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Univalence and starlikeness of nonlinear integral transform of certain class of analytic functions

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Abstract. Let $\mathcal{U}(\lambda, \mu)$ denote the class of all normalized analytic functions f in the unit disk $|z| < 1$ satisfying the condition

$$\frac{f(z)}{z} \neq 0 \quad \text{and} \quad \left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < \lambda, \quad |z| < 1.$$

For $f \in \mathcal{U}(\lambda, \mu)$ with $\mu \leq 1$ and $0 \neq \mu_1 \leq 1$, and for a positive real-valued integrable function φ defined on $[0, 1]$ satisfying the normalized condition $\int_0^1 \varphi(t)\, \mathrm{d}t = 1$, we consider the transform $G_\varphi f(z)$ defined by

$$G_\varphi f(z) = z\left[\int_0^1 \varphi(t)\left(\frac{zt}{f(tz)}\right)^{\mu} \mathrm{d}t\right]^{-1/\mu_1}, \quad z \in \Delta.$$

In this paper, we find conditions on the range of parameters λ and μ so that the transform $G_\varphi f$ is univalent or star-like. In addition, for a given univalent function of certain form, we provide a method of obtaining functions in the class $\mathcal{U}(\lambda, \mu)$.

Keywords. Univalent; Bazilevič; star-like and spiral-like functions; integral transforms.

1. Introduction

Let $\Delta := \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the set of all functions analytic in Δ with the usual normalization $f(0) = 0 = f'(0) - 1$, and let $\mathcal{A}_0 = \{f(z)/z: f \in \mathcal{A}\}$. Also, for $n \geq 1$, we let $\mathcal{A}_n = \{f \in \mathcal{A}: f(z) = z + a_{n+1}z^{n+1} + \cdots\}$ and $\mathcal{S} = \{f \in \mathcal{A}: f$ is univalent in $\Delta\}$. Observe that $\mathcal{A}_1 = \mathcal{A}$. One of the fundamental and independent directions in function theory is that of finding sufficient conditions for univalency (and for several other classical geometric subclasses of $\mathcal{S}$). The central place of investigation has been the development of analytic-geometric directions: the analytic description of subclasses of $\mathcal{S}$ determined, for example, by geometric property. This has led to appearance of new classes of univalent analytic functions which become fundamental in the development of many special topics such as the principle of subordination, the theory of duality mainly developed by Ruscheweyh [19], the geometrization of non-analytic conditions, etc. For instance, if $f \in \mathcal{S}$ maps Δ onto a star-like domain (with respect to the origin), i.e. $tw \in f(\Delta)$ whenever $t \in [0, 1]$ and $w \in f(\Delta)$, then we say that f is a star-like function. The class of all starlike functions is denoted by $\mathcal{S}^*$. Analytically, the class $\mathcal{S}^*$ is characterized as follows (see [4]):

$$\mathcal{S}^* := \left\{ f \in \mathcal{A}\colon \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \Delta \right\}.$$

With the help of Loewner–Kufarev differential equation (see Theorem 3.4 and Exercise 5 in p. 116 of [4]), Bazilevič [3] in 1955 was able to obtain a structural formula for a sufficiently wide class of functions from $\mathcal{S}$. More precisely, Bazilevič showed that the class of functions

$$f(z) = z\left\{(\alpha + i\beta)z^{-(\alpha+i\beta)} \int_0^z \left(\frac{g(t)}{t}\right)^{\alpha} p(t)t^{\alpha+i\beta-1}\,\mathrm{d}t\right\}^{1/(\alpha+i\beta)},$$

where $z \neq \Delta\backslash\{0\}$, $g \in \mathcal{S}^*$ and $\mathrm{Re}\,(\mathrm{e}^{i\gamma}p(z)) > 0$ in Δ for some real γ, β and $\alpha > 0$, forms a subclass of $\mathcal{S}$. The symbol $(g(t)/t)^{\alpha}$ denotes an analytic α power of $g(t)/t$ in Δ. All the powers inside the brackets are chosen as principal values, and the integral is taken along the line segment from 0 to z. The power outside the brackets is the analytic $(\alpha + i\beta)^{-1}$ power of the function inside the brackets which approaches 1 as $z \to 0$. The collection of univalent functions of the above form is called Bazilevič of type (α, β), and is often denoted by $\mathcal{B}(\alpha, \beta)$. We set $\mathcal{B} = \cup_{\alpha,\beta}\mathcal{B}(\alpha, \beta)$ and call $f \in \mathcal{B}$ a Bazilevič function. Particular choices of the parameters α, β, and the functions $g(z)$ and $p(z)$ yield the convex, star-like, close-to-convex and spiral-like functions (for a discussion on these classes, we refer to the book of Duren [4] and Goodman [6]). The Bazilevič class $\mathcal{B}$ is the largest known subclass of univalent functions in the unit disk Δ given by an explicit representation formula. In 1971, Sheil-Small [20] (see also [21]) gave a geometric proof of Bazilevič theorem and presented an intrinsic characterization of it along the lines of Kaplan's characterization of the close-to-convex functions [8]. A justification for allowing $\alpha = 0$ case has been discussed in [20] and an alternate approach to Bazilevič functions were also presented. In the special situation ($g(z) = z$, $\beta = 0 = \gamma$ and $\alpha = -\mu$), it is easy to see that the Bazilevič theorem takes the following simple form: *if $f \in \mathcal{A}$ satisfies the condition*

$$\mathrm{Re}\left(\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right) > 0, \quad z \in \Delta, \tag{1.1}$$

for some $\mu \leq 0$, then f is univalent in Δ. The class of functions f defined by (1.1), denoted simply by $f \in \mathcal{B}_1(-\mu)$, has been studied extensively (see for eg. [9,7,14,23] and references therein). In Thoerem 2.3 of [21], Sheil-Small obtained that if f satisfies condition

$$\left|\arg\left(\mathrm{e}^{i\gamma}\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right)\right| \leq \frac{(1-2\mu)\pi}{2}, \quad z \in \Delta, \tag{1.2}$$

for a suitable $\gamma \in \mathbb{R}$, then $f \in \mathcal{S}$ for $\mu \leq 1/2$ and for $\mu = 1/2$, f is spiral-like (see Corollary 3.4 of [21]). We note that the range of μ has been extended from $\mu \leq 0$ to $\mu \leq 1/2$. So, from (1.2), a natural question is to determine the region Ω of complex plane such that

$$\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) \in \Omega, \quad z \in \Delta, \tag{1.3}$$

implies that f is univalent. The first two authors, in particular, consider this problem for example with $\Omega = \{w \in \mathbb{C}\colon |w-1| < \lambda\}$ and determined condition on λ so that f satisfying the condition (1.3) is star-like, if $0 < \mu \leq 1$. Thus, one can still produce univalent functions f having a clear analytical description without being in the class $\mathcal{B}$, which has been considered to be the largest known subclass of univalent functions in the unit disk given by an explicit representation formula. In the following theorem, we see that we can achieve an even sharper result with μ being a complex parameter (see Theorem A below). Therefore, it is worth studying properties of the class $\mathcal{U}(\lambda, \mu)$ defined below:

$$\mathcal{U}(\lambda, \mu) := \left\{ f \in \mathcal{A}\colon \left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \Delta \right\}.$$

