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On the Matlis duals of local cohomology modules and modules of generalized fractions

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring with non-zero identity, $\mathfrak{a}$ a proper ideal of R and M a finitely generated R-module with $\mathfrak{a}M \neq M$. Let $D(-) := \mathrm{Hom}_R(-, E)$ be the Matlis dual functor, where $E := E(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. In this paper, by using a complex which involves modules of generalized fractions, we show that, if $x_1, \ldots, x_n$ is a regular sequence on M contained in $\mathfrak{a}$, then $H^n_{(x_1,\ldots,x_n)R}D(H^n_{\mathfrak{a}}(M)))$ is a homomorphic image of $D(M)$, where $H^i_{\mathfrak{b}}(-)$ is the i-th local cohomology functor with respect to an ideal $\mathfrak{b}$ of R. By applying this result, we study some conditions on a certain module of generalized fractions under which $D(H^n_{(x_1,\ldots,x_n)R}(D(H^n_{\mathfrak{a}}(M)))) \cong D(D(M))$.

Keywords. Local cohomology module; Matlis dual functor; module of generalized fractions; filter regular sequence.

1. Introduction

For an ideal $\mathfrak{a}$ of a commutative Noetherian local ring $(R, \mathfrak{m})$, we denote the n-th local cohomology functor with respect to $\mathfrak{a}$ by $H^n_{\mathfrak{a}}(-)$ and the Matlis dual functor $\mathrm{Hom}_R(-, E)$ by $D(-)$, where E is the injective hull of the field $R/\mathfrak{m}$. Also, for a sequence $\underline{x} := x_1, \ldots, x_n$ of elements of R, we use $\underline{x}R$ or $(x_1, \ldots, x_n)R$ to denote the ideal $\sum_{i=1}^{n} x_i R$ of R.

Recently, there has been some work on modules $D(H^n_{\mathfrak{a}}(R))$ and $H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(R)))$, where $\underline{x} := x_1, \ldots, x_n$ is a sequence of elements of $\mathfrak{a}$, and on some problems related to these modules (see for example, conjecture $(*)$ in [3] and [4] and Question 3.8 in [5]). Also, Hellus and Stückrad, in [5], showed that studying the Matlis dual of local cohomology modules is a useful tool for the description of the endomorphism ring of local cohomology modules. Moreover, they raised the following question: if R is a commutative Noetherian complete local ring and $\underline{x} := x_1, \ldots, x_n$ is a regular sequence on R contained in $\mathfrak{a}$, when exactly is $J_{\underline{x},\mathfrak{a},R} := D(H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(R))))$ zero?

On the other hand, Sharp and Zakeri, in [12], over an arbitrary commutative ring, introduced the concept of modules of generalized fractions. It was shown that this concept has many interactions with topics of recent and current interest in commutative algebra. In particular, there are strong links between modules of generalized fractions and local cohomology modules (cf. [13] and [7]).

In this paper, for a finitely generated R-module M, we show that whenever $\underline{x} := x_1, \ldots, x_n$ is a regular sequence on M in $\mathfrak{a}$, there exists an exact sequence

$$H^{n-1}_{\underline{x}R}(D(K)) \longrightarrow D(M) \longrightarrow H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(M))) \longrightarrow 0,$$

where K is the kernel of a differential map in a certain complex which involves modules of generalized fractions. Moreover, we show that

$$J_{\underline{x},\mathfrak{a},M} := D(H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(M)))) \cong D(D(M))$$

if $H^n_{\underline{x}R}(D(U^{-n-1}M)) = 0$, where $U^{-n-1}M$ is a module of generalized fractions of M with respect to a certain triangular subset U of R^{n+1}. In this paper, we study the vanishing of $J_{\underline{x},\mathfrak{a},M}$ (without any restriction on R). Our original goal of this paper is to find some relations between the theory of the generalized fractions and local cohomology theory. Although we can gain some technical results such as Theorem 3.4, we hope that, these can be useful in future.

Throughout this paper, we will generally assume that R is a commutative ring with non-zero identity and $\mathfrak{a}$ is an ideal of R. We shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers. Our terminology follows the textbook [1] on local cohomology.

