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Dirichlet expression for $L(1, \chi)$ with general Dirichlet character

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Abstract. In the famous work of Dirichlet on class number formula, $L(s, \chi)$ at $s = 1$ has been expressed as a finite sum, where $L(s, \chi)$ is the Dirichlet L-series of a real Dirichlet character. We show that this expression with obvious modification is valid for the general primitive Dirichlet character χ.

Keywords. Hurwitz zeta function; Dirichlet character; Dirichlet L-series; primitive character.

1. Introduction

In Dirichlet's famous work dealing with class number formula, the value of $L(1, \chi)$ is expressed in terms of finite sums as follows:

(I) $L(1, \chi) = -\frac{\pi}{|d|^{3/2}} \sum_{a=1}^{|d|} a \left(\frac{d}{a}\right)$, if $d < 0$.
(II) $L(1, \chi) = \frac{-1}{d^{1/2}} \sum_{a=1}^{d} \left(\frac{d}{a}\right) \log \sin \frac{\pi a}{d}$, if $d > 0$.

Here $L(s, \chi)$ is the Dirichlet's L-series associated with the real primitive character (d/a), where (d/a) is Kronecker's symbol. The question arises as to whether the corresponding expression for $L(1, \chi)$ holds for any general primitive Dirichlet character $\chi(\text{mod } q)$, where $q \geq 2$ is an integer.

We shall show that the answer is in affirmative. Our result is new.

2. Notation

For a Dirichlet character $\chi(\text{mod } q)$, we write $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e^{\frac{2\pi i a}{q}}$. A character $\chi(\text{mod } q)$ is said to be even or odd character depending upon $\chi(-1) = 1$ or -1 respectively.

We shall use the fact $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$, where $\Gamma(s)$ is gamma function.

Note that $L(s, \chi) = q^{-s} \sum_{a=1}^{q} \chi(a) \zeta\left(s, \frac{a}{q}\right)$. Here s is a complex variable and $\zeta(s, \alpha)$ is the Hurwitz zeta function defined by $\zeta(s, \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s}$ for Re $s > 1$ and its analytic continuation, given that $\alpha > 0$ is a real variable.

We also note that $\zeta(0, \alpha) = \frac{1}{2} - \alpha$ for $0 < \alpha \leq 1$.

Next we state our theorem.

Theorem. *For an integer $q > 1$, let $\chi \pmod q$ be a primitive Dirichlet character. Then we have*

(I) $L(1,\chi) = \frac{-1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a) \log \sin \frac{\pi a}{q}$, *if $\chi \pmod q$ is even.*
(II) $L(1,\chi) = \frac{-\pi i}{\tau(\bar{\chi})} \sum_{a=1}^{q} \frac{a}{q} \bar{\chi}(a)$, *if $\chi \pmod q$ is odd.*

Proof. Let $\chi \pmod q$ be an even primitive character so that we have the functional equation

$$L(s,\chi) = q^{-s} 2^s \pi^{s-1} \Gamma(1-s) \tau(\chi) \sin \frac{\pi s}{2} L(1-s,\bar{\chi}).$$

As $L(s,\chi)$ is analytic at $s = 1$, this gives $L(0,\chi) = 0$.

Using L'Hospital's rule and letting $s \to 0$ through real values, we have

$$\lim_{s \to 0} \frac{L(s,\chi)}{\sin \frac{\pi s}{2}} = \lim_{s \to 0} q^{-s} 2^s \pi^{s-1} \Gamma(1-s) \tau(\chi) L(1-s,\bar{\chi}).$$

This gives

$$\lim_{s \to 0} \frac{L'(s,\chi)}{\frac{\pi}{2} \cos \frac{\pi s}{2}} = \frac{\tau(\chi)}{\pi} L(1,\bar{\chi}),$$

where

$$L'(s,\chi) = \frac{\mathrm{d}}{\mathrm{d}s} L(s,\chi).$$

That is

$$\frac{L'(0,\chi)}{\pi/2} = \frac{\tau(\chi)}{\pi} L(1,\bar{\chi}).$$

Thus $L(1,\bar{\chi}) = \frac{2}{\tau(\chi)} L'(0,\chi)$ so that $L(1,\chi) = \frac{2}{\tau(\bar{\chi})} L'(0,\bar{\chi})$.