Here we have ignored the rotation factor $e^{i\gamma}$ in this definition as this has not much to do with our investigation. Set $\mathcal{U}(\lambda) := \mathcal{U}(\lambda, 1)$ and $\mathcal{U} := \mathcal{U}(1)$. From $f \in \mathcal{U}(\lambda, \mu)$, it follows that $f(z)/z \neq 0$ for $z \in \Delta$ and it is well-known that $\mathcal{U}(\lambda) \subsetneq \mathcal{U} \subsetneq \mathcal{S}$ whenever $0 < \lambda \leq 1$ (see [1,13]). In the recent years, the class $\mathcal{U}(\lambda)$ has been studied extensively. More recently, Fournier and Ponnusamy (Theorems 1 and 2 of [5]) proved as follows.

Theorem A. *Let* $\mu \in \mathbb{C}$ *with* $\operatorname{Re}(\mu) < 1$. *Then*

(1) $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^*$ *iff* $0 \leq \lambda \leq \frac{|1-\mu|}{\sqrt{|1-\mu|^2+|\mu|^2}}$.

(2) $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}p$ *iff* $0 \leq \lambda \leq \min\left(1, \frac{|1-\mu|}{|\mu|}\right)$, *where* $\mathcal{S}p$ *is characterized as follows*:

$$\mathcal{S}p := \{f \in \mathcal{A} \mid \operatorname{Re}(e^{i\theta} z f'(z)/f(z)) > 0, z \in \mathbb{D},$$
$$\textit{for some } \theta \in (-\pi/2, \pi/2)\}.$$

It is well-known that the class $\mathcal{S}p$ of spiral-like functions is included in $\mathcal{S}$ (see [4]). This theorem clearly gives conditions under which $\mathcal{U}(\lambda, \mu)$ has geometric significance (such as star-likeness and spiral-likeness). Part 2 of Theorem A yields the following lemma (worth comparing with (1.2)) which because of its independent interest we state it separately.

Lemma B. Functions $\mathcal{U}(1, \mu)$ *are included in the class* $\mathcal{S}p$ *of all spiral-like functions if and only if* $\operatorname{Re}\mu \leq 1/2$.

The class $\mathcal{U}(1, \mu)$ has many interesting properties (see the end of this section). We refer to [10–12] for many interesting results for the case $\mu = 1$.

Also, for $\mu \leq n$, we consider $\mathcal{U}_n(\lambda, \mu) = \mathcal{U}(\lambda, \mu) \cap \mathcal{A}_n$. We also introduce

$$\mathcal{B}_n = \{w \in \mathcal{H}\colon |w(z)| < 1 \quad \text{and} \quad w^{(k)}(0) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1\},$$

where $\mathcal{H}$ denotes the class of all analytic functions in Δ. By Schwarz' lemma, one has $|w(z)| \leq |z|^n$. The main aim of this article is to extend the recent work of Obradović and Ponnusamy [10] by discussing geometric properties of certain integral transform of functions in $\mathcal{U}(\lambda, \mu)$. Also, we state and prove several other related results for functions in $\mathcal{U}(\lambda, \mu)$ which miss their initial coefficients in their Taylor series expansion.

For the proof of our results, we also need the following lemmas.

Lemma C (*Corollary* 3.2 *of* [16]). *If* $f \in \mathcal{U}_n(\lambda, \mu)$ *with* $0 < \mu < n$ *and*

$$0 < \lambda \leq \frac{n-\mu}{\sqrt{(n-\mu)^2 + \mu^2}},$$

then $f \in \mathcal{S}^*$.

Lemma D (*Corollary* 3.1 *of* [17]). *If* $f \in \mathcal{U}_n(\lambda, n)$ *and*

$$0 < \lambda \leq \frac{-n^2|a_{n+1}| + \sqrt{1 + n^2(1 - |a_{n+1}|^2)}}{1 + n^2},$$

then $f \in \mathcal{S}^*$.

The upper bound on the range of λ in Lemmas C and D can be easily shown to be sharp using the ideas from [5].

Lemma E (*Lemma* 3 *of* [18]). *Let* $0 \leq \mu < 1$ *and* $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *be a non-vanishing analytic function in* Δ *satisfying the coefficient condition*

$$\sum_{n=1}^{\infty} (n-\mu)|b_n| \leq \lambda\mu.$$

Then the function f *defined by the equation* $(z/f(z))^{\mu} = \phi(z)$ *is in* $\mathcal{U}(\lambda, \mu)$.

We recall that the Hadamard product $f * g$ of two power series $f(z) := \sum_{n=0}^{\infty} a_n(f) z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g) z^n$ in $\mathcal{H}$ is the power series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n(f) a_n(g) z^n.$$

It is easy to observe that $(f * g)(z)$ is also analytic in the unit disk Δ. The following lemma will be used in the proof of Theorem 1.

Lemma F [19]. *Let* $c \in \mathbb{C}$ *with* $\operatorname{Re} c < 1$ *and*

$$F_c(z) := \sum_{n=1}^{\infty} \frac{1-c}{n-c} z^{n-1} \in \mathcal{H},$$

where $\mathcal{H}$ *denote the class of all functions analytic in the unit disk* Δ. *Then*

$$\sup_{z \in \Delta} |f * F_c(z)| \leq \sup_{z \in \Delta} |f(z)|, \quad \textit{for any } f \in \mathcal{H}.$$

It is worth ending this section with some facts about the class $\mathcal{U}(\lambda, 1)$ that are known from the recent work of the first two authors. For instance, for the details, we refer to [11,12] and the references therein.

- Although the Koebe function $k(z) = z/(1-z)^2 \in \mathcal{U}$ is extremal for the class $\mathcal{S}^*$, $\mathcal{U}$ is not included in $\mathcal{S}^*$. However, $\mathcal{U} \subset \mathcal{S}$ and so, it is interesting to investigate the properties of $\mathcal{U}(\lambda, 1)$ and more generally the class $\mathcal{U}(\lambda, \mu)$ in relation to integral transform, in particular.

