Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 1, February 2010, pp. 1–6.


# On the general Dedekind sums and its reciprocity formula

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MS received 28 April 2009; revised 3 July 2009

**Abstract.** In this paper, we prove an interesting reciprocity formula for a certain case of a general Dedekind sums using analytic methods and the Fourier expansion of the Bernoulli polynomials.

**Keywords.** General Dedekind sums; Fourier expansion; Bernoulli polynomial; reciprocity formula.

## 1. Introduction

For a positive integer $q$ and an arbitrary integer $h$, the Dedekind sums $S(h, q)$ is defined as:

$$S(h,q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ha}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The various properties of $S(h, q)$ were investigated by many authors. Maybe the most famous property of Dedekind sums is the reciprocity formula (see [2–4]):

$$S(h,q) + S(q,h) = \frac{h^2+q^2+1}{12hq} - \frac{1}{4} \tag{1}$$

for all $(h, q) = 1$, $q > 0$, $h > 0$. The main purpose of this paper is to introduce a general Dedekind sum:

$$S(h,n,m,q) = \sum_{a=1}^{q} \bar{B}_n\left(\frac{a}{q}\right)\bar{B}_m\left(\frac{ah}{q}\right), \tag{2}$$

where

$$\bar{B}_n(x) = \begin{cases} B_n(x - [x]), & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

$B_n(x)$ is the $n$-th Bernoulli polynomial, $\bar{B}_n(x)$ is defined as the $n$-th Bernoulli periodic function in the interval (0, 1]. If $m = n$, we write $S(h, n, n, q) = S(h, n, q)$. Especially for $n = m = 1$, $S(h, 1, 1, q) = S(h, q)$ is the classical Dedekind sum. As an extension of (1), using the analytic method and the Fourier expansion of the Bernoulli polynomials, we obtain an interesting reciprocity formula for the general Dedekind sums $S(h, 1, 3, q)$. That is, we prove the following:

**Theorem.** *Let $h$ and $q$ are positive integers with $(h, q) = 1$. Then we have the reciprocity formula*

$$\frac{h}{q} S(q, 1, 3, h) + \frac{q}{h} S(h, 1, 3, q) = \frac{1}{24} - \frac{h^2}{120q^2} - \frac{q^2}{120h^2} - \frac{1}{40q^2h^2}.$$

*Note.* Using this method, we can also give a new proof for the reciprocity formula of the classical Dedekind sum. For general positive integers $m$ and $n$, whether there exists a reciprocity formula for $S(h, m, n, q)$ is an unsolved problem.

## 2. Proof of the theorem

Now we use the Fourier expansion of the Bernoulli polynomials to prove the theorem directly. For $0 < x \le 1$, from Theorem 12.19 of [1] we have the identity

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{e(rx)}{r^n}$$

Especially for $n = 1$, we write

$$B_1(x) = x - \frac{1}{2} = -\frac{1}{(2\pi i)} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{e(rx)}{r}, \tag{3}$$

where $e(y) = e^{2\pi i y}$. Note the trigonometric sum

$$\sum_{a=1}^{q} e\left(\frac{am}{q}\right) = \begin{cases} q, & \text{if } q|m; \\ 0, & \text{otherwise.} \end{cases}$$

From the definition of $S(h, 1, 3, q)$ we have

$$\begin{aligned} S(h, 1, 3, q) &= \sum_{a=1}^{q} \bar{B}_1\left(\frac{a}{q}\right) \bar{B}_3\left(\frac{ah}{q}\right) \\ &= \sum_{a=1}^{q-1} \frac{1!3!}{(2\pi i)^{1+3}} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{+\infty} \frac{e\left(\frac{ra}{q} + \frac{sha}{q}\right)}{rs^3} \\ &= \frac{6}{(2\pi i)^4} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{+\infty} \frac{1}{rs^3} \sum_{a=1}^{q-1} e\left(\frac{a(r + sh)}{q}\right) \end{aligned}$$

$$= \frac{6}{(2\pi i)^4}\left[ q \sum_{\substack{r=-\infty\\ r\neq 0\\ r+sh\equiv 0(q)}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{rs^3} - \sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{rs^3}\right]$$

$$= \frac{6}{(2\pi i)^4}\left[ q \sum_{\substack{r=-\infty\\ r\neq 0\\ r+sh\equiv 0(q)}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{rs^3}\right], \tag{4}$$

since we have $\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty} \frac{1}{r} = 0$ and

$$\sum_{a=1}^{q-1} \sum_{r>N} \frac{e(\frac{ra}{q})}{r} \ll \frac{q^2}{N} \to 0 \ \ (N \to \infty).$$

