm{d}\mu_f(u)$, for any $\sigma \in G$.

The above result gives a new interpretation of a theorem of Serre (see [Ba]). Let $f \in \mathcal{C}(\mathbb{Z}_p \simeq U_1, O_K)$ be a continuous function. Then $f \in \varprojlim O_K\big[\big(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\big)\big] \simeq O_K[[T]]$ if and only if there exists a measure μ_f on U_1 with values in O_K (so $\|\mu_f\| \le 1$) such that $f(s) = \int_{u\in U_1} u^s \mathrm{d}\mu_f(u)$.

4. A special class of measures

In this paragraph x is not necessarily a normal element of $\mathbb{C}_p$, but we preserve the notations from the previous sections.

Let $\mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p)$ be the set of $G_{(p)}$-equivariant measures on $O(x)$ with values in $\bar{\mathbb{Q}}_p$, which means the measures μ such that $\mu(\sigma(B)) = \sigma(\mu(B))$, for any ball B in $O(x)$. Also, we consider

$$\mathbb{Q}_p\langle X\rangle := \left\{ \sum_{n\geq 0} a_n X^n \colon \{a_n\}_{n\geq 0} \text{ is a bounded sequence of } \mathbb{Q}_p \right\}.$$

On $\mathbb{Q}_p\langle X\rangle$ we have a norm defined as follows:

$$\left\| \sum_{n\geq 0} a_n X^n \right\| := \sup_{n\geq 0} |a_n|,$$

and in such a way $\mathbb{Q}_p\langle X\rangle$ becomes a Banach algebra with respect to this norm.

We have the following result

Theorem 3. *Let x be an element of $\mathbb{C}_p$. We have an isomorphism of Banach spaces between $\mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p)$ and $\mathbb{Q}_p\langle X\rangle$.*

Proof. Let $\theta\colon \mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p) \to \mathbb{Q}_p\langle X\rangle$ be defined as follows:

$$\theta(\mu) := \sum_{n\geq 0} \left(\int_{O(x)} M_n(t)\mathrm{d}\mu(t) \right) X^n,$$

where the M_n's were defined in Proposition 1 of §2. From Theorem 2 [APVZ], the map $\mu \to \varphi_\mu$, $\varphi_\mu(f) := \int_{O(x)} f\mathrm{d}\mu$ is a functional on $\mathcal{C}_{G_{(p)}}(O(x), \mathbb{C}_p)$ with values in $\mathbb{Q}_p$, so $\varphi_\mu(M_n) = \int_{O(x)} M_n(t)\mathrm{d}\mu(t) \in \mathbb{Q}_p$. We have

$$\left| \int_{O(x)} M_n(t)\mathrm{d}\mu(t) \right| \leq |M_n(x)| \cdot \|\mu\| \leq \|\mu\|,$$

for any $n \geq 0$, and we infer that θ is well defined.

The surjectivity of θ. Let $\sum_{n\geq 0} a_n X^n \in \mathbb{Q}_p\langle X\rangle$ be such that $\{a_n\}_{n\geq 0}$ is a bounded sequence in $\mathbb{Q}_p$. By Proposition 7 [APVZ], there exists a unique $\varphi\colon \mathcal{C}_{G_{(p)}}(O(x), \mathbb{C}_p) \to \mathbb{Q}_p$ defined by $\varphi(M_n(x)) = a_n$ and if $f = \sum_{n\geq 0} \alpha_n M_n$, $\alpha_n \to 0$, $\alpha_n \in \mathbb{Q}_p$, we put

$$\varphi(f) := \sum_{n\geq 0} \alpha_n a_n \in \mathbb{Q}_p.$$

From Theorem 1 [APVZ] we obtain $\mu_\varphi \in \mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p)$ and

$$\theta(\mu_\varphi) = \sum_{n\geq 0} \left(\int_{O(x)} M_n(t)\mathrm{d}\mu_\varphi(t) \right) X^n = \sum_{n\geq 0} a_n X^n.$$

The injectivity of θ. Let $\mu_1, \mu_2 \in \mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p)$ be such that $\theta(\mu_1) = \theta(\mu_2)$. Then $\int_{O(x)} M_n(t)\mathrm{d}\mu_1(t) = \int_{O(x)} M_n(t)\mathrm{d}\mu_2(t)$, so $\varphi_{\mu_1}(M_n) = \varphi_{\mu_2}(M_n)$, for any $n \geq 0$, that is bounded in $\mathbb{Q}_p$ and then

$$\varphi_{\mu_1}\Big|_{\mathcal{C}_{G_{(p)}}(O(x),\mathbb{C}_p)} = \varphi_{\mu_2}\Big|_{\mathcal{C}_{G_{(p)}}(O(x),\mathbb{C}_p)},$$

so $\tilde{\varphi}_{\mu_1} = \tilde{\varphi}_{\mu_2}$ implies $\mu_1 = \mu_2$ from the uniqueness of the representation, and the theorem is proved. ■

On $\mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p)$ we have the norm $\|\mu\| := \sup_{B\in\Omega(x)} |\mu(B)| < \infty$, where $\Omega(x)$ is the set of open balls in $O(x)$. One has the following result.