- For analytic functions f in Δ of form $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$, a sufficient condition for f to be in the class $\mathcal{U}$ is that $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$ and even more, a necessary condition for f to be in $\mathcal{S}$ is that $\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1$. On the other hand, no such simple necessary condition for functions f in the standard normalized form $f(z) = z + a_2 z^2 + \cdots$ to be in $\mathcal{U}$ seems to be known in the literature except the Bieberbach estimate $|a_n| \leq n$ with equality for the Koebe function.
- The sufficient condition stated above is useful especially for rational functions.
- The class of convex function is included in the class of star-like functions. We also have a simple analog of it concerning the class $\mathcal{U}$: if $f \in \mathcal{A}$ satisfies the condition

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2, \quad z \in \Delta,$$

then $f \in \mathcal{U}$.
- If $f, g \in \mathcal{S}$ then the function F defined by $\frac{z}{F(z)} = \frac{z}{f(z)} * \frac{z}{g(z)}$ is in $\mathcal{U}$ whenever $\frac{z}{f(z)} * \frac{z}{g(z)} \neq 0$ in Δ. The analog of this property does not exist for the convolution $f(z) * g(z)$. For example, convolution of the Koebe function by itself is not univalent in Δ.

Apparently, the class $\mathcal{U}(\lambda, \mu)$ is consistent with the Bazilevič class and enables one to proceed with a new approach to the theory of univalent functions although there does not seem to be direct geometric meaning! It is sometimes convenient to consider functions $f \in \mathcal{U}(\lambda, \mu)$ of the form $\left(\frac{z}{f(z)}\right)^{\mu} = 1 + b_1 z + b_2 z^2 + \cdots$, because the condition $\sum_{n=1}^{\infty} (n-\mu)|b_n|^2 \leq \mu$ is necessary for f to be in the class $\mathcal{S}$ (see Theorem 11, p. 193, vol. 2 of [6]).

2. Univalency and star-likeness of $(G_\varphi f)(z)$

In order to state and prove our results, we need to introduce

$$F_\varphi(z) = \int_0^1 \frac{\varphi(t)}{1 - tz}\, \mathrm{d}t, \quad z \in \Delta,$$

where φ is a positive real-valued function satisfying the normalized condition $\int_0^1 \varphi(t)\, \mathrm{d}t = 1$. Many classical functions are closed related to this operator. For example, Lerch function

$$\Phi_p(a; z) := \sum_{n=1}^{\infty} \frac{z^n}{(n+a)^p} \quad (p > 0, a > -1)$$

which includes the *polylogarithms* as special case ($\mathrm{Li}_p(z) = \Phi_p(0; z)$), where $\Phi_p(a; z)/z$ assumes the above form with

$$\varphi(t) = \frac{(\log 1/t)^{p-1} t^a}{\Gamma(p)}.$$

We refer to [15] for various other examples with its geometric root through the principle of duality or convolution.

Now, for $f \in \mathcal{U}(\lambda, \mu)$ with $\mu \leq 1$ and $0 \neq \mu_1 \leq 1$, we first consider the integral transform $G(z) = (G_\varphi f)(z)$ defined by

$$G(z) = z\left[\int_0^1 \varphi(t)\left(\frac{zt}{f(tz)}\right)^{\mu} \mathrm{d}t\right]^{-1/\mu_1}, \quad z \in \Delta. \tag{2.1}$$

When $\mu_1 = 0$, we set $(G_\varphi f)(z) = z$.

Now, we state our first result.

Theorem 1. *Let $f \in \mathcal{U}(\lambda, \mu)$ such that $(z/f(z))^{\mu} * F_\varphi(z) \neq 0$ in Δ, where $F_\varphi(z)$ is as above and $G := G_\varphi f$ is as defined in* (2.1) *with $0 \neq \mu_1 \leq 1$. Then $G \in \mathcal{U}(\lambda_1, \mu_1)$, where*

$$\lambda_1 = \begin{cases} \frac{(1-\mu_1)\mu a_2}{|\mu_1|}\int_0^1 t\varphi(t)\,\mathrm{d}t + 2\lambda\left|\frac{\mu}{\mu_1}\right|\left(1+\frac{|\mu-\mu_1|}{1-\mu}\right)\int_0^1 t^2\varphi(t)\,\mathrm{d}t, & \text{if } \mu < 1\\ \left|\frac{a_2}{\mu_1}\right|(1-\mu_1)\int_0^1 t\varphi(t)\,\mathrm{d}t + \frac{\lambda(2-\mu_1)}{|\mu_1|}\int_0^1 t^2\varphi(t)\,\mathrm{d}t, & \text{if } \mu = 1. \end{cases}$$

Proof.

Case 1. Let $f \in \mathcal{U}(\lambda, \mu)$ with $f(z) = z + a_2z^2 + \cdots$ and $\mu \leq 1$. Then

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} = 1 + \lambda w(z) = 1 + (1-\mu)a_2 z + \cdots, \quad |z| < 1 \tag{2.2}$$

for some $w(z) = w_1z + w_2z^2 + \cdots$ with $w \in \mathcal{B}_1$. From the above formulation, we see that

$$\lambda w'(0) = (1-\mu)a_2$$

so that either $\mu = 1$ or $a_2 = 0$ if and only if $w'(0) = 0$ and thus, in this case $w(z)$ would be in $\mathcal{B}_2$. Since $\mu \leq 1$, using principal branch of possible multiple-valued power functions (if necessary), we see that the function $q(z)$ defined by

$$q(z) = \left(\frac{z}{f(z)}\right)^{\mu} = 1 + q_1z + \cdots$$

is analytic for $|z| < 1$ and it is easy to see that (2.2) is equivalent to

$$q(z) - \frac{1}{\mu}zq'(z) = 1 + \lambda w(z).$$

By an algebraic computation, we find that

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \lambda\int_0^1 \frac{w(t^{1/\mu}z)}{t^2}\,\mathrm{d}t = 1 - \lambda\mu\sum_{n=1}^{\infty}\frac{w_nz^n}{n-\mu},$$

where, in the case $w'(0) = w_1 = 0$, the summation starts from $n = 2$ instead of $n = 1$. From (2.1), it is clear that

$$\left(\frac{z}{G(z)}\right)^{\mu_1} = \int_0^1 \varphi(t)\left(\frac{tz}{f(tz)}\right)^{\mu}\mathrm{d}t. \tag{2.3}$$