2. Matlis dual of local cohomology modules

Let M be an R-module. The construction of a module of generalized fractions of M requires a (positive integer n and a) triangular subset (see Definition 2.1 of [12]) $U \subseteq R^n$; the construction produces a module $U^{-n}M$, called the module of generalized fractions of M with respect to U, whose elements, called generalized fractions, have the form $\frac{m}{(u_1,\ldots,u_n)}$, where $m \in M$ and $(u_1, \ldots, u_n) \in U$. The concept of a chain of triangular subsets on R is explained in p. 420 of [8]. Such a chain $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ determines a complex of modules of generalized fractions

$$0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} U_1^{-1}M \longrightarrow \cdots \longrightarrow U_i^{-i}M \xrightarrow{d^i} U_{i+1}^{-i-1}M \longrightarrow \cdots,$$

in which $d^0(m) = m/(1)$ for all $m \in M$ and $d^i(m/(u_1, \ldots, u_i)) = m/(u_1, \ldots, u_i, 1)$ for all $i \in \mathbb{N}$, $m \in M$ and $(u_1, \ldots, u_i) \in U_i$. We shall denote this complex by $C(\mathcal{U}, M)$. The reader is referred to §2 of [12] and [8] for more details of the above constructions.

In the rest of the paper, we assume that M is a finitely generated R-module.

Lemma 2.1 (*See Proposition* 3.3 *of* [2])

(i) *Let U be a triangular subset of R such that $(1) \in U$. Then $U \times \{1\}$ is a triangular subset of R^2, and there is an exact sequence*

$$M \xrightarrow{d^0} U^{-1}M \xrightarrow{\omega} (U \times \{1\})^{-2}M \longrightarrow 0,$$

in which d^0 is the natural homomorphism and $\omega\left(\frac{m}{(u)}\right) = \frac{m}{(u,1)}$ for each $m \in M$ and $(u) \in U$.

(ii) *Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a chain of triangular sets on R. Choose $n \in \mathbb{N}$. Then $U_{n+1} \times \{1\}$ is a triangular subset of R^{n+2}, and there is an exact sequence*

$$U_n^{-n}M \xrightarrow{d^n} U_{n+1}^{-n-1}M \xrightarrow{\omega_{n+1}} (U_{n+1} \times \{1\})^{-n-2}M \longrightarrow 0,$$

in which d^n is defined as above and $\omega_{n+1}\left(\frac{m}{(u_1,\ldots,u_{n+1})}\right) = \frac{m}{(u_1,\ldots,u_{n+1},1)}$ for each $m \in M$ and $(u_1,\ldots,u_{n+1}) \in U_{n+1}$.

Notation 2.2. Let $\underline{x} := x_1, \ldots, x_n$ be a sequence of elements of R. For each $i \in \mathbb{N}$, set

$$U(\underline{x})_i := \{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}) : \text{thereexists } j \text{ with } 0 \leqslant j \leqslant i \text{ such that}$$
$$\alpha_1, \ldots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \cdots = \alpha_i = 0\},$$

where x_r is interpreted as 1 whenever $r > n$. It is easy to see that, for each $i \in \mathbb{N}$, $U(\underline{x})_i$ is a triangular subset of R^i. We use $\mathcal{R}(\underline{x})$ to denote the family $(U(\underline{x})_i)_{i\in\mathbb{N}}$. Hence $\mathcal{R}(\underline{x})$ is a chain of triangular subsets on R. Write the associated complex $C(\mathcal{R}(\underline{x}), M)$ as

$$0 \xrightarrow{d^{-1}_{\underline{x},M}} M \xrightarrow{d^{0}_{\underline{x},M}} U(\underline{x})_1^{-1}M \longrightarrow \cdots \longrightarrow U(\underline{x})_i^{-i}M \xrightarrow{d^{i}_{\underline{x},M}} U(\underline{x})_{i+1}^{-i-1}M \longrightarrow \cdots.$$

It will be convenient to allow $U(\underline{x})_0^{-0}M$ and $U(\underline{x})_{-1}^{-(-1)}M$ to denote M and 0 respectively.

Now, suppose that $(R, \mathfrak{m})$ is a local ring and $E := E_R(R/\mathfrak{m})$ denotes an injective hull of the field $R/\mathfrak{m}$. By $D(-)$ we denote the Matlis dual functor $\mathrm{Hom}_R(-, E)$ (cf. [10]).

