Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 4, September 2009, pp. 521–529.

Weighted composition operators between different Bergman spaces of bounded symmetric domains

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MS received 3 November 2007; revised 26 June 2009

Abstract. In this paper, we consider the boundedness and compactness of the weighted composition operators between different Bergman spaces of bounded symmetric domains in terms of the Carleson measure. As an application, we study the multipliers between different Bergman spaces.

Keywords. Weighted composition operator; Bergman space; bounded symmetric domains; Carleson measure.

1. Introduction

Let Ω be a bounded symmetric domain in $\mathbf{C}^n$ with Bergman kernel $K(z, w)$. We assume that Ω is in its Harish-Chandra realization and the volume measure $\mathrm{d}v$ of Ω is normalized so that $K(z, 0) = K(0, w) = 1$ for all z and w in Ω. Define the Bergman matrix of Ω to be

$$G_z = (g_{ij}(z)) = \frac{1}{2}\left(\frac{\partial^2}{\partial z_i \partial z_j} \log K(z, z)\right), \quad 1 \le i, j \le n.$$

For γ: $[0, 1] \to \Omega$, a piecewise smooth C^1 curve, we set

$$l(\gamma) = \int_0^1 \left(\sum_{i,j=1}^n g_{ij}(\gamma(t))\gamma_i^{'}(t)\overline{\gamma_j^{'}(t)}\right)^{\frac{1}{2}} \mathrm{d}t.$$

The Bergman distance function on Ω is defined as

$$d(z, w) = \inf\{l(\gamma)\colon \gamma\colon [0, 1] \to \Omega, \gamma(0) = z, \gamma(1) = w\}.$$

For $z \in \Omega$ and $r > 0$, introduce the ball $E(z, r) = \{w \in \Omega\colon d(z, w) < r\}$, and denote $|E(z, r)|$ the normalized volume of $E(z, r)$, that is, $|E(z, r)| = \int_{E(z,r)} \mathrm{d}v(w)$.

Let $H(\Omega)$ be the family of all holomorphic functions on Ω, set $\mathrm{d}v_\alpha(z) = K(z, z)^{1-\alpha}$ $\mathrm{d}v(z)$, where $\alpha > \frac{N-1}{N}$, N is the genus of Ω, then $\mathrm{d}v_\alpha$ is a finite measure on Ω (see [12]). For $0 < p < \infty$, the weighted Bergman space $L_a^p(\Omega, \mathrm{d}v_\alpha)$ is the space of all functions $f \in H(\Omega)$ for which

$$\|f\|_{p,\alpha} = \left(\int_\Omega |f(z)|^p \mathrm{d}v_\alpha(z)\right)^{\frac{1}{p}} < \infty.$$

For any $z \in \Omega$, set

$$k_z(w) = \frac{K(w,z)}{\sqrt{K(z,z)}}, \quad w \in \Omega,$$

Then k_z is the normalized reproducing kernel for $L_a^2(\Omega, \mathrm{d}v)$, and k_z^α is a unit vector of $L_a^2(\Omega, \mathrm{d}v_\alpha)$. For any $r > 0$, there exists some $C > 0$ such that for all $z \in \Omega$ and $w \in E(z,r)$,

$$C^{-1} \le |E(z,r)||k_z(w)|^2 \le C.$$

Taking $w = z$, we get

$$C^{-1} \le |E(z,r)|K(z,z) \le C, \quad z \in \Omega.$$

Furthermore, for any fixed $r > 0$, $s > 0$, $R > 0$, there exists some $C > 0$ (depending only on r, s, R) such that

$$C^{-1} \le \frac{|E(z,r)|}{|E(w,s)|} \le C,$$

for every $z, w \in \Omega$ with $d(z,w) \le R$ (see [2] for proofs).