Lemma 1. $\frac{1}{p} \cdot \|\mu\| \leq \|\theta(\mu)\| \leq \|\mu\|$.

Proof. We have $\|\theta(\mu)\| = \sup_{n\geq 0} \left| \int_{O(x)} M_n(t)\mathrm{d}\mu(t)\right|$ and $\|\mu\| = \|\varphi_\mu\|$, where

$$\varphi_\mu \colon \mathcal{C}_{G_{(p)}}(O(x), \mathbb{C}_p) \to \mathbb{Q}_p,\ \varphi_\mu(f) := \int_{O(x)} f(t)\mathrm{d}\mu(t).$$

Because $\|\varphi_\mu\| = \sup_{||f||\leq 1} \left| \int_{O(x)} f(t)\mathrm{d}\mu(t)\right|$ and $\frac{1}{p} < |M_n(x)| \leq 1$, one has

$$\left| \int_{O(x)} M_n(t)\mathrm{d}\mu(t)\right| \leq \sup_{||f||\leq 1} \left| \int_{O(x)} f(t)\mathrm{d}\mu(t)\right| = \left\|\varphi_\mu\right\| = \|\mu\|,$$

so $\|\theta(\mu)\| \leq \|\mu\|$.

On the other hand, $f = \sum_{n\geq 0} \alpha_n M_n$ with $\alpha_n \in \mathbb{Q}_p$, $\alpha_n \to 0$ and $\|f\| = \sup_{n\geq 0} \left|\alpha_n M_n(x)\right| \leq 1$. One obtains

$$\begin{aligned} &\left| \int_{O(x)} f(t)\mathrm{d}\mu(t)\right| \\ &\quad = \left| \int_{O(x)} \left(\sum_{n\geq 0} \alpha_n M_n(t)\right)\mathrm{d}\mu(t)\right| \\ &\quad = \left| \sum_{n\geq 0} \alpha_n \int_{O(x)} M_n(t)\mathrm{d}\mu(t)\right| \leq \sup_{n\geq 0} |\alpha_n| \cdot \left| \int_{O(x)} M_n(t)\mathrm{d}\mu(t)\right|. \end{aligned}$$

Because $\left| \int_{O(x)} M_n(t)\mathrm{d}\mu(t)\right| \leq \|\theta(\mu)\|$, for any $n \geq 0$ and $\frac{1}{p} < |M_n(x)| \leq 1$ we have $\frac{1}{p}|\alpha_n| < |\alpha_n M_n| \leq 1$, so $|\alpha_n| < p$, for any n that means $\left| \int_{O(x)} f(t)\mathrm{d}\mu(t)\right| \leq p\,\|\theta(\mu)\|$. From this we obtain $\|\varphi_\mu\| = \|\mu\| = p\,\|\theta(\mu)\|$ and the lemma is proved. ■

In case x is normal one has the following:

Remark 8.

(1) From Theorems 1 and 2 [APVZ], $\mathcal{M}_{G_{(p)}}(O(x), \bar{\mathbb{Q}}_p) \simeq \mathrm{Hom}_{\mathrm{cont}}(\widetilde{\mathbb{Q}_p[x]}, \mathbb{Q}_p)$ and by Theorems 2 and 3, we have an isomorphism of $\mathbb{Q}_p$-vector spaces $\Lambda^{(*)}_{O(x)}(\mathbb{Q}_p) \simeq \mathbb{Q}_p\langle X\rangle$.

(2) Let μ, ν be $G_{(p)}$-equivariant measures on $O(x)$ with values in $\bar{\mathbb{Q}}_p$ and φ, ψ the corresponding functionals of $\mathrm{Hom}_{\mathrm{cont}}(\widetilde{\mathbb{Q}_p[x]}, \mathbb{Q}_p)$ via the isomorphism defined in Remark 8(1). From Theorem 2 we obtain $\tilde{\varphi}, \tilde{\psi}$, the elements of the Iwasawa algebra that correspond to φ, ψ. The product $\tilde{\varphi} \cdot \tilde{\psi}$ in $\Lambda^{(*)}_{O(x)}(\mathbb{Q}_p)$ corresponds to the convolution product $\mu \star \nu$, which is bounded as a measure on $O(x)$, so the linear map defined on $\tilde{K}$ with values in $\mathbb{Q}_p$ that corresponds to $\tilde{\varphi} \cdot \tilde{\psi}$ is continuous. It follows that

$$\int_{O(x)} M_n(t)\mathrm{d}(\mu \star \nu)(t) \in \mathbb{Q}_p.$$

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