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Real moments of the restrictive factor

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Abstract. Let λ be a real number such that $0 < \lambda < 1$. We establish asymptotic formulas for the weighted real moments $\sum_{n \le x} R^{\lambda}(n)(1 - n/x)$, where $R(n) = \prod_{\nu=1}^{k} p_{\nu}^{\alpha_{\nu}-1}$ is the Atanassov strong restrictive factor function and $n = \prod_{\nu=1}^{k} p_{\nu}^{\alpha_{\nu}}$ is the prime factorization of n.

Keywords. Strong restrictive factor; arithmetic functions; Dirichlet series; Riemann zeta-function.

0. Introduction

In [2] and [3], Atanassov introduced two interesting arithmetic functions,

$$I(n) = \prod_{\nu=1}^{k} \sqrt[\alpha_{\nu}]{p_{\nu}}$$

and

$$R(n) = \prod_{\nu=1}^{k} p_{\nu}^{\alpha_{\nu}-1},$$

where $n = \prod_{\nu=1}^{k} p_{\nu}^{\alpha_{\nu}}$ is the prime factorization of n. Atanassov called $I(n)$ the irrational factor function and $R(n)$ the strong restrictive factor function. These functions are multiplicative and satisfy the inequality

$$I(n)R^2(n) \ge n,$$

with equality exactly when n is square-free. In [4], Panaitopol showed that

$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} < e^2$$

and that

$$G(n) = \prod_{\nu=1}^{n} \sqrt[n]{I(\nu)}$$

satisfies the inequalities

$$e^{-7}n < G(n) < n,$$

for each $n \geq 1$. He then asked whether the sequence $\{G(n)n^{-1}\}_{n\geq 1}$ is convergent. In [1], Alkan *et al.* proved that there exists a positive absolute constant c_1 such that, as n tends to infinity,

$$G(n) = c_1 n + O(\sqrt{n}).$$

In the present paper, we study the strong restrictive factor function $R(n)$. We establish asymptotic formulas for some weighted real moments of the function $R(n)$. To be more precise, we prove the following result.

Theorem. *Let λ be a real number such that $0 < \lambda < 1$, and let $\epsilon > 0$. Also, define*

$$R^{\lambda}(n) = \prod_{\nu=1}^{k} p_{\nu}^{\lambda(\alpha_{\nu}-1)},$$

where $n = \prod_{\nu=1}^{k} p_{\nu}^{\alpha_{\nu}}$ is the prime factorization of n. Then there exists a positive absolute constant c_2 such that for all large x,

$$\sum_{n\leq x} R^{\lambda}(n)\left(1-\frac{n}{x}\right)$$

$$= \frac{\zeta(2-\lambda)}{2\zeta(2)} \prod_{p \text{ prime}} \frac{\left(1+\frac{1}{p-p^{\lambda}}\right)\left(1-\frac{1}{p^{2-\lambda}}\right)}{1+\frac{1}{p}} x$$

$$+ \frac{2\zeta\left(\frac{1+\lambda}{2}\right)}{(\lambda^2+4\lambda+3)\zeta(1+\lambda)} \prod_{p \text{ prime}} \frac{\left(1+\frac{1}{p^{\frac{1+\lambda}{2}}-p^{\lambda}}\right)\left(1-\frac{1}{p}\right)}{1+\frac{1}{p^{\frac{1+\lambda}{2}}}} x^{\frac{1+\lambda}{2}}$$

$$+ \begin{cases} O_{\lambda}(x^{\frac{1}{2}}\exp(-c_2(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}})), & \text{if } 0<\lambda<1/4, \\ O_{\lambda,\epsilon}(x^{\frac{1+2\lambda}{3}+\epsilon}), & \text{if } 1/4 \leq \lambda < 1, \end{cases}$$

where $\zeta(s)$ denotes the Riemann zeta-function.

