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On the complexity of labeled oriented trees

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Abstract. We define a notion of complexity for labeled oriented trees (LOTs) related to the bridge number in knot theory and prove that LOTs of complexity 2 are aspherical. We also present a class of LOTs of higher complexity which is aspherical, give an upper bound for the complexity of labeled oriented intervals and study the complexity of torus knots.

Keywords. Asphericity; 2-complexes; complexity; labeled oriented trees.

1. Introduction

A 2-complex K is called *aspherical* if its second homotopy group is trivial ($\pi_2(K) = 0$). The Whitehead conjecture (see [9]) states that any subcomplex of an aspherical 2-complex is itself aspherical. The Whitehead conjecture remains open up to this day despite considerable expense of effort (see [7] for an overview).

Let $P = \langle x_1, \ldots, x_m | R_1, \ldots, R_n \rangle$ be a finite presentation where each relator is of the form $x_i x_j = x_j x_k$. Such a presentation is called a *labeled oriented graph presentation*, or short, *LOG-presentation* because it is represented by a *labeled oriented graph T_P* in the following way: For each generator x_i of P, T_P has a vertex labeled x_i and for each relator $x_i x_j = x_j x_k$ (or equivalently $x_i = x_j x_k x_j^{-1}$), T_P has an oriented edge from the vertex x_i to the vertex x_k labeled by x_j. If T_P is a tree we call it a *labeled oriented tree* or *LOT* and P a *LOT-presentation*. The 2-complex modeled on P will be called a LOG (or LOT)-complex. A *labeled oriented interval* or *LOI* is a LOT where the underlying graph is an interval.

For example the LOI depicted in figure 1 encodes the presentation

$$P = \langle a, b, c, d, e, f \,|\, ac = cb,\ bf = fc,\ db = bc,\ df = fe,\ fa = ae \rangle$$

It was shown by Howie [1] that if K is a finite 2-complex that 3-deforms to a point and $e \in K$ is an open 2-cell, then $K - e$ 3-deforms to some LOT-complex. He shows in particular that, if the Andrews–Curtis conjecture is true (i.e. if each finite contractible 2-complex 3-deforms to a point) then the asphericity of LOT-complexes implies the Whitehead conjecture in the finite case.

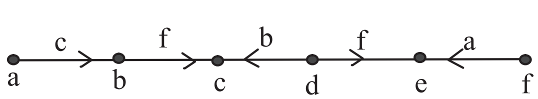

Figure 1. Example of a LOT.

We define a notation of complexity for LOTs which is related to the notion introduced by Ivanov (see [4]).

Given a LOG $P = \langle x_1, \ldots, x_m \mid R_1, \ldots, R_k \rangle$, we say that P has *complexity* n, provided there is a subset $S = \{x_{i_1}, \ldots, x_{i_n}\}$ of the set of generators X consisting of n elements, such that the following inductive process defines every generator of P to be good and there is no such set consisting of $n - 1$ elements:

1. The elements of S are good.
2. If $xy = yz$ or $zy = yx$ is a relator of P and x, y are good then so is z.

We say that P is *derived by* S and denote its complexity by $cx(P)$.

For example the LOT of figure 1 has complexity 2 because it can be derived by $\{b, f\}$. Certainly not all subsets of two elements derive this LOT, like for instance $\{a, c\}$ does not.

The complexity of a LOG is independent of the orientation of its edges. So any other orientation of the edges of the LOT of figure 1 has complexity 2 also.

In our context a LOG is called *reduced* if each relator involves three different generators. Any LOT can be transformed into a reduced LOT without changing the homotopy type of the corresponding 2-complex. If a LOT is reduced and has at least one relator, then its complexity is greater or equal to 2. The following theorem is a consequence of Lyndon's identity theorem for one-relator groups.

Theorem 1.1. *LOTs of complexity* 2 *are aspherical.*

As a corollary we show that LOTs with only two different edge labels are aspherical. We also present some classes of LOTs of complexity 2 and show that certain classes of LOTs with higher complexity are also aspherical by a 'product-lemma' for LOTs. There is an upper bound for the complexity of LOIs which is shown to be roughly half the number of generators. In the last section, we analyze the complexity of torus knots.

The author has written a small program to check the complexity of LOTs. Many 'small' LOTs seem to have low complexity. For instance all reduced LOIs with 8 generators which have all vertices of valence 1 appearing as conjugators were checked. There are

$$6^7 - 2 \cdot 6 \cdot 5^6 + 5 \cdot 5 \cdot 4^5 = 118036$$

of them without regarding the orientation of the edges. 77902 are of complexity 2, 40032 are of complexity 3 and 102 are of complexity 4. So almost $2/3$ of all these LOTs can be shown to be aspherical by using their complexity.