Note that $L(s,\chi) = q^{-s} \sum_{a=1}^{q} \chi(a) \zeta\left(s, \frac{a}{q}\right)$.

This gives

$$L'(s,\chi) = -(\log q) q^{-s} \sum_a \chi(a) \zeta\left(s, \frac{a}{q}\right) + q^{-s} \sum_a \chi(a) \zeta'\left(s, \frac{a}{q}\right)$$
$$= q^{-s} \sum_a \chi(a) \zeta'\left(s, \frac{a}{q}\right) - (\log q) L(s,\chi),$$

where $\zeta'(s,\alpha) = \frac{\partial}{\partial s} \zeta(s,\alpha)$.

Thus $L'(0,\chi) = \sum_a \chi(a) \zeta'\left(0, \frac{a}{q}\right)$, as $L(0,\chi) = 0$.

This gives $L'(0,\bar{\chi}) = \sum_a \bar{\chi}(a) \zeta'\left(0, \frac{a}{q}\right)$.

Noting $\zeta'(0,\alpha) = \log \frac{\Gamma(\alpha)}{\sqrt{2\pi}}$, we have

$$
\begin{aligned}
L'(0,\bar{\chi}) &= \sum_{a=1}^{q-1} \bar{\chi}(a) \log \Gamma\left(\frac{a}{q}\right) \\
&= \sum_{1\le a\le \frac{q-1}{2}} \bar{\chi}(a) \left(\log \Gamma\left(\frac{a}{q}\right) + \log\Gamma\left(1-\frac{a}{q}\right)\right) \\
&= \sum_{1\le a\le \frac{q-1}{2}} \bar{\chi}(a) \log \left(\Gamma\left(\frac{a}{q}\right) \Gamma\left(1-\frac{a}{q}\right)\right) \\
&= \sum_{a\le \frac{q-1}{2}} \bar{\chi}(a) \log \frac{\pi}{\sin \pi \frac{a}{q}} \\
&= -\sum_{a\le \frac{q-1}{2}} \bar{\chi}(a) \log \sin \frac{\pi a}{q} = -\frac{1}{2}\sum_{a} \bar{\chi}(a) \log\sin\frac{\pi a}{q}.
\end{aligned}
$$

Thus $L(1,\chi) = -\frac{1}{\tau(\bar{\chi})}\sum_a \bar{\chi}(a)\log\sin\frac{\pi a}{q}$

This completes the first case.

Next, let $\chi \pmod q$ be primitive and odd. Then, we have the functional equation

$$L(s,\chi) = (-i)2^s\pi^{s-1}q^{-s}\Gamma(1-s)\tau(\chi)\cos\frac{\pi s}{2}L(1-s,\bar{\chi}).$$

This gives $L(0,\chi) = \left(\frac{-i}{\pi}\right)\tau(\chi)L(1,\bar{\chi})$. Thus $L(1,\bar{\chi}) = \frac{\pi i}{\tau(\chi)}L(0,\chi)$. Hence,

$$
\begin{aligned}
L(1,\chi) &= \frac{\pi i}{\tau(\bar{\chi})}L(0,\bar{\chi}) \\
&= \frac{\pi i}{\tau(\bar{\chi})}\sum_a \bar{\chi}(a)\zeta\left(0,\frac{a}{q}\right) = \frac{\pi i}{\tau(\bar{\chi})}\sum_a \bar{\chi}(a)\left(\frac{1}{2}-\frac{a}{q}\right) \\
&= -\frac{\pi i}{\tau(\bar{\chi})}\sum_a \frac{a}{q}\bar{\chi}(a).
\end{aligned}
$$

This proves the theorem.