1. Proof of the theorem

The function $R^{\lambda}(n)$ satisfies the inequalities

$$R^{\lambda}(n) \leq R(n) \leq n,$$

for all $n \geq 1$. Since it is multiplicative, the associated Dirichlet series satisfies

$$\sum_{n=1}^{\infty} \frac{R^{\lambda}(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{p^{\lambda}}{p^{2s}} + \frac{p^{2\lambda}}{p^{3s}} + \cdots\right)$$

$$= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} \cdot \frac{1}{1 - \frac{1}{p^{s-\lambda}}}\right)$$

$$= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s}\right) \prod_{p \text{ prime}} \frac{1 + \frac{1}{p^s - p^{\lambda}}}{1 + \frac{1}{p^s}}$$

$$= \frac{\zeta(s)\zeta(2s-\lambda)K_{\lambda}(s)}{\zeta(2s)},$$

where

$$K_{\lambda}(s) = \prod_{p \text{ prime}} (1 + A_{\lambda,p}(s)),$$

and $A_{\lambda,p}(s)$ is obtained as follows. We have

$$1 + A_{\lambda,p}(s) = \frac{\left(1 + \frac{1}{p^s - p^{\lambda}}\right)\left(1 - \frac{1}{p^{2s-\lambda}}\right)}{1 + \frac{1}{p^s}},$$

where

$$\frac{1 + \frac{1}{p^s - p^{\lambda}}}{1 + \frac{1}{p^s}} = \frac{1 + \frac{1}{p^s}\left(1 + \frac{1}{p^{s-\lambda}} + O\left(\frac{1}{|p^{2(s-\lambda)}|}\right)\right)}{1 + \frac{1}{p^s}}$$

$$= 1 + \frac{1}{p^{2s-\lambda}\left(1 + \frac{1}{p^s}\right)} + O\left(\frac{1}{p^{3\sigma - 2\lambda}}\right),$$

for $\mathrm{Re}(s) = \sigma > \lambda > 0$. Since $|1/p^s| < 1/2^{\lambda} < 1$, we have $|1 + 1/p^s| > 1 - 1/2^{\lambda}$. Hence,

$$1 + A_{\lambda,p}(s) = 1 + \frac{1}{p^{2s-\lambda}\left(1 + \frac{1}{p^s}\right)} - \frac{1}{p^{2s-\lambda}} - \frac{1}{p^{4s-2\lambda}\left(1 + \frac{1}{p^s}\right)}$$

$$+ O\left(\frac{1}{p^{3\sigma - 2\lambda}}\right).$$

Here we note that

$$\frac{1}{p^{4s-2\lambda}\left(1+\dfrac{1}{p^s}\right)} \le \frac{1}{p^{4\sigma-2\lambda}\left(1-\dfrac{1}{2^\lambda}\right)}$$

and

$$|p^s+1| \ge p^\sigma - 1 \ge p^\sigma\left(1-\frac{1}{2^\lambda}\right),$$

since $\sigma > \lambda$. Consequently,

$$A_{\lambda,p}(s) = \frac{1}{p^{2s-\lambda}}\left(\frac{1}{1+\dfrac{1}{p^s}} - 1\right) + O\left(\frac{1}{p^{3\sigma-2\lambda}}\right),$$

from which

$$|A_{\lambda,p}(s)| \ll_\lambda \frac{1}{p^{3\sigma-2\lambda}}.$$

Then summing this over primes p, we obtain

$$\sum_{p \text{ prime}} |A_{\lambda,p}(s)| \ll_\lambda \sum_{p \text{ prime}} \frac{1}{p^{3\sigma-2\lambda}} \ll_\lambda 1.$$

Therefore, the product $K_\lambda(s)$ is uniformly bounded on any half-plane $\mathrm{Re}(s) \ge \sigma_0 > (1+2\lambda)/3$ (see §1.42, p. 15 of [8]).

Let us now fix a real number λ with $0 < \lambda < 1$. We employ a variant of Perron's formula. Precisely, using the line integral

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s = \max(x-1,0), \quad x > 0,$$

we write

$$\sum_{n\le x} R^\lambda(n)\left(1-\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^s\zeta(s)\zeta(2s-\lambda)K_\lambda(s)}{s(s+1)\zeta(2s)}\,\mathrm{d}s. \tag{1.1}$$

There are two cases here: $0 < \lambda < 1/4$ and $1/4 \le \lambda < 1$.