In the case of a knot the complexity of a LOI is related to the bridge number of the corresponding knot projection. Bridges in the knot projection correspond to edge labels in the LOI and the set of edge labels always derives the LOI (see Proposition 3.1). Nevertheless it remains an open question whether the bridge number of a LOI is always equal to its complexity.

2. Proof of theorem 1.1

Let T_P be a LOT with presentation $P = \langle x_1, \ldots, x_m \mid R_1, \ldots, R_{m-1} \rangle$ and generating set $X = \{x_1, \ldots, x_m\}$. Let $S = \{x_{i_1}, \ldots, x_{i_n}\}$ be a subset of the generators which derives P. The process of deriving all generators from S may be viewed in the tree T_P as follows: Start with the n vertices in T_P labeled by elements of $S_0 = S$. Let T_0 be the subgraph of

T_P having S_0 as vertex-set and with edge-set consisting of those edges, which have their endpoints and edge labels in S_0. If S is a set of minimal cardinality deriving T_P then T_0 has no edges.

Now there has to be an element $a \in S_0$ and an edge $e \in T_P$ with a in its boundary and edge label $c \in S_0$ such that its other boundary vertex $b \notin S_0$. Make up a new set $S_1 = S \cup \{b\}$.

We inductively get an increasing sequence $S = S_0 \subset S_1 \subset \cdots \subset S_{m-n} = X$. Let T_i be the subgraph of T_P having S_i as vertex-set and with edge-set consisting of those edges, which have its endpoints and edge label in S_i. T_0 consists of exactly n components if S is a set of minimal cardinality deriving P. Each T_i consists of at most n components and we have a sequence:

$$T_0 \subset T_1 \subset \cdots \subset T_{m-n} = T_P.$$

Lemma 2.1. *If a LOT has complexity n then it can be transformed into a presentation with n generators and* $n-1$ *relators by a sequence of* Q^{**} *operations* (*the analogue of* 3*-deformations on presentation level*).

Proof. Let $P = \langle x_1, \ldots, x_m \mid R_1, \ldots, R_{m-1} \rangle$ be a LOT with $cx(P) = n$ being derived by $S = \{x_{i_1}, \ldots, x_{i_n}\}$. If $m > n$ there must be a relator $r_1 = R_j$ of P of the form $xy = yz$ or $zy = yx$, where $x, y \in S$ and $z \notin S$. Replace all the occurrences of z in all other relators by $y^{\epsilon} x y^{-\epsilon}$ (where $\epsilon = \pm 1$ depending on r_1). Then delete z from the set of generators and r_1 from the set of relators of P to get a presentation P_1 and form a set $S_1 = \{x_{i_1}, \ldots, x_{i_n}, z\}$.

Inductively, if $S_i \neq X = \{x_1, \ldots, x_m\}$, then there is a generator $z \notin S_i$, such that there is a relator r_i of P of the form $xy = yz$ or $zy = yx$, where $x, y \in S_i$. Replace all the occurrences of z in all relators of P_i by a word deduced from (the refined) r_i consisting only of elements of S. Then delete z from the set of generators of P_i and r_i from the set of relators to get a presentation P_{i+1} and form a set $S_{i+1} = S \cup S_i$. Thus we continue, until $S_{m-n} = X$ and we have eliminated $m-n$ generators and $m-n$ relators. The presentation P_{m-n} has the generating set S and $m - 1 - (m-n) = n - 1$ relators and we have only performed Q^{**} moves. □

Lemma 2.2. *Let* $P = \langle x, y \mid w^n \rangle$ *be a one-relator presentation of a group G with* $G' = G/[G, G] = \mathbb{Z}$. *Then* $n = 1$.