Lemma 2.3.

(i) *Let $\mathfrak{a}$ be an ideal of R and $\underline{x} = x_1, \ldots, x_n$ be a sequence of elements of R such that $x_n \in \mathfrak{a}$. Then*

$$H^i_{\mathfrak{a}}(D(U(\underline{x})_n^{-n}M)) = 0$$

for all $i \in \mathbb{N}_0$.

(ii) *Let $\underline{x} := x_1, \ldots, x_n$ be a sequence of elements of R. Then, for every $x_{n+1} \in R$,*

$$H^n_{\underline{x}R}(D(\mathrm{Ker}\, d^n_{x_1,\ldots,x_{n+1},M})) = 0.$$

Proof.

(i) Set

$$V := \{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) : \alpha_1, \ldots, \alpha_n \in \mathbb{N}\}.$$

Then V is a triangular subset of R^n. Consider the R-monomorphism $\varphi\colon V^{-n}M \longrightarrow U(\underline{x})_n^{-n}M$ such that $\varphi(\frac{m}{(x_1^{\alpha_1},\ldots,x_n^{\alpha_n})}) = \frac{m}{(x_1^{\alpha_1},\ldots,x_n^{\alpha_n})}$ for all $m \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$. We show that φ is surjective. Let $\xi = \frac{m}{(x_1^{\beta_1},\ldots,x_n^{\beta_n})} \in U(\underline{x})_n^{-n}M$ for some $m \in M$ and $\beta_1, \ldots, \beta_n \in \mathbb{N}_0$. In view of Remark 3.3(ii) of [12], we may assume that

$\beta_1, \ldots, \beta_{n-1} \in \mathbb{N}$ and $\beta_n \in \mathbb{N}_0$. Hence $\xi = \frac{x_n m}{\left(x_1^{\beta_1}, \ldots, x_{n-1}^{\beta_{n-1}}, x_n^{\beta_n + 1}\right)} \in \operatorname{Im} \varphi$. Thus $V^{-n}M \cong U(\underline{x})_n^{-n}M$. Therefore it is enough to show that

$$H_{\mathfrak{a}}^i(D(V^{-n}M)) = 0$$

for all $i \in \mathbb{N}_0$. In view of Lemma 2.1 of [13], the multiplication by x_n provides an automorphism on $V^{-n}M$. Hence, by applying the functor $D(-)$, it is easy to see that the multiplication by x_n provides again an automorphism on $D(V^{-n}M)$. Since $x_n \in \mathfrak{a}$, it follows that $H_{\mathfrak{a}}^i(D(V^{-n}M)) = 0$ for all $i \in \mathbb{N}_0$.

(ii) Let $x_{n+1} \in R$ and use the exact sequence

$$0 \longrightarrow \operatorname{Ker} d_{\underline{y},M}^n \longrightarrow U(\underline{y})_n^{-n}M \longrightarrow \operatorname{Im} d_{\underline{y},M}^n \longrightarrow 0$$

to deduce the exact sequence

$$0 \longrightarrow D(\operatorname{Im} d_{\underline{y},M}^n) \longrightarrow D(U(\underline{y})_n^{-n}M) \longrightarrow D(\operatorname{Ker} d_{\underline{y},M}^n) \longrightarrow 0, \qquad (*)$$

where $\underline{y} := x_1, \ldots, x_{n+1}$. By using the construction of modules of generalized fractions, $U(\underline{y})_n^{-n}M \cong U(\underline{x})_n^{-n}M$. Hence it follows from (i) that $H_{\underline{x}}^i(D(U(\underline{y})_n^{-n}M)) = 0$ for all $i \in \mathbb{N}_0$. Now, applying the functor $\Gamma_{\underline{x}R}(-)$ on the exact sequence $(*)$ and Corollary 3.3.3 of [1] completes the proof. □

Note that the first part of the above lemma and its proof are rather closed to Proposition 2.4 of [6] and its proof.

In the next theorem, we use a natural generalization of regular sequences which is called filter regular sequences (cf. [11], [15] and [7]).