Given $\varphi, \psi \in H(\Omega)$ and $\varphi(\Omega) \subseteq \Omega$, the weighted composition operator $W_{\varphi,\psi}$ is defined as

$$W_{\varphi,\psi} f(z) = \psi(z)(f \circ \varphi)(z) = \psi(z) f(\varphi(z)), \quad f \in H(\Omega), \quad z \in \Omega.$$

It is obvious that $W_{\varphi,\psi}$ is a linear operator, which is closely related to the composition operator and multiplier. The behavior of these two operators have been studied extensively on various spaces of holomorphic function (see [5]). It is natural to consider the boundedness and compactness of weighted composition operators on Bergman spaces. In the unit disk and the unit ball, the problem has been studied by many authors (see [3], [4], [9] and [11]). In the bounded symmetric domain, Luo and Shi considered the composition operator C_φ on Bergman spaces in [8] and [9]. Recently, Sanjay and Kanwar [10] studied the boundedness and compactness of weighted composition operators on weighted Bergman spaces. Relatively to these papers, our work is to obtain the sufficient and necessary conditions on φ, ψ such that the operator $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ is bounded (or compact) for all $0 < p, q < \infty$, $\alpha, \beta > \frac{N-1}{N}$, where $\varphi, \psi \in H(\Omega)$ with $\varphi(\Omega) \subseteq \Omega$. This will extend the results in [10].

In what follows we always suppose $\varphi, \psi \in H(\Omega)$ $\varphi(\Omega) \subseteq \Omega$ and $\alpha, \beta > \frac{N-1}{N}$. C will stand for a positive constant whose value may change from line to line but is independent of the functions in $H(\Omega)$. The expression $A \simeq B$ means $C^{-1}A \le B \le CA$.

2. Boundedness and compactness of $W_{\varphi,\psi}$

First, we collect a few preliminary results that will be needed later in this paper. We begin with a result on compact weighted composition operators. For any $0 < p, q < \infty$, the operator $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ is compact if and only if

$$\lim_{k\to\infty} \|W_{\varphi,\psi} f_k\|_{q,\beta} = 0$$

for any norm bounded sequence $\{f_k\} \subseteq L_a^p(\Omega, \mathrm{d}v_\alpha)$ that converges to 0 uniformly on any compact subsets of Ω.

Suppose $q > 0$, $\psi \in L_a^q(\Omega, \mathrm{d}v_\beta)$, define the nonnegative measure $\mu_{\varphi,\psi,q,\beta}$ to be

$$\mu_{\varphi,\psi,q,\beta}(E) = \int_{\varphi^{-1}(E)} |\psi|^q \mathrm{d}v_\beta,$$

here E is a measurable subset of Ω. Using Theorem C in p. 163 of [6], we have the following change of variables formula:

$$\int_\Omega g \mathrm{d}\mu_{\varphi,\psi,q,\beta} = \int_\Omega |\psi|^q (g \circ \varphi) \mathrm{d}v_\beta, \tag{2.1}$$

where g is an arbitrary measurable positive function on Ω.

Suppose μ is a finite positive Borel measure on Ω. It is said to be a δ-Carleson measure if

$$\sup_{z\in\Omega} \frac{\mu(E(z,r))}{|E(z,r)|^\delta} < \infty,$$

where $\mu(E(z,r)) = \int_{E(z,r)} \mathrm{d}\mu(w)$. Moreover, if

$$\lim_{z\to\partial\Omega} \frac{\mu(E(z,r))}{|E(z,r)|^\delta} = 0,$$

then μ is called a vanishing δ-Carleson measure. It is well-known that δ-Carleson measure plays an important role in weighted Bergman space. More precisely, the following result holds.

Lemma 2.1. *Let μ be a finite positive Borel measure on Ω, $0 < p \le q < \infty$. Then the following statements are equivalent*:

(1) *There exists some C such that for any $f \in L_a^p(\Omega, \mathrm{d}v_\alpha)$, we have*

$$\left(\int_\Omega |f(z)|^q \mathrm{d}\mu(z)\right)^{\frac{1}{q}} \le C \left(\int_\Omega |f(z)|^p \mathrm{d}v_\alpha(z)\right)^{\frac{1}{p}};$$

(2) *μ is a $\frac{\alpha q}{p}$-Carleson measure*;

(3) $\sup_{z\in\Omega} \int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu(w) < \infty$.

Proof. This easily follows from Theorem 2.1 in [7], just taking $\eta = \frac{q}{p}$.