We treat first the case $1/4 \le \lambda < 1$. We fix an $\epsilon > 0$ and a real number β such that $(1+2\lambda)/3 < \beta < (1+2\lambda)/3 + \epsilon$. In addition, we fix $T \le x^2$, let $\alpha = 1 + c_3(\log x)^{-1}$, and then deform the line of integration into the path which consists of the union of line segments

$$\begin{cases} s = \alpha + it, & \text{if } |t| \ge T, \\ s = \sigma \pm iT, & \text{if } \beta \le \sigma \le \alpha, \\ s = \beta + it, & \text{if } |t| \le T. \end{cases}$$

The integrand is analytic on and within this modified contour, except at the points $s = (1+\lambda)/2$ and $s = 1$, where it has simple poles. By Cauchy's theorem, we have

$$\sum_{n\le x} R(n)^\lambda \left(1 - \frac{n}{x}\right) = \frac{\zeta(2-\lambda)K_\lambda(1)}{2\zeta(2)}x + \frac{2\zeta\left(\frac{1+\lambda}{2}\right)K_\lambda\left(\frac{1+\lambda}{2}\right)}{(\lambda^2+4\lambda+3)\zeta(1+\lambda)}x^{\frac{1+\lambda}{2}} + J_1^- + J_2^- + J_3 + J_2^+ + J_1^+, \tag{1.2}$$

the main contributions being due to the residues of the simple poles.

Along the line segments on which $\sigma = \alpha + it$, with $|t| \ge T$, we have $|\zeta(\alpha+it)| \ll \log x$, $|\zeta(2\alpha - \lambda + i2t)| \ll \log x$, and $1/|\zeta(2\alpha + i2t)| \ll 1$, so that

$$|J_1^\pm| \ll_{\lambda,\sigma_0} x(\log x)^2 \int_T^\infty \frac{\mathrm{d}t}{t^2} \ll_{\lambda,\sigma_0} \frac{x(\log x)^2}{T}. \tag{1.3}$$

Along the line segments on which $s = \sigma \pm iT$, with $\beta \le \sigma \le \alpha$, we have $|\zeta(\sigma+iT)| \ll T^{(1-\sigma)/2}\log T$ if $\beta \le \sigma \le 1$, $|\zeta(\sigma+iT)| \ll \log T$ if $1 \le \sigma \le \alpha$, $|\zeta(2\sigma - \lambda + i2T)| \ll T^{(1-2\sigma+\lambda)/2}\log T$ if $\beta \le \sigma \le 1$, $|\zeta(2\sigma-\lambda+i2T)| \ll \log T$ if $1 \le \sigma \le \alpha$ and $1/|\zeta(2\sigma + i2T)| \ll (\log T)^{2/3}(\log\log T)^{1/3}$, so that

$$\begin{aligned}|J_2^\pm| &\ll_{\lambda,\sigma_0} \frac{(\log T)^{\frac{8}{3}}(\log\log T)^{\frac{1}{3}}}{T}\left(\int_\beta^1 \left(\frac{x}{T}\right)^\sigma \mathrm{d}\sigma + \frac{1}{T}\int_1^\alpha x^\sigma\,\mathrm{d}\sigma\right)\\ &\ll_{\lambda,\sigma_0} \frac{x(\log T)^{\frac{8}{3}}(\log\log T)^{\frac{1}{3}}}{T^2}.\end{aligned} \tag{1.4}$$

Along the line $s = \beta + it$, with $|t| \le T$, we have $|\zeta(\beta+it)| \ll |t|^{(1-\beta)/2}\log(|t|+2)$, $|\zeta(2\beta - \lambda + i2t)| \ll |t|^{(1-2\beta+\lambda)/2}\log(|t|+2)$ and $1/|\zeta(2\beta+i2t)| \ll (\log(|t|+2))^{2/3}(\log\log(|t|+3))^{1/3}$, so that

$$\begin{aligned}|J_3| &\ll_{\lambda,\sigma_0} x^\beta \int_{-T}^T \frac{(|t|^{\frac{1-\beta}{2}+\frac{1-2\beta+\lambda}{2}}\log(|t|+2))^2(\log(|t|+2))^{\frac{2}{3}}(\log\log(|t|+3))^{\frac{1}{3}}}{1+t^2}\,\mathrm{d}t\\ &\ll_{\lambda,\sigma_0} x^\beta,\end{aligned} \tag{1.5}$$

since $(1-\beta)/2 + (1-2\beta+\lambda)/2 < 1$.