We say that a sequence $x_1, \ldots, x_n$ of elements of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M if

$$\operatorname{Supp}_R\left(\frac{(x_1, \ldots, x_{i-1})M :_M x_i}{(x_1, \ldots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. Also, we say that an element $x \in \mathfrak{a}$ is an $\mathfrak{a}$-filter regular element on M if $\operatorname{Supp}_R(0 :_M x) \subseteq V(\mathfrak{a})$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the one of a filter regular sequence which has been studied in [11], [15] and has led to some interesting results. Both concepts coincide if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of a local ring with maximal ideal $\mathfrak{m}$. Note that $x_1, \ldots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of Appendix Lemma 2(ii) of [15] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so that, if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for a positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{a}$ be a proper ideal of R. Let $\underline{x} := x_1, \ldots, x_n (n > 0)$ be a regular sequence on M contained in $\mathfrak{a}$. Then there is an exact sequence*

$$H_{\underline{x}R}^{n-1}(D(\operatorname{Ker} d_{\underline{y},M}^n)) \longrightarrow D(M) \longrightarrow H_{\underline{x}R}^n(D(H_{\mathfrak{a}}^n(M))) \longrightarrow 0,$$

for every $x_{n+1} \in \mathfrak{a}$ such that $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M.

Proof. Let x_{n+1} be an element of $\mathfrak{a}$ such that $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. Note that the existence of such element is explained in the above paragraph. Since $x_1, \ldots, x_n$ is a regular sequence on M, by using the exactness theorem for generalized fractions (see Theorem 3.1 of [14] or Theorem 3.3 of [8]), we have the exact sequence

$$0 \xrightarrow{d^{-1}_{\underline{y},M}} M \xrightarrow{d^{0}_{\underline{y},M}} U(\underline{y})^{-1}_{1} M \xrightarrow{d^{1}_{\underline{y},M}} \cdots \xrightarrow{d^{n-2}_{\underline{y},M}} U(\underline{y})^{-n+1}_{n-1} M \longrightarrow \operatorname{Im} d^{n-1}_{\underline{y},M} \longrightarrow 0.$$

Note that $U(\underline{y})^{-i}_{i} M \cong U(\underline{x})^{-i}_{i} M$ and the following diagram commutes:

$$\begin{array}{ccc} U(\underline{y})^{-i+1}_{i-1} M & \xrightarrow{d^{i}_{\underline{y},M}} & U(\underline{y})^{-i}_{i} M \\ \downarrow\cong & & \downarrow\cong \\ U(\underline{x})^{-i+1}_{i-1} M & \xrightarrow{d^{i}_{\underline{x},M}} & U(\underline{x})^{-i}_{i} M \end{array}$$

for all $i \in \mathbb{N}_0$ with $i \leqslant n$. Now, by breaking the above exact sequence into short exact sequences and applying the exact functor $D(-)$ on them, in conjunction with Lemma 2.3(i), we have the following isomorphisms.

$$\begin{aligned} H^{n-1}_{\underline{x}R}(D(\operatorname{Im} d^{n-1}_{\underline{y},M})) &\cong H^{n-2}_{\underline{x}R}(D(\operatorname{Im} d^{n-2}_{\underline{y},M})) \\ &\cong \ldots \\ &\cong H^{1}_{\underline{x}R}(D(\operatorname{Im} d^{1}_{\underline{y},M})) \\ &\cong H^{0}_{\underline{x}R}(D(M)) \end{aligned}$$

Since $D(M)$ is Artinian, $H^{0}_{\underline{x}R}(D(M)) \cong D(M)$ and so

$$H^{n-1}_{\underline{x}R}(D(\operatorname{Im} d^{n-1}_{\underline{y},M})) \cong D(M). \qquad (**)$$

Next, it follows from Consequences 1.3(i) of [7] that $H^{n}_{\mathfrak{a}}(M) \cong \operatorname{Ker} d^{n}_{\underline{y},M}/\operatorname{Im} d^{n-1}_{\underline{y},M}$. Hence, we can obtain the exact sequence

$$0 \longrightarrow \operatorname{Im} d^{n-1}_{\underline{y},M} \longrightarrow \operatorname{Ker} d^{n}_{\underline{y},M} \longrightarrow H^{n}_{\mathfrak{a}}(M) \longrightarrow 0,$$

which induces the exact sequence

$$0 \longrightarrow D(H^{n}_{\mathfrak{a}}(M)) \longrightarrow D(\operatorname{Ker} d^{n}_{\underline{y},M}) \longrightarrow D(\operatorname{Im} d^{n-1}_{\underline{y},M}) \longrightarrow 0.$$