Lemma 2.2 [8]. *Let μ be a finite positive Borel measure on Ω, $0 < q < p < \infty$. Then the following statements are equivalent*:

(1) *There exists some C such that for any $f \in L_a^p(\Omega, \mathrm{d}v_\alpha)$, we have*

$$\left(\int_\Omega |f(z)|^q \mathrm{d}\mu(z)\right)^{\frac{1}{q}} \le C \left(\int_\Omega |f(z)|^p \mathrm{d}v_\alpha(z)\right)^{\frac{1}{p}};$$

(2) $\frac{\mu(E(z,r))}{|E(z,r)|^\alpha} \in L^s(\Omega, \mathrm{d}v_\alpha)$, *where* $\frac{1}{s} + \frac{q}{p} = 1$, $L^s(\Omega, \mathrm{d}v_\alpha)$ *is the usual Lebesgue space.*

The following arguments give the characterizations of the boundedness and compactness of the weighted composition operator $W_{\varphi,\psi}$ from $L_a^p(\Omega, \mathrm{d}v_\alpha)$ to $L_a^q(\Omega, \mathrm{d}v_\beta)$.

Theorem 2.3. *Suppose* $0 < p \leq q < \infty$. *Then the following statements are equivalent*:

(1) $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ *is bounded*;
(2) $\mu_{\varphi,\psi,q,\beta}$ *is a* $\frac{\alpha q}{p}$*-Carleson measure*;
(3) $\sup_{z\in\Omega} \int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) < \infty$.

Proof.
(1)⇒(3). For $z \in \Omega$, set

$$g_z(w) = (k_z(w))^{\frac{2\alpha}{p}}, \quad w \in \Omega. \tag{2.2}$$

Then $g_z \in H(\Omega)$ and $\|g_z\|_{p,\alpha} \leq C$. By the boundedness of $W_{\varphi,\psi}$ and (2.1), we have

$$\begin{aligned}\int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) &= \int_\Omega |g_z(w)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) \\ &= \int_\Omega |\psi(w)|^q |(g_z \circ \varphi)(w)|^q \mathrm{d}v_\beta(w) \\ &= \|W_{\varphi,\psi} g_z\|_{q,\beta}^q \\ &\leq C.\end{aligned}$$

Hence,

$$\sup_{z\in\Omega} \int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) < \infty.$$

(3)⇒(2). The condition (3) implies

$$\begin{aligned}\frac{\mu_{\varphi,\psi,q,\beta}(E(z,r))}{|E(z,r)|^{\frac{\alpha q}{p}}} &= \frac{1}{|E(z,r)|^{\frac{\alpha q}{p}}} \int_{E(z,r)} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) \\ &\simeq \int_{E(z,r)} |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) \\ &\leq \int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w).\end{aligned}$$

Thus, $\mu_{\varphi,\psi,q,\beta}$ is a $\frac{\alpha q}{p}$-Carleson measure.
(2)⇒(1). Suppose $f \in L_a^p(\Omega, \mathrm{d}v_\alpha)$. Lemma 2.1 and (2) show

$$\begin{aligned}\|W_{\varphi,\psi} f\|_{q,\beta}^q &= \int_\Omega |\psi(w)|^q |(f \circ \varphi)(w)|^q \mathrm{d}v_\beta(w) \\ &= \int_\Omega |f(w)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) \\ &\leq C \int_\Omega |f(w)|^p \mathrm{d}v_\alpha(w).\end{aligned}$$

Hence, $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ is bounded. The proof is completed.

Theorem 2.4. *Suppose* $0 < p \le q < \infty$. *Then the following statements are equivalent*:

(1) $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ *is compact*;
(2) $\mu_{\varphi,\psi,q,\beta}$ *is a vanishing* $\frac{\alpha q}{p}$*-Carleson measure*;
(3) $\lim\limits_{z\to\partial\Omega} \int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) = 0$.