Further, for a given $g \in \mathcal{A}_0$, we have

$$g(z) * F_{\varphi}(z) = \int_0^1 \varphi(t) g(tz)\,\mathrm{d}t.$$

In view of the last observation,

$$\left(\frac{z}{f(z)}\right)^{\mu} * F_{\varphi}(z) = \int_0^1 \varphi(t) \left(\frac{tz}{f(tz)}\right)^{\mu} \mathrm{d}t$$

is a non-vanishing (by hypothesis) analytic function in Δ, which in series form is equivalent to

$$\left(\frac{z}{f(z)}\right)^{\mu} * F_{\varphi}(z) = 1 - \lambda\mu \sum_{n=1}^{\infty} \left(\frac{w_n}{n-\mu} \int_0^1 t^n \varphi(t)\,\mathrm{d}t\right) z^n.$$

Again we understood that the summation starts from $n = 2$ if either $\mu = 1$ or $a_2 = 0$ (because $w_1 = 0$). A comparison with (2.3) shows that

$$\left(\frac{z}{G(z)}\right)^{\mu_1} = \left(\frac{z}{f(z)}\right)^{\mu} * F_{\varphi}(z). \tag{2.4}$$

Differentiating (2.4), we get

$$(\mu_1 - \mu)\left(\frac{z}{G(z)}\right)^{\mu_1} + \mu \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) * F_{\varphi}(z) = \mu_1 p(z), \tag{2.5}$$

where

$$p(z) = \left(\frac{z}{G(z)}\right)^{\mu_1+1} G'(z).$$

Because the left-hand side of (2.5) is analytic in the unit disk Δ, it follows that $p(z)$ is also analytic in Δ. Further, from the representation of $G(z)$ and $\varphi(z)$, the above equation is equivalent to the following:

$$\mu_1 \left[\left(\frac{z}{G(z)}\right)^{\mu_1} - \left(\frac{z}{G(z)}\right)^{\mu_1+1} G'(z)\right] = -\lambda\mu \sum_{n=1}^{\infty} z^n \frac{n w_n}{n-\mu} \int_0^1 t^n \varphi(t)\,\mathrm{d}t$$

which in turn, from (2.4), implies that

$$\begin{aligned}\left(\frac{z}{G(z)}\right)^{\mu_1+1} G'(z) - 1 &= \frac{\lambda\mu}{\mu_1} \int_0^1 \varphi(t) \sum_{n=1}^{\infty} \left(\frac{n-\mu_1}{n-\mu}\right) w_n t^n z^n\,\mathrm{d}t \\ &= z\left(\frac{\mu(1-\mu_1)}{\mu_1}\right) a_2 \int_0^1 t\varphi(t)\,\mathrm{d}t \\ &\quad + \frac{\lambda\mu}{\mu_1} \int_0^1 \varphi(t) \sum_{n=2}^{\infty} \left(\frac{n-\mu_1}{n-\mu}\right) w_n t^n z^n\,\mathrm{d}t. \end{aligned} \tag{2.6}$$

When $\mu < 1$, we have the following:

$$\frac{\lambda\mu}{\mu_1}\int_0^1 \varphi(t)\sum_{n=2}^{\infty}\left(\frac{n-\mu_1}{n-\mu}\right) w_n t^n z^n \,\mathrm{d}t$$

$$= \frac{\lambda\mu}{\mu_1}\int_0^1 (w(tz)-w'(0)tz)\varphi(t)\,\mathrm{d}t + (\mu-\mu_1)\frac{\lambda\mu}{\mu_1}\int_0^1 \varphi(t)\sum_{n=2}^{\infty}\frac{w_n t^n z^n}{n-\mu}\,\mathrm{d}t$$

$$= \frac{\lambda\mu}{\mu_1}\int_0^1 (w(tz)-w'(0)tz)\varphi(t)\,\mathrm{d}t$$

$$+ \lambda\frac{\mu(\mu-\mu_1)}{\mu_1(1-\mu)}\int_0^1 \varphi(t)[(w(tz)-w'(0)tz) * z\,F_\mu(z)]\,\mathrm{d}t,$$

where $F_\mu(z)$ denotes the Gaussian hypergeometric function $F(1, 1-\mu; 2-\mu; z)$ (see [2,22]). Now, from Lemma F,

$$|(w(tz)-w'(0)tz) * tz\,F_\mu(tz)| \le \sup_{z\in\Delta}|(w(tz)-w'(0)tz)| \le 2t^2.$$

Indeed, as $(w(tz)-w'(0)tz)/2$ is again a Schwarz function with a zero of order two at the origin, it follows from the Schwarz lemma that

$$|w(tz)-w'(0)tz| \le 2t^2|z|^2.$$

Thus, (2.6) gives

$$\left|\left(\frac{z}{G(z)}\right)^{\mu_1+1} G'(z)-1\right|$$

$$\le \left|\frac{\mu(1-\mu_1)}{\mu_1}a_2 z\int_0^1 t\varphi(t)\,\mathrm{d}t\right| + \left|\frac{\lambda\mu}{\mu_1}\int_0^1 (w(tz)-w'(0)tz)\varphi(t)\,\mathrm{d}t\right|$$

$$+ \left|\lambda\frac{\mu(\mu-\mu_1)}{\mu_1(1-\mu)}\int_0^1 \varphi(t)[(w(tz)-w'(0)tz) * z\,F_\mu(z)]\,\mathrm{d}t\right|$$

$$< \left|\frac{\mu}{\mu_1}\right|(1-\mu_1)|a_2|\int_0^1 t\varphi(t)\,\mathrm{d}t + 2\lambda\left|\frac{\mu}{\mu_1}\right|\int_0^1 t^2\varphi(t)\,\mathrm{d}t$$

$$+ 2\lambda\left|\frac{\mu}{\mu_1}\right|\frac{|\mu-\mu_1|}{1-\mu}\int_0^1 t^2\varphi(t)\,\mathrm{d}t.$$

This gives the required result for $\mu < 1$.