Note that $(h, q) = 1$, from the method of proving (4) we also have

$$qS(h,2,q) = \frac{q}{4\pi^4}\left[ q \sum_{\substack{r=-\infty\\ r\neq 0\\ r+sh\equiv 0(q)}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{(rs)^2} - \sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{(rs)^2}\right]$$

$$= \frac{q}{4\pi^4}\left[ q \sum_{\substack{k=-\infty\\ kq\neq sh}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{(kq-sh)^2 s^2} - \sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{(rs)^2}\right]$$

$$= \frac{q^2}{4\pi^4} \sum_{\substack{k=-\infty\\ k\neq 0\\ kq\neq sh}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{h^2}{(1-s\frac{h}{kq})^2 (kq)^4 \left(s\frac{h}{kq}\right)^2}$$

$$+ \frac{2q^2}{4\pi^4 h^2}\zeta(4) - \frac{q}{4\pi^4} \sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty} \sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty} \frac{1}{(rs)^2}. \tag{5}$$

For convenience, let $c_j$ be constants such that

$$\frac{1}{x^2(1-x)^2} = \sum_{j=1}^{2} \frac{c_j}{x^j} + \sum_{j=1}^{2} \frac{c_j}{(1-x)^j}.$$

Then from (5) we have

$$
\begin{aligned}
qS(h,2,q) &= \frac{q^2}{4\pi^4}\sum_{j=1}^{2}\sum_{\substack{k=-\infty\\k\neq 0\\kq\neq sh}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{h^2c_j}{(1-s\frac{h}{kq})^j(kq)^4}\\
&\quad+\frac{q^2}{4\pi^4}\sum_{j=1}^{2}\sum_{\substack{k=-\infty\\k\neq 0\\kq\neq sh}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{h^2c_j}{(kq)^4\left(s\frac{h}{kq}\right)^j}\\
&\quad+\frac{2q^2}{4\pi^4h^2}\zeta(4)-\frac{q}{4\pi^4}\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{1}{(rs)^2}\\
&=\frac{q^2h^2}{4\pi^4}\sum_{j=1}^{2}\frac{c_j(-1)^j}{q^{4-j}}\sum_{\substack{k=-\infty\\k\neq 0\\kq\neq sh}}^{+\infty}\sum_{s=-\infty}^{+\infty}\frac{1}{(sh-kq)^jk^{4-j}}\\
&\quad-\frac{2h^2}{4\pi^4q^2}\zeta(4)\sum_{j=1}^{2}c_j+\frac{q^2h^2}{4\pi^4}\sum_{j=1}^{2}\frac{c_j}{h^jq^{4-j}}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{1}{k^{4-j}s^j}\\
&\quad-\frac{q^2h^2}{4\pi^4}\sum_{j=1}^{2}\frac{c_j}{h^jq^{4-j}}\sum_{\substack{k=-\infty\\k\neq 0\\kq=sh}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{1}{k^{4-j}s^j}\\
&\quad+\frac{2q^2}{4\pi^4h^2}\zeta(4)-\frac{q}{4\pi^4}\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{1}{(rs)^2}.
\end{aligned}
$$