Therefore, by applying the functor $\Gamma_{\underline{x}R}(-)$ to it, we have the exact sequence

$$\begin{aligned} H^{n-1}_{\underline{x}R}(D(\operatorname{Ker} d^{n}_{\underline{y},M})) \longrightarrow H^{n-1}_{\underline{x}R}(D(\operatorname{Im} d^{n-1}_{\underline{y},M})) &\longrightarrow H^{n}_{\underline{x}R}(D(H^{n}_{\mathfrak{a}}(M))) \\ &\longrightarrow H^{n}_{\underline{x}R}(D(\operatorname{Ker} d^{n}_{\underline{y},M})). \end{aligned}$$

The result now immediately follows from Lemma 2.3(ii) and (∗∗). □

The following corollary is an immediate consequence of Theorem 2.4.

COROLLARY 2.5 (Comp. Theorem 2.5 of [6])

Let $(R, \mathfrak{m})$ *be a Noetherian local ring and* $\mathfrak{a}$ *be a proper ideal of* R. *Let* $\underline{x} := x_1, \ldots, x_n$ *be a regular sequence on* R *contained in* $\mathfrak{a}$. *Then* $H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(R)))$ *is a homomorphic image of* E.

3. On the question of Hellus and Stückrad

In this section, we study a slight generalization of the ideal $J_{\underline{x},\mathfrak{a},R}$ of R. In this regard we need the following definition.

DEFINITIONS 3.1 (Comp. Definition 3.3 of [5])

Let $\mathfrak{a}$ be a proper ideal of a Noetherian local ring R such that $\mathfrak{a}M \neq M$. Let $\underline{x} := x_1, \ldots, x_n$ be a regular sequence on M contained in $\mathfrak{a}$. Set

$$J_{\underline{x},\mathfrak{a},M} := D(H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(M)))).$$

In [5], Hellus and Stückrad studied the ideal $J_{\underline{x},\mathfrak{a},R}$ in the case that $(R, \mathfrak{m})$ is a Noetherian complete local ring with respect to $\mathfrak{m}$-adic topology and $\underline{x} = x_1, \ldots, x_n \in \mathfrak{a}$ is a regular sequence on R and they asked when $J_{\underline{x},\mathfrak{a},R} := D(H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(R))))$ is exactly zero. In this section, by using the theory of modules of generalized fractions, we study the R-module $J_{\underline{x},\mathfrak{a},M}$ (without any restriction on R). In fact we show that if $\underline{y} := x_1, \ldots, x_n, x_{n+1} (n > 0)$ is an $\mathfrak{a}$-filter regular sequence on M such that $\underline{x} := x_1, \ldots, x_n$ is a regular sequence on M and $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M)) = 0$, then

$$J_{\underline{x},\mathfrak{a},M} \cong D(D(M)).$$

The following proposition is a consequence of Theorem 2.4.

PROPOSITION 3.2

Let $\mathfrak{a}$ *be a proper ideal of a Noetherian local ring* R. *Let* $\underline{x} := x_1, \ldots, x_n (n > 0)$ *be a regular sequence on* M *contained in* $\mathfrak{a}$. *Then there exists an exact sequence*

$$0 \longrightarrow J_{\underline{x},\mathfrak{a},M} \longrightarrow D(D(M)) \longrightarrow D(H^{n-1}_{\underline{x}R}(D(\mathrm{Ker}\, d^n_{\underline{y},M})))$$

for every $x_{n+1} \in \mathfrak{a}$ *such that* $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ *is an* $\mathfrak{a}$*-filter regular sequence on* M.

