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n-Colour even self-inverse compositions

YU-HONG GUO

Department of Mathematics, Hexi University, Gansu, Zhangye 734000,
People's Republic of China
E-mail: gyh7001@163.com

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Abstract. An n-colour even self-inverse composition is defined as an n-colour self-inverse composition with even parts. In this paper, we get generating functions, explicit formulas and recurrence formulas for n-colour even self-inverse compositions. One new binomial identity is also obtained.

Keywords. n-colour even self-inverse compositions; generating function; explicit formula; recurrence formula; binomial identity.

1. Introduction

In the classical theory of partitions, compositions were first defined by MacMahon [1] as ordered partitions. For example, there are 5 partitions and 8 compositions of 4. The partitions are 4, 31, 22, 21^2, 1^4 and the compositions are 4, 31, 13, 22, 21^2, 121, $1^2 2$, 1^4.

Agarwal and Andrews [2] defined an n-colour partition as a partition in which a part of size n can come in n different colours. They denoted different colours by subscripts: $n_1, n_2, \ldots, n_n$. Analogous to MacMahon's ordinary compositions Agarwal [3] defined an n-colour composition as an n-colour ordered partition. Thus, for example, there are 21 n-colour compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4,$$

$$3_1 1_1, 3_2 1_1, 3_3 1_1, 1_1 3_1, 1_1 3_2, 1_1 3_3,$$

$$2_1 2_1, 2_1 2_2, 2_2 2_2, 2_2 2_1,$$

$$2_1 1_1 1_1, 2_2 1_1 1_1, 1_1 2_1 1_1, 1_1 1_1 2_1, 1_1 2_2 1_1, 1_1 1_1 2_2,$$

$$1_1 1_1 1_1 1_1.$$

More properties of n-colour compositions were found in [4,5].

DEFINITION 1.1 [1]

A composition is said to be self-inverse when the parts of the composition read from left to right are identical with when read from right to left.

Analogous to the above definition of classical self-inverse compositions, Narang and Agarwal [6] defined an n-colour self-inverse composition as follows:

DEFINITION 1.2 [6]

An n-colour composition whose parts read from left to right are identical with when read from right to left and is called an n-colour self-inverse composition.

Thus, for example there are 8 n-colour self-inverse compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4,$$
$$2_1 2_1, 2_2 2_2, 1_1 2_1 1_1, 1_1 2_2 1_1.$$

More properties of n-colour self-inverse compositions were found in [6].

Guo in [7] defined n-colour even compositions and gave some properties.

DEFINITION 1.3 [7]

An n-colour even composition whose parts are even.

Thus, for example, there are 8 n-colour even compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4,$$
$$2_1 2_1, 2_1 2_2, 2_2 2_2, 2_2 2_1.$$

In this paper, we shall study n-colour even self-inverse compositions.

DEFINITION 1.4

An n-colour even composition whose parts read from left to right are identical with when read from right to left and is called an n-colour even self-inverse composition.

Thus, for example, there are 6 n-colour even self-inverse compositions of 4, viz.,

$$4_1, 4_2, 4_3, 4_4, 2_1 2_1, 2_2 2_2.$$

In §2 we shall give explicit formulas for n-colour even self-inverse compositions. We shall prove recurrence formulas in §3. And in §4, we shall give generating functions for n-colour even self-inverse compositions. In §5 we shall give one new binomial identity.

Guo [7] proved the following theorems.

Theorem 1.1 [7]. *Let $C(m, e, q)$ and $C(e, q)$ denote the enumerative generating functions for $C(m, e, \nu)$ and $C(e, \nu)$, respectively, where $C(m, e, \nu)$ is the number of n-colour even compositions of ν into m parts and $C(e, \nu)$ is the number of n-colour even compositions of ν. Then*

$$C(m, e, q) = \frac{2^m q^{2m}}{(1 - q^2)^{2m}}, \tag{1.1}$$

$$C(e, q) = \frac{2q^2}{1 - 4q^2 + q^4}, \tag{1.2}$$

$$C(m,e,\nu) = 2^m \binom{\frac{\nu}{2}+m-1}{2m-1}, \tag{1.3}$$

$$C(e,\nu) = \sum_{m=1}^{\frac{\nu}{2}} 2^m \binom{\frac{\nu}{2}+m-1}{2m-1}, \tag{1.4}$$

where ν is even.