Proof.
(1)⇒(3). For $z \in \Omega$, define g_z as in (2.2). Then $g_z \in H(\Omega)$ and $g_z \to 0$ weakly in $L_a^p(\Omega, \mathrm{d}v_\alpha)$ as $z \to \partial\Omega$. By Lemma 3.2 in [10], we get

$$\begin{aligned}\int_\Omega |k_z(w)|^{\frac{2\alpha q}{p}} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) &= \int_\Omega |g_z(w)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w)\\ &= \int_\Omega |\psi(w)|^q |(g_z \circ \varphi)(w)|^q \mathrm{d}v_\beta(w)\\ &= \|W_{\varphi,\psi} g_z\|_{q,\beta}^q\\ &\to 0 \quad (z \to \partial\Omega).\end{aligned}$$

(3)⇒(2). This is similar to the proof of (3)⇒(2) in Theorem 2.3.
(2)⇒(1). Notice that, for any $r > 0$, by Lemma 5 in [1], we can choose a sequence $\{a_j\} \subseteq \Omega$ with $a_j \to \partial\Omega$ as $j \to \infty$ satisfying (i) $\Omega = \bigcup_{j=1}^\infty E(a_j, r)$; (ii) there is a positive integer N such that each point $z \in \Omega$ belongs to at most N of the sets $E(a_j, 2r)$. Then for any $\varepsilon > 0$, by (2), we get

$$\frac{\mu_{\varphi,\psi,q,\beta}(E(a_j, r))}{|E(a_j, r)|^{\frac{\alpha q}{p}}} < \varepsilon, \tag{2.3}$$

if j is sufficiently large. Suppose $\{f_k\}$ is any norm bounded sequence in $L_a^p(\Omega, \mathrm{d}v_\alpha)$ and $f_k \to 0$ uniformly on each compact subsets of Ω. We claim $\lim\limits_{k\to\infty} \|W_{\varphi,\psi} f_k\|_{q,\beta} = 0$. In fact,

$$\begin{aligned}|f(z)|^p &\le \frac{C}{|E(z, r)|} \int_{E(z,r)} |f(w)|^p \mathrm{d}v(w)\\ &= \frac{1}{|E(z, r)|} \int_{E(z,r)} |f(w)|^p \frac{\mathrm{d}v_\alpha(w)}{K(w, w)^{1-\alpha}}\\ &\simeq \frac{1}{|E(z, r)|} \int_{E(z,r)} |f(w)|^p |E(w, r)|^{1-\alpha} \mathrm{d}v_\alpha(w)\\ &\simeq \frac{1}{|E(z, r)|^\alpha} \int_{E(z,r)} |f(w)|^p \mathrm{d}v_\alpha(w),\end{aligned}$$

the first inequality follows from Lemma 7 in [1]. Then

$$\begin{aligned}
&\sup\left\{|f(z)|^p : z \in E(a,r)\right\} \\
&\quad \leq \sup\left\{\frac{C}{|E(z,r)|^\alpha}\int_{E(z,r)}|f(w)|^p \mathrm{d}v_\alpha(w) : z \in E(a,r)\right\} \\
&\quad \leq \frac{C}{|E(a,r)|^\alpha}\int_{E(a,2r)}|f(w)|^p \mathrm{d}v_\alpha(w).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|W_{\varphi,\psi} f_k\|_{q,\beta}^q &= \int_\Omega |\psi(z)|^q |(f_k\circ\varphi)(z)|^q \mathrm{d}v_\beta(z) \\
&= \int_\Omega |f_k(z)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(z) \\
&\leq \sum_{j=1}^{\infty}\int_{E(a_j,r)} |f_k(z)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(z) \\
&\leq \sum_{j=1}^{\infty}\frac{\mu_{\varphi,\psi,q,\beta}(E(a_j,r))}{|E(a_j,r)|^\alpha}\sup\left\{|f_k(z)|^q : z\in E(a_j,r)\right\} \\
&\leq \sum_{j=1}^{\infty}\frac{\mu_{\varphi,\psi,q,\beta}(E(a_j,r))}{|E(a_j,r)|^{\frac{\alpha q}{p}}}\left(\int_{E(a_j,2r)}|f_k(z)|^p \mathrm{d}v_\alpha(z)\right)^{\frac{q}{p}} \\
&= \left(\sum_{j=1}^{J_0}+\sum_{j=J_0}^{\infty}\right)\frac{\mu_{\varphi,\psi,q,\beta}(E(a_j,r))}{|E(a_j,r)|^{\frac{\alpha q}{p}}}\left(\int_{E(a_j,2r)}|f_k(z)|^p \mathrm{d}v_\alpha(z)\right)^{\frac{q}{p}} \\
&= I_1 + I_2.
\end{aligned}$$