Lemma 3.3. *Let* $n \in \mathbb{N}_0$ *and* $\underline{z} := z_1, \ldots, z_{n+1}$ *be a sequence of elements of* R. *Then there exists the exact sequence*

$$H^n_{(z_1,\ldots,z_n)R}(D(H^{n+1}_{\underline{z}R}(M))) \longrightarrow H^n_{(z_1,\ldots,z_n)R}(D(U(\underline{z})^{-n-1}_{n+1}M))$$
$$\longrightarrow H^{n-1}_{(z_1,\ldots,z_n)R}(D(\mathrm{Ker}\, d^n_{\underline{z},M})) \longrightarrow 0.$$

Proof. In view of Lemma 2.1, there is an exact sequence

$$0 \longrightarrow \mathrm{Im}\, d^n_{\underline{z},M} \longrightarrow U(\underline{z})^{-n-1}_{n+1}M \longrightarrow U(\underline{z})^{-n-2}_{n+2}M \longrightarrow 0. \qquad (\dagger)$$

On the other hand, by Theorem 5.2.9 of [1], the $(n+1)$-th local cohomology module $H^{n+1}_{\underline{z}R}(M)$ can be interpreted as a direct limit of Koszul homology modules, and so, $H^{n+1}_{\underline{z}R}(M) \cong \varinjlim_{t\in\mathbb{N}} M/\sum_{j=1}^{n+1} z_j^t M$ with the map

$$M/\sum_{j=1}^{n+1} z_j^t M \longrightarrow M/\sum_{j=1}^{n+1} z_j^{t+1} M$$

being induced by multiplication by $z_1 \ldots z_{n+1}$. Thus, in view of Theorem 2.4 of [9],

$$H^{n+1}_{\underline{z}R}(M) \cong U(\underline{z})^{-n-2}_{n+2} M.$$

So (†) implies the exact sequence

$$0 \longrightarrow \operatorname{Im} d^n_{\underline{z},M} \longrightarrow U(\underline{z})^{-n-1}_{n+1} M \longrightarrow H^{n+1}_{\underline{z}R}(M) \longrightarrow 0,$$

which induces the exact sequence

$$0 \longrightarrow D(H^{n+1}_{\underline{z}R}(M)) \longrightarrow D(U(\underline{z})^{-n-1}_{n+1} M) \longrightarrow D(\operatorname{Im} d^n_{\underline{z},M}) \longrightarrow 0.$$

Therefore, by applying the functor $\Gamma_{(z_1,\ldots,z_n)R}(-)$ on the above exact sequence together with Corollary 3.3.3 of [1], one can obtain the exact sequence

$$H^n_{(z_1,\ldots,z_n)R}(D(H^{n+1}_{\underline{z}R}(M))) \longrightarrow H^n_{(z_1,\ldots,z_n)R}(D(U(\underline{z})^{-n-1}_{n+1} M))$$
$$\longrightarrow H^n_{(z_1,\ldots,z_n)R}(D(\operatorname{Im} d^n_{\underline{z},M})) \longrightarrow 0. \qquad (\ddagger)$$

Now, use the exact sequence

$$0 \longrightarrow \operatorname{Ker} d^n_{\underline{z},M} \longrightarrow U(\underline{z})^{-n}_{n} M \longrightarrow \operatorname{Im} d^n_{\underline{z},M} \longrightarrow 0$$

to deduce the exact sequence

$$0 \longrightarrow D(\operatorname{Im} d^n_{\underline{z},M}) \longrightarrow D(U(\underline{z})^{-n}_{n} M) \longrightarrow D(\operatorname{Ker} d^n_{\underline{z},M}) \longrightarrow 0.$$

Set $\underline{z}' := z_1, \ldots, z_n$. Note that $U(\underline{z})^{-i}_i M \cong U(\underline{z}')^{-i}_i M$ and the diagram

$$\begin{array}{ccc} U(\underline{z})^{-i+1}_{i-1} M & \stackrel{d^i_{\underline{z},M}}{\longrightarrow} & U(\underline{z})^{-i}_{i} M \\ \downarrow\cong & & \downarrow\cong \\ U(\underline{z}')^{-i+1}_{i-1} M & \stackrel{d^i_{\underline{z}',M}}{\longrightarrow} & U(\underline{z}')^{-i}_{i} M \end{array}$$

commutes for all $i \in \mathbb{N}_0$ with $i \leqslant n$. Thus, by Lemma 2.3,

$$H^n_{(z_1,\ldots,z_n)R}(D(\operatorname{Im} d^n_{\underline{z},M})) \cong H^{n-1}_{(z_1,\ldots,z_n)R}(D(\operatorname{Ker} d^n_{\underline{z},M})).$$

The result now follows from (‡). □

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{a}$ be a proper ideal of R. Let $\underline{x} := x_1, \ldots, x_n (n > 0)$ be a regular sequence on M in $\mathfrak{a}$. Suppose that there exists $x_{n+1} \in \mathfrak{a}$ such that $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M and $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M)) = 0$. Then*

$$J_{\underline{x},\mathfrak{a},M} \cong D(D(M)).$$

Proof. It follows from Lemma 3.3 and Proposition 3.2. □

The following corollary is a natural generalization of the implication (i) ⇒ (v) of Theorem 3.7 in [5] in a more general case.