Proof. Assume w has exponent sum k in x and l in y. Then w^n has exponent sum kn in x and ln in y. If $k = l = 0$ then $G' = \mathbb{Z} \times \mathbb{Z}$ in contradiction to the assumption, so $k > 0$ or $l > 0$. If $n > 1$ then x or y would have order a multiple of n and G' would contain a finite cyclic factor. □

Proof of Theorem 1.1. We start with a LOT P such that $cx(P) = 2$. P can be transformed into a one-relator presentation P' with 2 generators by Lemma 2.1. This is done by Q^{**} moves which do not change the homotopy type of the corresponding 2-complex. Since all LOTs abelianize to the infinite cyclic group, we know that the relator of P' is not a proper power by Lemma 2.2. Lyndon [5] has shown that 2-complexes corresponding to one-relator presentations where the relator is not a proper power are aspherical. □

3. LOTs of different complexities

It would be interesting to know, which LOTs have which complexity. Certainly there are infinitely many LOTs with complexity 2 given for example by the following class of presentations:

$$\langle x_1, \ldots, x_n \mid x_1 x_n = x_n x_2,$$
$$x_2 x_1 = x_1 x_3, x_3 x_2 = x_2 x_4, \ldots, x_{n-1} x_{n-2} = x_{n-2} x_n \rangle$$

which can all be derived by $\{x_1, x_n\}$.

Another class of LOTs of small complexity is given by the following proposition:

PROPOSITION 3.1

If the number of different edge labels of the LOT P is n, then $cx(P) \leq n$.

The proof is clear: Any LOT may be derived by its edge labels. A relation to the bridge number for knots can be seen here: For a LOI which is the Wirtinger presentation of a knot the number of different edge labels is equal to the bridge number of the knot projection. Using Theorem 1.1 we get as an immediate consequence of Proposition 3.1:

COROLLARY 3.2

A LOT with at most two different edge labels is aspherical.

On the other hand, it is easy to construct for any given positive integer $n \geq 3$ LOTs of complexity n:

Example 3.3. Take $n - 1$ LOTs of complexity 2

$$P_i = \langle x_{i,1}, \ldots, x_{i,m_i} \mid r_{i,1}, \ldots, r_{i,m_i-1} \rangle$$

with $1 \leq i < n$ derived by the sets $S_i = \{x_{i,1}, x_{i,2}\}$. Make up a new LOT P by identifying $x_{1,1} = x_{2,1} = \cdots = x_{n-1,1}$. This LOT certainly has complexity n derived by $S = \{x_{1,1}, x_{1,2}, x_{2,2}, x_{3,2}, \ldots, x_{n-1,2}\}$ because it cannot be derived with fewer generators since any 'branch' P_i needs an extra generator in the set S. These LOTs are aspherical.

This follows by induction from Theorem 1.1 and the following lemma, which is well known:

Lemma 3.4. *Assume $P_1 = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$ and $P_2 = \langle y_1, \ldots, y_m \mid s_1, \ldots, s_{m-1} \rangle$ are two LOTs, where the corresponding 2-complexes are aspherical. Let Q be the LOT given by the union of P_1 and P_2 by identifying some x_k with some y_j (this is an amalgamation of the corresponding groups over the integers). Then the corresponding 2-complex K_Q is aspherical.*

Proof. Let K_i be the 2-complex modeled on P_i for $i = 1, 2$. $K_1 \cap K_2 = S^1$ is π_1 injective in each of the K_i. The result then follows from a theorem of Whitehead (see [8]). □

Many small LOTs which do not contain a *proper sub-LOT* (i.e. a subtree which itself is a LOT) are of complexity 2 and therefore aspherical (which fits to the class of aspherical LOTs presented in [2]). But not all LOTs without a sub-LOT are of complexity 2 which can be seen in the following example:

Example 3.5. For any orientation of its edges, the LOT in figure 2 has complexity 3 (for example, derived from $\{a, g, e\}$) and does not contain a proper sub-LOT. It cannot be derived by any pair of generators which can be seen by checking all possibilities.

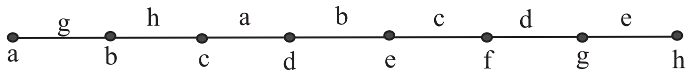

Figure 2. Example of a LOT

There is an orientation of the LOT in Example 3.5 which may be realized as a Wirtinger-presentation of an alternating knot in 3-space.

It is also easy to construct a LOT which contains a proper sub-LOT but is of complexity 2. Add the generators g, h and the relators $gh = hb, hg = gf$ to the LOT P of figure 1. The resulting LOT has complexity 2 (derived from $\{g, h\}$) and contains a proper sub-LOT, namely the LOT P. In general, the following is true:

PROPOSITION 3.6

Let P be a proper sub-LOT of the LOT Q. Then $cx(P) \leq cx(Q)$.