Theorem 1.2 [7]. *Let e_ν denote the number of n-colour even compositions of 2ν. Then*

$$e_1 = 2, e_2 = 8 \quad \textit{and} \quad e_\nu = 4e_{\nu-1} - e_{\nu-2} \textit{ for } \nu > 2.$$

Remark. In this paper, we will study mainly even numbers.

2. Explicit formulas

In this section, we shall prove the following explicit formulas for n-colour even self-inverse compositions.

Theorem 2.1. *For ν even, let $B(e,\nu)$ denote the number of n-colour even self-inverse compositions of ν. Then*

(1) $$B(e,4\nu+2) = (4\nu+2) + \sum_{t=2}^{4\nu-2} \sum_{m=1}^{\frac{4\nu+2-t}{4}} t2^m \binom{\frac{4\nu+2-t}{4}+m-1}{2m-1},$$

where $\nu = 0,1,2,\ldots;\ t = 4k+2,\ k = 0,1,2,\ldots,\nu-1.$

(2) $$B(e,4\nu) = 4\nu + \sum_{t=4}^{4\nu-4} \sum_{m=1}^{\frac{4\nu-t}{4}} t2^m \binom{\frac{4\nu-t}{4}+m-1}{2m-1} + \sum_{m=1}^{\nu} 2^m \binom{\nu+m-1}{2m-1},$$

where $\nu = 1,2,\ldots;\ t = 4k,\ k = 1,2,\ldots,\nu-1.$

Proof.

(1) Obviously, an even number which is $4\nu+2$ ($\nu = 0,1,2,\ldots$) can have even self-inverse n-colour compositions only when the number of parts is odd. There are $4\nu+2$ n-colour even self-inverse compositions when the number of parts is only one. An even self-inverse composition of $4\nu+2$ into $2m+1$ ($m \geq 1$) parts can be read as a central part, say, t (where $t = 4k+2$, $k = 0,1,2,\ldots,\nu-1$) and two identical even n-colour compositions of $\frac{4\nu+2-t}{2}$ into m parts on each side of the central part. The number of even n-colour compositions of $\frac{4\nu+2-t}{2}$ into m parts is $C\left(m,e,\frac{4\nu+2-t}{2}\right)$ by (1.3). Now the central part can appear in t ways. Therefore, the number of n-colour even self-inverse compositions of $4\nu+2$ is:

$$\begin{aligned} B(e,4\nu+2) &= (4\nu+2) + \sum_{t=2}^{4\nu-2} \sum_{m=1}^{\frac{4\nu+2-t}{4}} tC\left(m,e,\frac{4\nu+2-t}{2}\right) \\ &= (4\nu+2) + \sum_{t=2}^{4\nu-2} \sum_{m=1}^{\frac{4\nu+2-t}{4}} t2^m \binom{\frac{4\nu+2-t}{4}+m-1}{2m-1}. \end{aligned}$$

(2) For even numbers 4ν ($\nu = 1, 2, \dots$), we can have even self-inverse n-colour compositions irrespective of the parity of parts. When the number of parts is odd, analogous to the proof of (1) we see that there are $\sum_{t=4}^{4\nu-4}\sum_{m=1}^{\frac{4\nu-t}{4}} t2^m\binom{\frac{4\nu-t}{4}+m-1}{2m-1}+4\nu$ (where $t = 4k, k = 1, 2, \dots, \nu-1$) n-colour even self-inverse compositions of 4ν. When the number of parts is even, we see that the two identical even n-colour compositions are exactly even n-colour compositions of 2ν, the number of which is $\sum_{m=1}^{\nu} 2^m\binom{\nu+m-1}{2m-1}$ by (1.4). Hence, the total number of n-colour even self-inverse compositions of 4ν is:

$$B(e, 4\nu) = 4\nu + \sum_{t=4}^{4\nu-4}\sum_{m=1}^{\frac{4\nu-t}{4}} t2^m\binom{\frac{4\nu-t}{4}+m-1}{2m-1} + \sum_{m=1}^{\nu} 2^m\binom{\nu+m-1}{2m-1}.$$

We complete the proof.