Case 2. Let $\mu = 1$. In this case $f \in \mathcal{U}(\lambda, 1)$ represented by (2.2) takes the form

$$\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' = f'(z)\left(\frac{z}{f(z)}\right)^2 = 1+\lambda w(z),$$

for some $w \in \mathcal{B}_2$. This gives that

$$\frac{z}{f(z)} = 1 - a_2 z - \lambda\int_0^1 \frac{w(tz)}{t^2}\,\mathrm{d}t,$$

where $|w(z)| \leq |z|^2$. Substituting this in (2.3), and proceeding exactly as in the previous case, we see that the corresponding (2.6) reduces to

$$\begin{aligned}
&\left(\frac{z}{G(z)}\right)^{\mu_1+1} G'(z) - 1 \\
&= z\left(\frac{1-\mu_1}{\mu_1}\right) a_2 \int_0^1 t\varphi(t)\,\mathrm{d}t + \frac{\lambda}{\mu_1}\int_0^1 \varphi(t)\sum_{n=2}^{\infty}\left(\frac{n-\mu_1}{n-1}\right) w_n t^n z^n\,\mathrm{d}t \\
&= z\left(\frac{1-\mu_1}{\mu_1}\right) a_2 \int_0^1 t\varphi(t)\,\mathrm{d}t + \frac{\lambda}{\mu_1}\int_0^1 \varphi(t) w(tz)\,\mathrm{d}t \\
&\quad + \frac{\lambda(1-\mu_1)}{\mu_1}\int_0^1\int_0^1 \varphi(t)\frac{w(tsz)}{s^2}\,\mathrm{d}s\,\mathrm{d}t, \quad w \in \mathcal{B}_2.
\end{aligned}$$

We observe that (since $w \in \mathcal{B}_2$)

$$\begin{aligned}
\left|\int_0^1\int_0^1 \varphi(t)\frac{w(tsz)}{s^2}\,\mathrm{d}s\,\mathrm{d}t\right| &\leq \int_0^1\int_0^1 \varphi(t)\frac{|w(tsz)|}{s^2}\,\mathrm{d}s\,\mathrm{d}t \\
&\leq \int_0^1\int_0^1 \varphi(t)t^2|z|^2\,\mathrm{d}s\,\mathrm{d}t \\
&< \int_0^1 \varphi(t)t^2\,\mathrm{d}t
\end{aligned}$$

and therefore, by the triangle inequality, we easily see that

$$\begin{aligned}
\left|\left(\frac{z}{G(z)}\right)^{\mu_1+1} G'(z) - 1\right| &< \left|\frac{a_2}{\mu_1}\right|(1-\mu_1)\int_0^1 t\varphi(t)\,\mathrm{d}t + \frac{\lambda}{|\mu_1|}\int_0^1 t^2\varphi(t)\,\mathrm{d}t \\
&\quad + \lambda\frac{1-\mu_1}{|\mu_1|}\int_0^1 t^2\varphi(t)\,\mathrm{d}t.
\end{aligned}$$

The desired conclusion follows. ■

As a consequence of Lemma B, we have the following result for the univalency of G defined by (2.1).

COROLLARY 1

Let $f \in \mathcal{U}(\lambda,\mu)$ with $f''(0) = 0$, where λ and μ are related by

$$\lambda = \begin{cases} \frac{1-\mu}{2|\mu|(3-4\mu)\int_0^1 t^2\varphi(t)\,\mathrm{d}t} & \text{if } \mu \leq 1/2 \\ \frac{1-\mu}{2\mu\int_0^1 t^2\varphi(t)\,\mathrm{d}t} & \text{if } 1/2 \leq \mu < 1 \\ \frac{1}{3\int_0^1 t^2\varphi(t)\,\mathrm{d}t} & \text{if } \mu = 1. \end{cases}$$

If, in addition, $(z/f(z))^{\mu} * F_{\varphi}(z) \neq 0$ in Δ where $F_{\varphi}(z)$ is defined as in Theorem 1, then $G := G_{\varphi}f$ defined by (2.1) belongs to $\mathcal{U}(1, 1/2)$ (in particular, $G_{\varphi}f$ is spiral-like).

Proof. The conclusion follows from Theorem 1 if we choose $\lambda_1 = 1$ and $\mu_1 = 1/2$, and finally apply Lemma B. ■

Taking $\lambda_1 = 1 = \mu_1$ in Theorem 1, we have the following:

COROLLARY 2

Let $f \in \mathcal{U}(\lambda, \mu)$ with $f''(0) = 0$, where λ and μ are related by

$$\lambda = \begin{cases} \frac{1}{4|\mu| \int_0^1 t^2 \varphi(t)\,\mathrm{d}t}, & \text{if } \mu < 1 \\ \frac{1}{\int_0^1 t^2 \varphi(t)\,\mathrm{d}t}, & \text{if } \mu = 1. \end{cases}$$

Then $G := G_\varphi f$ defined by (2.1) belongs to $\mathcal{U}(1, 1)$; in particular, G is univalent.

If we take $\varphi(t) = ct^{c-1}$ for $c > 0$, $\mu = 1 = \mu_1$, then the transform $G := G_\varphi f$ defined by (2.1) has the following simple form:

$$G(z) = z\left(c\int_0^1 \frac{tz}{f(tz)} t^{c-1}\,\mathrm{d}t\right)^{-1}, \quad z \in \Delta.$$

Consequently, Corollary 2 reduces to the following result obtained in Theorem 1 of [10].

COROLLARY 3

Let $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$, and $c > 0$ be such that

$$\frac{z}{f(z)} * F(1, c; c+1; z) \neq 0 \text{ in } \Delta,$$

and $G = G_f^c$ be the transform defined by

$$G(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)} = z\left(c\int_0^1 \frac{tz}{f(tz)} t^{c-1}\,\mathrm{d}t\right)^{-1}, \quad z \in \Delta.$$

Then, $G \in \mathcal{U}(1, 1)$ whenever $0 < \lambda \le (c+2)/c$. In particular, G is univalent.

Here $F(a, b; c; z)$ denotes the Gaussian hypergeometric function and, for a discussion on hypergeometric functions with some modern approach, we refer to [2,22]. Now we state and prove general results for functions with missing coefficients.

Theorem 2. *Let $\mu \le n$ and $f \in \mathcal{U}_n(\lambda, \mu)$ be such that $(z/f(z))^\mu * F_\varphi(z) \neq 0$ in Δ where $F_\varphi(z)$ is defined as in Theorem 1. Then $G := G_\varphi f$ defined by (2.1), with $0 \neq \mu_1 \le n$, belongs to the class $\mathcal{U}_n(\lambda_1, \mu_1)$ where λ_1 satisfies*

$$\lambda_1 = \begin{cases} \frac{\mu(n-\mu_1)|a_{n+1}|}{|\mu_1|} \int_0^1 t^n \varphi(t)\,\mathrm{d}t + \frac{2\lambda|\mu|}{|\mu_1|}\left(1 + \frac{|\mu-\mu_1|}{n-\mu}\right) \int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t, & \text{if } \mu < n \\ \frac{n(n-\mu_1)|a_{n+1}|}{|\mu_1|} \int_0^1 t^n \varphi(t)\,\mathrm{d}t + \frac{\lambda n(n+1-\mu_1)}{|\mu_1|} \int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t, & \text{if } \mu = n. \end{cases}$$

Proof. The proof follows if we proceed exactly as in the proof of Theorem 1 but with $w \in \mathcal{B}_n$ in the case of $\mu < n$, and with $w \in \mathcal{B}_{n+1}$ in the case of $\mu = n$. So we omit the details. ■

Using Theorem 2, we can find λ such that $G := G_\varphi f$ defined by (2.1) is in $\mathcal{S}^*$. Again, we need to consider two cases, namely, $\mu < n$ and $\mu = n$. In the case of $\mu < n$, we use Lemmas C, D and Theorem 2 and obtain the following general result.