On the one hand, when $1 \leq j \leq J_0$, $E(a_j, 2r)$ is a compact subset of Ω, if k is sufficiently large. Then

$$I_1 \leq C\varepsilon^q.$$

On the other hand, (2.3) yields

$$I_2 \leq CN\varepsilon\left(\int_\Omega |f_k(z)|^p \mathrm{d}v_\alpha(z)\right)^{\frac{q}{p}} \leq CN\varepsilon\|f_k\|_{p,\alpha}^q \leq C\varepsilon.$$

Therefore, $\lim_{k\to\infty}\|W_{\varphi,\psi} f_k\|_{q,\beta} = 0$. The proof is completed.

When $p = q$, Theorems 2.3 and 2.4 is just the main results in [10].

Theorem 2.5. *Suppose* $0 < q < p < \infty$. *Then the following statements are equivalent*:

(1) $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ *is bounded*;
(2) $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ *is compact*;
(3) $\frac{\mu_{\varphi,\psi,q,\beta}(E(z,r))}{|E(z,r)|^\alpha} \in L^s(\Omega, \mathrm{d}v_\alpha)$, *where* $\frac{1}{s} + \frac{q}{p} = 1$.

Proof. The implication (2)⇒(1) is trivial.

(1)⇔(3). The operator $W_{\varphi,\psi}\colon L_a^p(\Omega, \mathrm{d}v_\alpha) \to L_a^q(\Omega, \mathrm{d}v_\beta)$ is bounded if and only if, there exists some C such that for any $f \in L_a^p(\Omega, \mathrm{d}v_\alpha)$,

$$\begin{aligned} \|W_{\varphi,\psi} f\|_{q,\beta}^q &= \int_\Omega |\psi(w)|^q |(f \circ \varphi)(w)|^q \mathrm{d}v_\beta(w) \\ &= \int_\Omega |f(w)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(w) \\ &\le C \int_\Omega |f(w)|^p \mathrm{d}v_\alpha(w). \end{aligned} \tag{2.4}$$

By Lemma 2.2, the necessary and sufficient condition for (2.4) is

$$\frac{\mu_{\varphi,\psi,q,\beta}(E(z,r))}{|E(z,r)|^\alpha} \in L^s(\Omega, \mathrm{d}v_\alpha),$$

where $\frac{1}{s} + \frac{q}{p} = 1$.

Now, we will show (3)⇒(2). Suppose $\{f_k\}$ is any norm bounded sequence in $L_a^p(\Omega, \mathrm{d}v_\alpha)$ and $f_k \to 0$ uniformly on each compact subsets of Ω. Since $\chi_{E(z,r)}(w) = \chi_{E(w,r)}(z)$, then

$$\begin{aligned} \|W_{\varphi,\psi} f_k\|_{q,\beta}^q &= \int_\Omega |\psi(z)|^q |(f_k \circ \varphi)(z)|^q \mathrm{d}v_\beta(z) \\ &= \int_\Omega |f_k(z)|^q \mathrm{d}\mu_{\varphi,\psi,q,\beta}(z) \\ &\le \int_\Omega \frac{1}{|E(z,r)|^\alpha} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(z) \int_{E(z,r)} |f_k(w)|^q \mathrm{d}v_\alpha(w) \\ &= \int_\Omega \frac{1}{|E(z,r)|^\alpha} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(z) \int_\Omega \chi_{E(z,r)}(w) |f_k(w)|^q \mathrm{d}v_\alpha(w) \\ &= \int_\Omega |f_k(w)|^q \mathrm{d}v_\alpha(w) \int_\Omega \frac{\chi_{E(w,r)}(z)}{|E(z,r)|^\alpha} \mathrm{d}\mu_{\varphi,\psi,q,\beta}(z) \\ &= \int_\Omega |f_k(w)|^q \mathrm{d}v_\alpha(w) \int_{E(w,r)} \frac{\mathrm{d}\mu_{\varphi,\psi,q,\beta}(z)}{|E(z,r)|^\alpha} \\ &\simeq \int_\Omega |f_k(w)|^q \frac{\mu_{\varphi,\psi,q,\beta}(E(w,r)}{|E(w,r)|^\alpha} \mathrm{d}v_\alpha(w) = I. \end{aligned}$$