Proof. Let n be the complexity of Q derived by $S_0 = \{x_{i_1}, \ldots, x_{i_n}\}$. The corresponding subgraph $T_0 \subset T_Q$ (see the beginning of §2) consists of exactly n components all of which are only vertices and each $T_i \subset T_Q$ consists of at most n components. Since T_Q is a tree, we have that $T_i \cap T_P$ consists of at most n components for all i. For each of these components, there must be a smallest index j such that $T_j \cap T_P$ consists of exactly one vertex. P is derived by these at most n vertices because no edge or vertex label of $T_Q - T_P$ occurs as edge label in T_P. □

Let $P = \langle X \mid R \rangle$ be a LOT with generating set $X = \{x_1, \ldots, x_n\}$. Let $Q = \langle X, y \mid R, yx_i = x_i x_j \rangle$ for some $x_i, x_j \in X$. Certainly the corresponding 2-complexes K_P and K_Q are homotopy equivalent. We say that P comes from Q by *removing an unessential edge*.

Lemma 3.7. *If the LOT P comes from the LOT Q by removing an unessential edge, then $cx(P) = cx(Q)$.*

Proof. This follows from Proposition 3.6 and the fact that any subset of the set X which derives P also derives Q. □

Theorem 3.8. *Let P be a LOI with n vertices. Then*

$$cx(P) \leq \frac{n+1}{2}. \tag{1}$$

Proof. The proof uses induction on the number n of generators. Observe first that all LOTs with up to 2 generators satisfy the inequality (1). If P has only one generator, then there are no edges in T_P and the complexity is one. In the case of two generators the relator has the form $aa = ab$ and the complexity is one also.

Any LOI where no unessential edges are removable has a vertex of valence 2 which does not appear as edge label. We assume now that all LOIs with at most $n-1$ generators having a vertex of valence 2 which is not an edge label satisfy inequality (1). Let $P = \langle X|R \rangle$ be a LOI with generating set $X = \{x_1, \ldots, x_n\}$. If P has unessential edges we remove them and by induction and Lemma 3.7, inequality (1) is satisfied. So we assume there are no unessential edges. Then there is a vertex x_k in the corresponding tree T_P of valence 2 which is not used as an edge label. Let d_i, d_j be the two edges with x_k in their boundary having

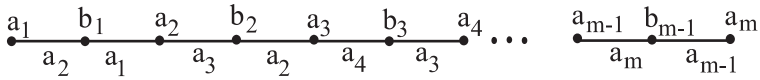

Figure 3. LOIs which are of maximal complexity

as second boundary vertex x_i, x_j respectively. Let Q' be the LOI with d_i, d_j removed and x_i and x_j identified to the vertex y. Let Q be the LOI Q' after removing unessential edges. Q has at most $n-2$ generators so by induction and Lemma 3.7 it can be derived by a set W of at most $(n-1)/2$ generators.

If $y \in W$, then $W \backslash \{y\} \cup \{x_i, x_j\}$ derives P and is of the required cardinality. If $y \notin W$ then the process of deriving the generators of Q out of W described at the beginning of §2 must prove y to be good at some step. Reproducing these steps with P shows one of x_i or x_j to be good at the same step, say x_i. Then $W \cup \{x_j\}$ clearly derives P and is of the required cardinality. □

Example 3.9. For each odd $n > 0$ there are LOIs such that the inequality (1) is an equality: The LOIs in figure 3 for any orientation of the edges are derived by $\{a_1, \ldots, a_m\}$ and are of complexity m having $2m-1$ generators.

Moritz Christmann has generalized Theorem 3.8 for all LOTs.

4. Torus knots

Let $K_{p,q}$ be the torus knot of type p, q which wraps around the solid torus in the longitudinal direction p times and in the meridinal direction q times. It is known that the corresponding knot group has a 1-relator presentation (see for instance [6]):

$$\pi_1(S^3 - K_{p,q}) = \langle a, b \mid a^p = b^q \rangle.$$

Given any tame knot $K \in S^3$, the Wirtinger-presentation of the knot space gives rise to a LOG, which is a circle. Any one of the relators is redundant, so we may represent the homotopy type of the knot space by a LOI T_K. With a slight abuse of notation we call T_K 'the' Wirtinger-presentation of the knot.

The following theorem implies that the complexity does not detect the minimal number of generators needed to present a LOT-group.

Theorem 4.1. *Let $p, q \geq 2$ be coprime integers. Let $T_{p,q}$ be the Wirtinger presentation of the torus knot $K_{p,q}$. Then $cx(T_{p,q}) = \min\{p, q\}$.*

Schubert has shown that the bridge number of the torus knot $K_{p,q}$ is also $\min\{p, q\}$. We will show that $T_{p,q}$ is derived by its edge labels.