COROLLARY 3.5 (Comp. Theorem 3.7 of [5])

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{a}$ be a proper ideal of R. Let $\underline{x} := x_1, \ldots, x_n$ be a regular sequence on M contained in $\mathfrak{a}$ such that $\sqrt{\mathfrak{a}} = \sqrt{\underline{x}R}$. Then $J_{\underline{x},\mathfrak{a},M} \cong D(D(M))$.

Proof. Let x_{n+1} be an element in $\mathfrak{a}$ such that $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. We show that $U(\underline{y})^{-n-1}_{n+1}M = 0$. To achieve this, let $\frac{m}{(x_1^{\alpha_1},\ldots,x_{n+1}^{\alpha_{n+1}})}$ be an arbitrary element of $U(\underline{y})^{-n-1}_{n+1}M$ for some $m \in M$ and $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{N}_0$. Since $\sqrt{\mathfrak{a}} = \sqrt{\underline{x}R} = \sqrt{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})R}$, there exists $r_1, \ldots, r_n \in R$ such that $x^{\beta}_{n+1} = r_1 x_1^{\alpha_1} + \cdots + r_n x_n^{\alpha_n}$ for some $\beta \in \mathbb{N}$. So, in view of Remark 3.3(ii) of [12],

$$\frac{m}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, x_{n+1}^{\alpha_{n+1}})} = \frac{x^{\beta}_{n+1}m}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, x_{n+1}^{\alpha_{n+1}+\beta})} = 0.$$

The result now follows from Theorem 3.4. □

In the following proposition, we study the R-module $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M))$.

PROPOSITION 3.6

Let $(R, \mathfrak{m})$ be a Noetherian local ring and n be a positive integer. Suppose that $x_1, \ldots, x_{n+1}$ is a sequence of elements of R. Set $\underline{x} := x_1, \ldots, x_n$ and $\underline{y} := x_1, \ldots, x_{n+1}$. Then we have the following isomorphisms:

(i) $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M)) \cong H^n_{\underline{x}R}(R) \otimes_R \mathrm{Hom}_R(U(\underline{y})^{-n-1}_{n+1}R, D(M))$.
(ii) $D(H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M))) \cong \mathrm{Hom}_R(H^n_{\underline{x}R}(R), D(D(U(\underline{y})^{-n-1}_{n+1}M)))$.

Proof. In view of Lemma 3.16 of [14], $U(\underline{y})^{-n-1}_{n+1}M \cong U(\underline{y})^{-n-1}_{n+1}R \otimes_R M$. Hence

$$\begin{aligned} H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M)) &\cong H^n_{\underline{x}R}(R) \otimes_R D(U(\underline{y})^{-n-1}_{n+1}M) \\ &\cong H^n_{\underline{x}R}(R) \otimes_R \mathrm{Hom}_R(U(\underline{y})^{-n-1}_{n+1}R, D(M)). \end{aligned}$$

Also, this is a fact that

$$D(H^n_{\underline{x}R}(R) \otimes_R D(U(\underline{y})^{-n-1}_{n+1}M)) \cong \mathrm{Hom}_R(H^n_{\underline{x}R}(R), D(D(U(\underline{y})^{-n-1}_{n+1}M))),$$

which provides the isomorphism (ii). □

Remark 3.7. In view of Proposition 3.6 and Remark 10.2.2(ii) in [1], one can replace the condition $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M)) = 0$ by each of the following conditions:

(i) $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1}M)) = 0$.
(ii) $H^n_{\underline{x}R}(R) \otimes_R \mathrm{Hom}_R(U(\underline{y})^{-n-1}_{n+1}R, D(M)) = 0$.
(iii) $\mathrm{Hom}_R(H^n_{\underline{x}R}(R), D(D(U(\underline{y})^{-n-1}_{n+1}M))) = 0$.

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