3. Recurrence formulas

In this section, we shall prove the following theorem.

Theorem 3.1. *Let s_ν and r_ν denote the number of n-colour even self-inverse compositions for $4\nu+2$ and 4ν, respectively. Then*

(1) $s_0 = 2,\ s_1 = 10$ *and* $s_\nu = 4s_{\nu-1} - s_{\nu-2}$ *for* $\nu > 1$,

(2) $r_1 = 6,\ s_2 = 24$ *and* $r_\nu = 4r_{\nu-1} - r_{\nu-2}$ *for* $\nu > 2$.

Proof (*Combinatorial*).

(1) To prove that $s_\nu = 4s_{\nu-1} - s_{\nu-2}$, we split the n-colour even self-inverse compositions enumerated by $s_\nu + s_{\nu-2}$ into three classes:

(A) enumerated by s_ν and having 2_1 or 2_2 on both extremes.
(B) enumerated by s_ν and having h_t on both extremes, $h > 2$, $1 \le t \le h-2$ and n-colour even self-inverse compositions of $4\nu+2$ of the form $(4\nu+2)_t$, $1 \le t \le 4\nu-2$.
(C) enumerated by s_ν and having h_t on both extremes, $h > 2$, $h-1 \le t \le h$, $(4\nu+2)_{t_1}$, $4\nu-1 \le t_1 \le 4\nu+2$ and those enumerated by $s_{\nu-2}$.

We transform the n-colour even self-inverse compositions in class (A) by deleting 2_1 or 2_2 on both extremes. This produces two n-colour even self-inverse compositions enumerated by $s_{\nu-1}$. Conversely, given any n-colour even self-inverse composition enumerated by $s_{\nu-1}$ we add 2_1 or 2_2 on both extremes to produce the elements of the class (A). In this way we establish that there are exactly $2s_{\nu-1}$ elements in class (A).

Next, we transform the n-colour even self-inverse compositions in class (B) by subtracting 2 from h, that is, replacing h_t by $(h-2)_t$ and subtracting 4 from $4\nu+2$ of $(4\nu+2)_t$, $1 \le t \le 4\nu-2$. This transformation also establishes the fact that there are exactly $s_{\nu-1}$ elements in class (B).

Finally, we transform the elements in class (C) as follows: Subtract 2_2 from h_t on both extremes, that is, replace h_t by $(h-2)_{(t-2)}$, $h > 2, h-1 \le t \le h$. We will get those n-colour even self-inverse compositions of $4(\nu-1)+2$ whose extremes are h_t, $h-1 \le t \le h$ except self-inverse even compositions in one part only. Also replace $(4\nu+2)_{t_1}$ by $(4\nu-2)_{t_1-4}$, $4\nu-1 \le t_1 \le 4\nu+2$. To get the remaining n-colour even compositions from $s_{\nu-2}$ we add 2 to both extremes, that is, replace h_t by $(h+2)_t$. For n-colour even self-inverse compositions into one part we add 4, that is, replace $(4\nu-6)_t$ by $(4\nu-2)_t$, $1 \leq t \leq 4\nu-6$. We see that the number of n-colour even self-inverse compositions in class (C) is also equal to $s_{\nu-1}$. Hence, we have $s_\nu + s_{\nu-2} = 4s_{\nu-1}$, viz., $s_\nu = 4s_{\nu-1} - s_{\nu-2}$.

The recurrence relation (2) can be proved similarly and hence is omitted. Thus, we complete the proof.