Theorem 3. *Let $\mu < n$, $f \in \mathcal{U}_n(\lambda, \mu)$ be such that $(z/f(z))^\mu * F_\varphi(z) \neq 0$ in Δ and $G := G_\varphi f$ be as defined in* (2.1). *Then $G \in \mathcal{S}^*$ whenever* $0 < \lambda \leq \lambda^*$, *where*

$$\lambda^* = \begin{cases} \left[\frac{1}{\sqrt{(n-\mu_1)^2+\mu_1^2}} - \frac{|\mu|\,|a_{n+1}|}{|\mu_1|}\int_0^1 t^n\varphi(t)\,\mathrm{d}t\right]\frac{|\mu_1|(n-\mu_1)(n-\mu)}{2|\mu|(n-\mu+|\mu-\mu_1|)\int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t}, & \text{if } \mu_1 < n \\ \left[\frac{-n^2|a_{n+1}|+\sqrt{1+n^2(1-|a_{n+1}|^2)}}{1+n^2}\right]\frac{n}{4|\mu|\int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t}, & \text{if } \mu_1 = n. \end{cases}$$

Proof. Suppose that $f \in \mathcal{U}_n(\lambda, \mu)$ with $\mu < n$. We need to deal with two cases.

Case i. Let $\mu_1 < n$. In this case, Theorem 2 implies that $G := G_\varphi f$ defined by (2.1) belongs to $\mathcal{U}_n(\lambda_1, \mu_1)$. As an application of Lemma C, we conclude that $G \in \mathcal{S}^*$ whenever λ_1 in Theorem 2 satisfies the inequality

$$\lambda_1 \leq \frac{n-\mu_1}{\sqrt{(n-\mu_1)^2+\mu_1^2}}$$

which, by the definition of λ_1 with $\mu < n$, is equivalent to the inequality

$$\frac{\mu(n-\mu_1)|a_{n+1}|}{|\mu_1|}\int_0^1 t^n\varphi(t)\,\mathrm{d}t + \frac{2\lambda|\mu|}{|\mu_1|}\left(1+\frac{|\mu-\mu_1|}{n-\mu}\right)\int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t \leq \frac{n-\mu_1}{\sqrt{(n-\mu_1)^2+\mu_1^2}}.$$

A simplification gives the condition on λ, namely $0 < \lambda \leq \lambda^*$, and so the first part of the theorem follows.

Case ii. Let $\mu_1 = n$. Using Theorem 2 and Lemma D, we see that $G \in \mathcal{S}^*$ whenever λ_1 in Theorem 2 satisfies the inequality

$$\lambda_1 \leq \frac{-n^2|a_{n+1}|+\sqrt{1+n^2(1-|a_{n+1}|^2)}}{1+n^2}$$

which, by the definition of λ_1 with $\mu = n$, is equivalent to

$$\frac{n(n-\mu_1)|a_{n+1}|}{|\mu_1|}\int_0^1 t^n\varphi(t)\,\mathrm{d}t + \frac{\lambda n(n+1-\mu_1)}{|\mu_1|}\int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t \leq \frac{-n^2|a_{n+1}|+\sqrt{1+n^2(1-|a_{n+1}|^2)}}{1+n^2}.$$

This inequality gives the desired range for λ. ■

Now, we are in a position to present an analog of Theorem 3 dealing with the case $\mu = n$. Since its proof follows exactly as in the proof of the last theorem with $\mu = n$, we omit the details.

Theorem 4. *Let $f \in \mathcal{U}_n(\lambda, n)$ such that $(z/f(z))^\mu * F_\varphi(z) \neq 0$ in Δ, and $G := G_\varphi f$ be as defined in* (2.1). *Then $G \in \mathcal{S}^*$ whenever $0 < \lambda \leq \lambda^*$, where*

$$\lambda^* = \begin{cases} \left[\frac{1}{\sqrt{(n-\mu_1)^2+\mu_1^2}} - \frac{n\,|a_{n+1}|}{|\mu_1|}\int_0^1 t^n\varphi(t)\,\mathrm{d}t\right]\frac{|\mu_1|(n-\mu_1)}{n(n+1-\mu_1)\int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t}, & \text{if } \mu_1 < n \\ \left[\frac{-n^2|a_{n+1}|+\sqrt{1+n^2(1-|a_{n+1}|^2)}}{1+n^2}\right]\frac{1}{\int_0^1 t^{n+1}\varphi(t)\,\mathrm{d}t}, & \text{if } \mu_1 = n. \end{cases}$$

The last two theorems provide conditions so that functions in $\mathcal{U}(\lambda, \mu)$ produces star-like functions (in particular, univalent functions) via the integral transform defined by (2.1). It is natural to ask when the converse be true? Now, we have an affirmative answer to this question by finding a sufficient condition for functions to be in $\mathcal{U}(\lambda, \mu)$ in terms of the Taylor coefficients.

Theorem 5. *Let $f \in \mathcal{S}$ be of the form $(z/f(z))^\mu = 1 + \sum_{n=1}^\infty b_n z^n$, where $0 < \mu \leq 1$. Define*

$$H_f(z) = \frac{z}{\left(\left(\frac{z}{f(z)}\right)^\mu * h(z)\right)^{1/\mu}}, \tag{2.7}$$

where $h(z) = 1 + \sum_{n=1}^\infty c_n z^n$ is such that

$$\mu|a_2c_1| + \sqrt{\mu(1-\mu|a_2|^2(1-\mu))}\left(\sum_{n=2}^\infty \frac{|c_n|^2}{n-\mu}\right)^{1/2} \leq 1 \tag{2.8}$$

and

$$\left[\sum_{n=1}^\infty ((n/\mu)-1)|c_n|^2\right]^{1/2} \leq \lambda < \infty, \tag{2.9}$$

where $a_2 = f''(0)/2$. Then $H_f \in \mathcal{U}(\lambda, \mu)$.