Then, by our choice of $T = x$ and by virtue of (1.1) and (1.2) and estimates (1.3) through (1.5), we get the desired result for $1/4 \le \lambda < 1$.

We treat next the case $0 < \lambda < 1/4$. To do this, we make use of the Vinogradov–Korobov type zero-free region (see p. 135 of [9])

$$\sigma \ge 1 - c_4(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}, \quad t \ge t_0,$$

in which

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}.$$

Note that, in this case, the Dirichlet series $\sum_{n=1}^{\infty} R^{\lambda}(n)n^{-s}$ has meromorphic continuation to the half-plane $\mathrm{Re}(s) > (1+2\lambda)/3$. We fix T, $U > 0$ such that $T < U < x^2$. We then let $\alpha = 1 + c_4(\log x)^{-1}$ and $\beta = 1/2 - c_5(\log 2T)^{-2/3}(\log\log 2T)^{-1/3}$, where $c_5 = c_4/2$, so that for large T, we have $\beta \geq \sigma_0$ with $(1+2\lambda)/3 < \sigma_0 < 1/2$. The line of integration may be deformed into a path which consists of the union of line segments

$$\begin{cases} s = \alpha + it, & \text{if } |t| \geq U, \\ s = \sigma \pm iU, & \text{if } 1/2 \leq \sigma \leq \alpha, \\ s = 1/2 + it, & \text{if } T \leq |t| \leq U, \\ s = \sigma \pm iT, & \text{if } \beta \leq \sigma \leq 1/2, \\ s = \beta + it, & \text{if } |t| \leq T. \end{cases}$$

Then the integrand is analytic on and within this modified contour, except for the simple poles at the points $s = (1+\lambda)/2$ and $s = 1$. Thus, we have

$$\sum_{n \leq x} R^{\lambda}(n)\left(1 - \frac{n}{x}\right) = \frac{\zeta(2-\lambda)K_{\lambda}(1)}{2\zeta(2)}x + \frac{4\zeta\left(\frac{1+\lambda}{2}\right)K_{\lambda}\left(\frac{1+\lambda}{2}\right)}{(\lambda^2+4\lambda+3)\zeta(1+\lambda)}x^{\frac{1+\lambda}{2}}$$
$$+ J_1^- + J_2^- + J_3^- + J_4^- + J_5 + J_4^+ + J_3^+ + J_2^+ + J_1^+. \qquad (1.6)$$

To estimate the integrals on the right side above, we apply the well-known estimates, for $t \geq t_0 > 0$ uniformly in σ,

$$|\zeta(\sigma + it)| \ll \begin{cases} t^{\frac{1-\sigma}{2}} \log t, & \text{if } 0 \leq \sigma \leq 1, \\ \log t, & \text{if } 1 \leq \sigma \leq 2, \\ 1, & \text{if } \sigma \geq 2. \end{cases}$$

Along the line segments on which $\sigma = \alpha + it$, with $|t| \geq U$, we have $|\zeta(\alpha + it)| \ll \log x$, $|\zeta(2\alpha - \lambda + i2t)| \ll \log x$, and $1/|\zeta(2\alpha + i2t)| \ll 1$, so that

$$|J_1^{\pm}| \ll_{\lambda,\sigma_0} x(\log x)^2 \int_U^{\infty} \frac{\mathrm{d}t}{t^2} \ll_{\lambda,\sigma_0} \frac{x(\log x)^2}{U}. \qquad (1.7)$$

Along the line segments on which $s = \sigma \pm iU$, with $1/2 \leq \sigma \leq \alpha$, we have $|\zeta(\sigma + iU)| \ll U^{(1-\sigma)/2}\log U$ if $1/2 \leq \sigma \leq 1$, $|\zeta(\sigma + iU)| \ll \log U$ if $1 \leq \sigma \leq 2$, $|\zeta(2\sigma - \lambda + i2U)| \ll U^{(1-2\sigma+\lambda)/2}\log U$ if $\lambda/2 \leq \sigma \leq (1+\lambda)/2$, $|\zeta(2\sigma - \lambda + i2U)| \ll \log U$ if $(1+\lambda)/2 \leq \sigma \leq (\alpha+\lambda)/2$ and $1/|\zeta(2\sigma + i2U)| \ll \log U \ll \log x$, so that

$$|J_2^{\pm}| \ll_{\lambda,\sigma_0} \log x \left(\int_{\frac{1}{2}}^{1} \frac{x^{\sigma}(U^{\frac{1-2\sigma+\lambda}{2}}\log U)^2}{U^2}\,\mathrm{d}\sigma + \int_{\frac{1}{2}}^{\alpha} \frac{x^{\sigma}(\log U)^2}{U^2}\,\mathrm{d}\sigma \right)$$
$$\ll_{\lambda,\sigma_0} \frac{x(\log x)^3}{U^2}. \qquad (1.8)$$