4. Generating functions

In this section, we will prove the following theorem by recurrence relations in §3.

Theorem 4.1.

(1) $\sum_{\nu=0}^{\infty} s_\nu q^\nu = \frac{2+2q}{1-4q+q^2}$,

(2) $\sum_{\nu=1}^{\infty} r_\nu q^\nu = \frac{6q}{1-4q+q^2}$.

Proof.

(1) We have

$$\begin{aligned}\sum_{\nu=0}^{\infty} s_\nu q^\nu &= s_0 + s_1 q + \sum_{\nu=2}^{\infty} s_\nu q^\nu \\ &= 2 + 10q + \sum_{\nu=2}^{\infty}(4s_{\nu-1} - s_{\nu-2})q^\nu \\ &= 2 + 10q + 4\sum_{\nu=2}^{\infty} s_{\nu-1}q^\nu - \sum_{\nu=2}^{\infty} s_{\nu-2}q^\nu \\ &= 2 + 10q + 4\sum_{\nu=1}^{\infty} s_\nu q^{\nu+1} - \sum_{\nu=0}^{\infty} s_\nu q^{\nu+2} \\ &= 2 + 10q + 4\sum_{\nu=0}^{\infty} s_\nu q^{\nu+1} - 4 \times s_0 q - \sum_{\nu=0}^{\infty} s_\nu q^{\nu+2} \\ &= 2 + 2q + 4\sum_{\nu=0}^{\infty} s_\nu q^{\nu+1} - \sum_{\nu=0}^{\infty} s_\nu q^{\nu+2} \\ &\times (q^2 - 4q + 1)\sum_{\nu=0}^{\infty} s_\nu q^\nu = 2 + 2q.\end{aligned}$$

Thus

$$\sum_{\nu=0}^{\infty} s_\nu q^\nu = \frac{2+2q}{1-4q+q^2}.$$

(2) We have

$$\sum_{\nu=1}^{\infty} r_\nu q^\nu = r_1 + r_2 q + \sum_{\nu=3}^{\infty} r_\nu q^\nu$$

$$= 6q + 24q^2 + \sum_{\nu=3}^{\infty}(4r_{\nu-1} - r_{\nu-2})q^{\nu}$$

$$= 6q + 24q^2 + 4\sum_{\nu=2}^{\infty} r_\nu q^{\nu+1} - \sum_{\nu=1}^{\infty} r_\nu q^{\nu+2}$$

$$= 6q + 24q^2 + 4\sum_{\nu=1}^{\infty} r_\nu q^{\nu+1} - 24q^2 - \sum_{\nu=1}^{\infty} r_\nu q^{\nu+2}$$

$$\times\ (q^2 - 4q + 1)\sum_{\nu=1}^{\infty} r_\nu q^{\nu} = 6q.$$

So

$$\sum_{\nu=1}^{\infty} r_\nu q^{\nu} = \frac{6q}{1 - 4q + q^2}.$$

We complete the proof.

5. New binomial identity

We first give the following theorem for the generating function of e_ν in Theorem 1.2.

Theorem 5.1.

$$\sum_{\nu=1}^{\infty} e_\nu q^{\nu} = \frac{2q}{1 - 4q + q^2}.$$

From Theorem 1.2 and proof of Theorem 4.1, this theorem can be proved similarly. We shall get the following binomial identity:

Theorem 5.2. *For* $\nu \geq 1$,

$$4\nu + \sum_{t=4}^{4\nu-4} \sum_{m=1}^{\frac{4\nu-t}{4}} t2^m \binom{\frac{4\nu-t}{4} + m - 1}{2m - 1} = 2\sum_{m=1}^{\nu} 2^m \binom{\nu + m - 1}{2m - 1}.$$

Proof. By comparing generating functions for e_ν and r_ν, we have

$$r_\nu = 3e_\nu \quad \text{for} \quad \nu \geq 1.$$

From the definitions of r_ν and e_ν, we get this identity easily.

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