Proof. Using the given power series representation, we have

$$\left(\frac{z}{f(z)}\right)^\mu * h(z) = 1 + \sum_{n=1}^\infty b_n c_n z^n,$$

where $b_1 = -\mu a_2$. Further, from the representation of $f \in \mathcal{S}$, it follows from the well-known area theorem (Theorem 11 in p. 193 of vol. 2 of [6]) that

$$\sum_{n=1}^\infty (n-\mu)|b_n|^2 \leq \mu, \quad \text{i.e.} \quad \sum_{n=2}^\infty (n-\mu)|b_n|^2 \leq \mu - (1-\mu)\mu^2|a_2|^2.$$

Now, by the triangle inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^{\mu} * h(z)\right| &\geq 1-|b_1c_1|\,|z| - \sum_{n=2}^{\infty}\sqrt{n-\mu}|b_n|\frac{|c_n|}{\sqrt{n-\mu}}|z|^n \\ &> 1-\mu|a_2c_1| - \left(\sum_{n=2}^{\infty}(n-\mu)|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-\mu}\right)^{1/2} \\ &\geq 1-\mu|a_2c_1| - \sqrt{\mu(1-\mu|a_2|^2(1-\mu))}\sqrt{\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-\mu}},\end{aligned}$$

where the last term on the right is nonnegative, by the hypothesis. Therefore, $H_f(z)$ is well-defined and analytic in the unit disk. In order to complete the proof, we need to apply Lemma E. By Lemma *E*, it suffices to prove that

$$\sum_{n=1}^{\infty}(n-\mu)|b_n|\,|c_n| \leq \lambda\mu.$$

Again, the Cauchy-Schwarz inequality gives

$$\begin{aligned}\sum_{n=1}^{\infty}(n-\mu)|b_n|\,|c_n| &= \sum_{n=1}^{\infty}(\sqrt{n-\mu}|b_n|)(\sqrt{n-\mu}|c_n|) \\ &\leq \left(\sum_{n=1}^{\infty}(n-\mu)|b_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}(n-\mu)|c_n|^2\right)^{1/2} \\ &\leq \sqrt{\mu}\,\sqrt{\mu}\,\lambda,\end{aligned}$$

since $\sum_{n=1}^{\infty}(n-\mu)|c_n|^2 \leq \mu\lambda^2$ by hypothesis. This gives the required result. ■

Remark. The case $\mu = 1$ of Theorem 5 has been dealt in [10]. However, Theorem 5 does have some interesting applications. ■

COROLLARY 4

Let $f \in \mathcal{S}$ be of the form $(z/f(z))^{\mu} = 1 + \sum_{n=2}^{\infty} b_n z^n$ where $0 < \mu \leq 1/2$, and $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be such that $\sum_{n=1}^{\infty}(n-\mu)|c_n|^2 \leq \mu$. Then the function H_f defined by (2.7) *belongs to the class $\mathcal{S}p$* (*and hence H_f is univalent in Δ*).

Proof. From the representation of f, we observe that $a_2 = f''(0)/2 = 0$. Next we observe that $n-\mu \geq 1-\mu$ for all $n \geq 1$. Using this, we obtain that

$$\begin{aligned}\mu\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-\mu} &= \mu\left[\sum_{n=1}^{\infty}\frac{(n-\mu)|c_n|^2}{(n-\mu)^2} - \frac{|c_1|}{1-\mu}\right] \\ &\leq \frac{\mu}{(1-\mu)^2}(\mu-(1-\mu)|c_1|) \leq 1,\end{aligned}$$

where the last inequality is clear, because $\mu/(1-\mu) \leq 1$. Thus, both (2.8) and (2.9) hold. Thus, by Theorem 5, it follows that $H_f \in \mathcal{U}(1,\mu)$ where $0 < \mu \leq 1/2$. The desired conclusion follows from Lemma B. ■

Example 1. Choose $h(z) = 1/(1 - az)$ with $0 < |a| = r \leq 1/\sqrt{3}$ and $\mu = 1/2$. Then $c_n = a^n$ and it is a simple exercise to see that

$$\sum_{n=1}^{\infty}(2n-1)|c_n|^2 = \sum_{n=1}^{\infty}(2n-1)r^{2n} \leq \sum_{n=1}^{\infty}(2n-1)\frac{1}{3^n} = 1.$$

Thus, by Corollary 4, we conclude that the function

$$H_f(z) = \frac{z}{\left(\sqrt{\frac{z}{f(z)}} * \frac{1}{1-az}\right)^2} \quad (|a| \leq 1/\sqrt{3})$$

belongs to $\mathcal{U}(1, 1/2)$ and hence $H_f(z)$ is spiral-like in Δ if $f \in \mathcal{S}$ with $f''(0) = 0$. ■

Again, choose $h(z)$ in Theorem 5 as $h(z) = 1/(1 - az)$ with $0 < |a| = r < 1$. Then $c_n = a^n$ and so

$$\lambda^2 = \sum_{n=1}^{\infty}((n/\mu) - 1)|c_n|^2 = \sum_{n=1}^{\infty}((n/\mu) - 1)r^2$$

which, by an elementary computation, shows that the condition (2.9) takes the form

$$\begin{aligned}\lambda &= \frac{r}{\sqrt{\mu}}\sqrt{\sum_{n=1}^{\infty}(n-1)r^{2(n-1)} + (1-\mu)\sum_{n=1}^{\infty}r^{2(n-1)}}\\ &= \frac{r\sqrt{r^2 + (1-\mu)(1-r^2)}}{(1-r^2)\sqrt{\mu}}.\end{aligned}$$

Further, the inequality (2.8) is equivalent to

$$r\left[\mu|a_2| + \sqrt{\mu - |a_2|(1-\mu)}\left(\sum_{n=2}^{\infty}\frac{r^{2(n-1)}}{n-\mu}\right)^{1/2}\right] \leq 1.$$

To simplify the last expression in compact form, we may express

$$\begin{aligned}\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{r^{2(n-1)}}{(n/\mu)-1} &= \frac{1}{\mu}\sum_{n=1}^{\infty}r^{2(n-1)}\int_0^1 t^{(n/\mu)-2}\,\mathrm{d}t\\ &= \frac{1}{\mu}\int_0^1 \frac{\sum_{n=1}^{\infty}r^{2n}(t^{1/\mu})^n}{r^2t^2}\,\mathrm{d}t\\ &= \frac{1}{\mu}\int_0^1 \frac{t^{(1/\mu)-2}}{1-r^2t^{1/\mu}}\,\mathrm{d}t\\ &= \int_0^1 \frac{T^{-\mu}}{1-r^2T}\,\mathrm{d}T\\ &= \frac{1}{1-\mu}F(1, 1-\mu; 2-\mu; r^2)\end{aligned}$$

so that

$$\sum_{n=2}^{\infty}\frac{r^{2(n-1)}}{n-\mu}=\frac{1}{1-\mu}F(1,1-\mu;2-\mu;r^2)-\frac{1}{1-\mu}.$$

The above discussion leads to the following.