Along the line segments on which $s = 1/2 + it$, with $T \le |t| \le U$, we have $|\zeta(1-\lambda+i2t)| \ll |t|^{(1-(1-\lambda))/2}\log|t|$ and $1/|\zeta(1+i2t)| \ll \log|t| \ll \log U$, so that by Ramachandra's mean-value formula

$$\int_X^{2X} \left|\zeta\left(\frac{1}{2}+it\right)\right| \mathrm{d}t = O(X(\log X)^{\frac{1}{4}})$$

(see [5] and [6]; also, see Theorem 2, p. 146 of [7]), we have

$$|J_3^{\pm}| \ll_{\lambda,\sigma_0} x^{\frac{1}{2}}(\log U)^2 \int_T^U \frac{|\zeta(1/2+it)||\zeta(1-\lambda+i2t)|}{t^2}\,\mathrm{d}t$$

$$\ll_{\lambda,\sigma_0} x^{\frac{1}{2}}(\log x)^2 \sum_{\frac{T}{2}\le 2^k\le U} \frac{k^{\frac{1}{4}}}{(2^{1-\frac{\lambda}{2}})^k} \ll_{\lambda,\sigma_0} \frac{x^{\frac{1}{2}}(\log x)^2(\log T)^{\frac{1}{4}}}{T^{\frac{7}{8}}}. \quad (1.9)$$

Along the line segments on which $s = \sigma \pm iT$, with $\beta \le \sigma \le 1/2$, we have $|\zeta(\sigma+iT)| \ll T^{(1-\beta)/2}\log T$, $|\zeta(2\sigma-\lambda+i2T)| \ll T^{(1-2\sigma+\lambda)/2}\log T \ll T^{(1+\lambda-2\beta)/2}\log T$, and $1/|\zeta(2\sigma+i2T)| \ll (\log T)^{2/3}(\log\log T)^{1/3}$, so that

$$|J_4^{\pm}| \ll_{\lambda,\sigma_0} \int_\beta^{\frac{1}{2}} \frac{x^\sigma(T^{\frac{1-\sigma}{2}}\log T)^2(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}{T^2}\,\mathrm{d}\sigma$$

$$\ll_{\lambda,\sigma_0} \frac{x^{\frac{1}{2}}(\log T)^2}{T^{1-\frac{\lambda}{2}+\frac{3\beta}{2}}}. \quad (1.10)$$

Along the line segment on which $s = \beta + it$, with $|t| \le T$, we have $|\zeta(\beta+it)| \ll |t|^{(1-\beta)/2}\log(|t|+2)$, $|\zeta(2\beta-\lambda+i2t)| \ll |t|^{(1-2\beta+\lambda)/2}\log(|t|+2)$, and $1/|\zeta(2\beta+i2t)| \ll \log(|t|+2)^{2/3}(\log\log(|t|+3))^{1/3}$, so that

$$|J_5| \ll_{\lambda,\sigma_0} x^\beta \int_{-T}^{T} \frac{(|t|^{\frac{1-\beta}{2}+\frac{1-2\beta+\lambda}{2}}\log(|t|+2))^2(\log(|t|+2))^{\frac{2}{3}}(\log\log(|t|+3))^{\frac{1}{3}}}{1+t^2}\,\mathrm{d}t$$

$$\ll_{\lambda,\sigma_0} x^\beta, \quad (1.11)$$

since $(1-\beta)/2 + (1-2\beta+\lambda)/2 < 1$.

Finally, by our choices of $T = \exp(c_6(\log x)^{3/5}(\log\log x)^{-1/5})$ and $U = x$, and by virtue of (1.1) and (1.6) and estimates (1.7) through (1.11), we obtain the required result for $0 < \lambda < 1/4$. This completes the proof of the theorem.

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