The proof of Theorem 4.1 relies on the following lemma:

Lemma 4.2. *Let $P = \langle X, Y \mid R \rangle$ be a reduced LOT-presentation with edge labels $X = \{x_1, \ldots, x_q\}$ and additional generators $Y = \{y_1, \ldots, y_k\}$ which do not occur as edge labels in T_P. Assume that for any pair $x_i, x_j \in X$, $i \neq j$ there are at least two edges labeled x_i and x_j on the path from the vertex x_i to the vertex x_j in T_P. Then $cx(P) = q$ and P may be derived by X.*

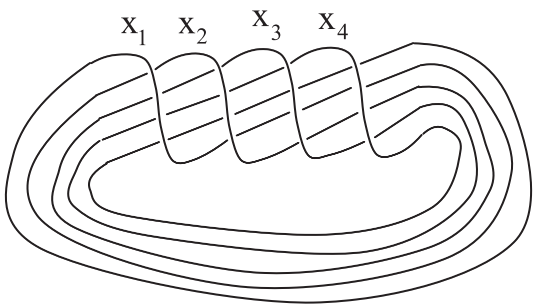

Figure 4. The torus knot $T_{5,4}$.

Proof. Proposition 3.1 implies $cx(P) \leq q$. For the proof of $cx(P) \geq q$ let S be a subset of the generators $X \cup Y$ which derives P of minimal cardinality. So the number of elements of S is less or equal to q.

We claim that if S contains an element $y = y_s \in Y$, then there is a set S' which derives P such that $S' = S \backslash \{y\} \cup \{x_j\}$ for some $x_j \in X$ for the following reason: Let $W = \{x_1, \ldots, x_m\} \subseteq X$ (after renaming generators) be the subset of those vertices in T_P such that each $x_i \in W$ can be reached from the vertex labeled y by a path w_i in T_P which has, apart from x_i, only elements of Y as vertex labels. One of the elements $x_j \in W$ cannot be in S. This follows from the fact that the vertex y, since S is minimal, has to reach at least one of the vertices of W, say x_j, by the process described at the beginning of §2.

In this process it is possible to add all edges starting in y and reaching x_j one after the other in the sequence of $T_0, T_1, \ldots$. So it is just as well possible to start in x_j and reach y because no vertex on the path from y to x_j occurs as edge label. But then $S' = S \backslash \{y\} \cup \{x_j\}$ also derives T_P.

By induction S can be replaced by a set $\bar{S}$ such that $\bar{S}$ derives T_P, $|S| = |\bar{S}|$ and $\bar{S} \subseteq X$. If $\bar{S} = X$ we are done so assume $\bar{S} \neq X$. Then an element $x \in X \backslash S$ cannot be derived by the elements in $\bar{S}$ by the following argument: Let $V \subseteq X$ be the subset of those vertices in T_P such that each $x_k \in V$ can be reached from the vertex labeled x by a path in T_P which has, apart from x_k and x, only elements of Y as vertex labels. By assumption for each $x_i \in V$ there is an edge e_i labeled x on the path from the vertex x_i to the vertex x in T_P. The subtree consisting of all e_i and the paths between them cannot be derived by $\bar{S}$. So it follows that $\bar{S} = X$. □

Proof of Theorem 4.1. There is a projection of the torus knot $K_{p,q}$ with exactly q strands $x_1, \ldots, x_q$ which overcross $p-1$ strands each (see the torus knot $T_{5,4}$ in figure 4). So only $X = \{x_1, \ldots, x_q\}$ occur as edge labels of $T_{p,q}$. If $p > q$ then for any pair $x_i, x_j \in X$, $i \neq j$ and for any $x_l \in X$ there is an edge labeled x_l on the path from x_i to x_j in $T_{p,q}$. So Lemma 4.2 gives the desired result.

For $q > p$ take a meridinal disk in the solid torus carrying $K_{p,q}$ which cuts exactly p strands $z_1, \ldots, z_p$ of $K_{p,q}$. The corresponding generators are easily seen to derive $T_{p,q}$ which implies $cx(T_{p,q}) \leq p$. For the proof of $cx(T_{p,q}) \geq p$ one can show quite similar to the proof of Lemma 4.2 that $T_{p,q}$ may be derived by a subset of the set X only. Then one shows that at least p elements of X are necessary to derive $T_{p,q}$. We omit the details, since the proof is very much the same as the proof of Lemma 4.2. □

Acknowledgement

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