COROLLARY 5

Let $f\in\mathcal{S}$ *with* $a_2=f''(0)/2$, $0<\mu\le 1$, $\lambda>0$ *and* $0<r<1$ *satisfy the conditions*

$$\lambda=\frac{r\sqrt{r^2+(1-\mu)(1-r^2)}}{(1-r^2)\sqrt{\mu}}$$

and

$$\mu[1-\mu|a_2|^2(1-\mu)]\frac{F(1,1-\mu;2-\mu;r^2)-1}{1-\mu}\le\left(\frac{1}{r}-\mu|a_2|\right)^2.$$

(*If* $\mu=1$, *the last inequality may be treated as a limiting case.*) *Then the function*

$$H_f(z)=\frac{z}{\left(\left(\frac{z}{f(z)}\right)^{\mu}*\frac{1}{1-az}\right)^{1/\mu}}$$

belongs to $\mathcal{U}(\lambda,\mu)$. *Also, we have*

(1) $H_f(z)\in\mathcal{S}^*$ *if* $\lambda\le\frac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}$ *and* $0<\mu<1$.

(2) $H_f(z)\in\mathcal{S}^*$ *if* $\lambda\le\frac{-|a_2|+\sqrt{2-|a_2|^2}}{2}$ *and* $\mu=1$.

For our next result, we consider $h(z)=1+\sum_{k=1}^{n}r^kz^k$ so that $c_k=r^k$ for $1\le k\le n$ and $c_k=0$ for $k>n$. Then

$$\left(\frac{z}{H_f(z)}\right)^{\mu}=\left(\frac{z}{f(z)}\right)^{\mu}*h(z)=s_n((rz/f(rz))^{\mu}),$$

where

$$s_n(g(z))=1+\sum_{k=1}^{n}\frac{g^{(k)}(0)}{k!}z^k$$

represents partial sum of the power series representation of $g(z)$. Further, to use (2.9), we compute

$$\mu\lambda^2=\sum_{k=1}^{\infty}(k-\mu)|c_k|^2=\sum_{k=1}^{n}kr^{2k}-\mu\sum_{k=1}^{n}r^{2k}$$

and it is a simple exercise to see that

$$\lambda\sqrt{\mu}=r\frac{\sqrt{1-\mu+\mu r^2-(n+1-\mu)r^{2n}+(n-\mu)r^{2n+2}}}{1-r^2}. \tag{2.10}$$

Again the condition (2.8) is easily seen to be equivalent to

$$\sqrt{(\mu - \mu^2|a_2|^2(1-\mu))r^{2\mu} \int_0^{r^2} \frac{t^{1-\mu} - t^{n-\mu}}{1-t}\, dt} \leq 1 - \mu r|a_2|. \tag{2.11}$$

Thus, we have the following result.

COROLLARY 6

Suppose that $f \in \mathcal{S}, 0 < \mu \leq 1$ *and* $\lambda > 0$ *are related by* (2.10) *and* (2.11). *Then, for* $r \in (0, 1)$, *each partial sum* $s_n((rz/f(rz))^\mu)$ *belongs to* $\mathcal{U}(\lambda, \mu)$.

We have an interesting observation from Theorem 5 and Corollary 6.

Example 2. Consider $h(z) = 1 + rz + r^2z^2 + r^3z^3$, $r > 0$ and $f \in \mathcal{S}$ is of the form $z/f(z) = 1 + \sum_{n=2}^\infty b_n z^n$ so that H_f takes the form

$$H_f(z) = \frac{z}{1 + b_2r^2z^2 + b_3r^3z^3}.$$

A direct calculation shows that the conditions (2.8) and (2.9) are satisfied (since $\mu = 1$, $f''(0) = 0$, $c_k = r^k$ for $1 \leq k \leq 3$, $c_k = 0$ for $k \geq 4$) with $\lambda = r^2\sqrt{1+2r^2}$ and

$$r^2\sqrt{2 + r^2} \leq \sqrt{2}, \quad \text{i.e. if } r \leq r_0 = 0.9161.$$

Consequently, we obtain that $H_f \in \mathcal{U}(\lambda)$. In particular, we have

(1) $H_f \in \mathcal{U}(1)$ for $r \leq r_1 \approx 0.8107$, where r_1 is the unique positive root of the equation

$$1 = r^2\sqrt{1+2r^2}.$$

(2) $H_f \in \mathcal{U}(1/\sqrt{2}) \subset \mathcal{S}^*$ for $r \leq r_2 \approx 0.7071$, where r_2 is the unique positive root of

$$1 = \sqrt{2}r^2\sqrt{1+2r^2}.$$

■

To state our next result, consider $c_1 = 0$ and for $n \geq 2$, $c_n = 1/n^{s/2}$ for $s > 2$. Then, we see that

$$\sum_{n=2}^\infty (n-\mu)|c_n|^2 = \zeta(s-1) - 1 - \mu(\zeta(s) - 1) = \mu$$

whenever $\mu = (\zeta(s-1) - 1)/\zeta(s)$ and so, (2.9) holds with $\lambda = 1$. Here $\zeta(s)$ denotes the usual zeta function. Further, we find that for $s > 2$,

$$\begin{aligned} \mu(1 - \mu|a_2|^2(1-\mu)) \sum_{n=2}^\infty \frac{1}{(n-\mu)n^s} &\leq \frac{\mu(1-\mu|a_2|^2(1-\mu))}{2-\mu}(\zeta(s) - 1) \\ &< \frac{\mu}{2-\mu} \quad (\text{since } 0 < \zeta(s) - 1 < 1) \\ &< 1 \quad (\text{since } \mu \leq 1/2) \end{aligned}$$

which shows that the condition (2.8) is satisfied. Moreover, a computation shows that $\mu = (\zeta(s-1) - 1)/\zeta(s) \leq 1/2$ holds if $s > 3.0557$ (approx.) Thus, Theorem 5 and Lemma B give the following result.

COROLLARY 7

For $s > 3.0557$, *let* $\mu = (\zeta(s-1)-1)/\zeta(s)$. *Let* $h(z) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{s/2}} z^n$ *and* $f \in \mathcal{S}$. *Then*, $H_f(z)$ *defined by* (2.7) *belongs to* $\mathcal{U}(1, \mu)$. *In particular*, H_f *is univalent* (*in fact spiral-like*).

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