From $(h,q)=1$ we know that $q|s$ and $h|k$ if $kq=sh$, so we can assume $k=mh$ and $s=mq$. Then we obtain

$$
\begin{aligned}
qS(h,2,q) &= \frac{q^2h^2}{4\pi^4}\sum_{j=1}^{2}\frac{c_j(-1)^j}{q^{4-j}}\sum_{\substack{k=-\infty\\k\neq 0\\kq\neq sh}}^{+\infty}\sum_{s=-\infty}^{+\infty}\frac{1}{(sh-kq)^jk^{4-j}}-\frac{2h^2}{4\pi^4q^2}\zeta(4)\sum_{j=1}^{2}c_j\\
&\quad+\frac{q^2h^2}{4\pi^4}\sum_{j=1}^{2}\frac{c_j}{h^jq^{4-j}}\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{1}{k^{4-j}s^j}-\frac{q^2h^2}{4\pi^4}\sum_{j=1}^{2}\frac{c_j}{h^4q^4}\sum_{\substack{m=-\infty\\m\neq 0}}^{+\infty}\frac{1}{m^4}\\
&\quad+\frac{2q^2}{4\pi^4h^2}\zeta(4)-\frac{q}{4\pi^4}\sum_{\substack{r=-\infty\\r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\s\neq 0}}^{+\infty}\frac{1}{(rs)^2}.
\end{aligned}
\tag{6}
$$

Arguing as in (4) and (5) we can also get the identity

$$S(q,j,4-j,h)=\frac{j!(4-j)!}{(2\pi i)^4}\left[h\sum_{\substack{r=-\infty\\ r\neq 0\\ r+kq\equiv 0(h)}}^{+\infty}\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}\frac{1}{r^j s^{4-j}}-\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{r^j s^{4-j}}\right]$$

$$=\frac{j!(4-j)!}{(2\pi i)^4}\left[h\sum_{\substack{s=-\infty\\ sh\neq kq}}^{+\infty}\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}\frac{1}{(sh-kq)^j k^{4-j}}-\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{r^j s^{4-j}}\right].$$

That is,

$$\sum_{\substack{s=-\infty\\ sh\neq kq}}^{+\infty}\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}\frac{1}{(sh-kq)^j k^{4-j}}$$

$$=\frac{(2\pi i)^4}{j!(4-j)!h}S(q,j,4-j,h)+\frac{1}{h}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{r^j s^{4-j}}. \tag{7}$$

Combining (6) and (7) we get

$$qS(h,2,q)=4h\sum_{j=1}^{2}\frac{c_j(-1)^j}{q^{2-j}j!(4-j)!}S(q,j,4-j,h)$$

$$+\frac{h}{4\pi^4}\sum_{j=1}^{2}\frac{c_j(-1)^j}{q^{2-j}}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{r^j s^{4-j}}$$

$$-\frac{2h^2}{4\pi^4 q^2}\zeta(4)\sum_{j=1}^{2}c_j+\frac{1}{4\pi^4}\sum_{j=1}^{2}\frac{c_j h^{2-j}}{q^{2-j}}\sum_{\substack{k=-\infty\\ k\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{k^{4-j}s^j}$$

$$-\frac{2}{q^2h^2 4\pi^4}\zeta(4)\sum_{j=1}^{2}c_j+\frac{2q^2}{4\pi^4h^2}\zeta(4)-\frac{q}{4\pi^4}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{(rs)^2}. \tag{8}$$

Note that $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, $c_1=2$ and $c_2=1$. From (8) we can easily get

$$qS(h,2,q)-hS(q,2,h)$$

$$=\frac{1}{36}-\frac{4h}{3q}S(q,1,3,h)+\frac{h}{36}-\frac{q}{36}+\frac{q^2}{180h^2}-\frac{h^2}{60q^2}-\frac{1}{60q^2h^2}. \tag{9}$$

Changing the position of $h$ and $q$ in (9), we get

$$hS(q,2,h) - qS(h,2,q)$$
$$= \frac{1}{36} - \frac{4q}{3h}S(h,1,3,q) + \frac{q}{36} - \frac{h}{36} + \frac{h^2}{180q^2} - \frac{q^2}{60h^2} - \frac{1}{60q^2h^2}. \quad (10)$$

Adding (9) and (10), we immediately deduce that

$$\frac{h}{q}S(q,1,3,h) + \frac{q}{h}S(h,1,3,q) = \frac{1}{24} - \frac{h^2}{120q^2} - \frac{q^2}{120h^2} - \frac{1}{40q^2h^2}.$$

This completes the proof of the theorem.

## Acknowledgment

The author expresses her gratitude to the referee for his helpful and detailed comments. This work is supported by N.S.F. (10601039) People's Republic of China.

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