For any $\varepsilon > 0$, (3) implies, there exists some $r \in (0, 1)$ such that

$$\int_{\Omega \backslash r\Omega} \left(\frac{\mu_{\varphi,\psi,q,\beta}(E(w,r))}{|E(w,r)|^{\alpha}} \right)^{s} \mathrm{d}v_{\alpha}(w) < \varepsilon^{sq}, \tag{2.5}$$

where $r\Omega = \{rz \colon z \in \Omega\}$ is a compact subset of Ω. Hence, using the Hölder inequality, we have

$$\begin{aligned} I &= \left(\int_{\Omega \backslash r\Omega} + \int_{r\Omega} \right) |f_k(w)|^q \frac{\mu_{\varphi,\psi,q,\beta}(E(w,r))}{|E(w,r)|^{\alpha}} \mathrm{d}v_{\alpha}(w) \\ &\leq \left(\int_{\Omega} |f_k(w)|^p \mathrm{d}v_{\alpha}(w) \right)^{\frac{q}{p}} \cdot \left(\int_{\Omega \backslash r\Omega} \left(\frac{\mu_{\varphi,\psi,q,\beta}(E(w,r))}{|E(w,r)|^{\alpha}} \right)^{s} \mathrm{d}v_{\alpha}(w) \right)^{\frac{1}{s}} \\ &\quad + \left(\int_{r\Omega} |f_k(w)|^p \mathrm{d}v_{\alpha}(w) \right)^{\frac{q}{p}} \cdot \left(\int_{\Omega} \left(\frac{\mu_{\varphi,\psi,q,\beta}(E(w,r))}{|E(w,r)|^{\alpha}} \right)^{s} \mathrm{d}v_{\alpha}(w) \right)^{\frac{1}{s}} \\ &< C\varepsilon^{q}, \end{aligned}$$

if k is sufficiently large. The last inequality is obtained by (2.5) and the fact $f_k \to 0$ uniformly on $r\Omega$. Thus, $\lim_{k\to\infty} \|W_{\varphi,\psi} f_k\|_{q,\beta} = 0$, which means that $W_{\varphi,\psi} \colon L_a^p(\Omega, \mathrm{d}v_{\alpha}) \to L_a^q(\Omega, \mathrm{d}v_{\beta})$ is compact.

3. Application

Let X and Y be two spaces of holomorphic function. We call ψ a pointwise multiplier from X to Y if $M_{\psi} f = \psi f \in Y$ for every $f \in X$. The collection of all point-wise multipliers from X to Y is denoted by $M(X, Y)$. Setting $\varphi(z) = z$, the weighted composition operator $W_{\varphi,\psi}$ is just the multiplication operator M_{ψ}. The multiplication operators have been studied by many authors. By the main results in §2, we can obtain the property of ψ such that $\psi \in M(L_a^p(\Omega, \mathrm{d}v_{\alpha}), L_a^q(\Omega, \mathrm{d}v_{\beta}))$.

Theorem 3.1. *Suppose* $0 < p \leq q < \infty$. *Then the following statements are equivalent*:

(1) $\psi \in M(L_a^p(\Omega, \mathrm{d}v_{\alpha}), L_a^q(\Omega, \mathrm{d}v_{\beta}))$;
(2) $\sup_{z\in\Omega} \frac{\int_{E(z,r)} |\psi(w)|^q \mathrm{d}v_{\beta}(w)}{|E(z,r)|^{\frac{\alpha q}{p}}} < \infty$;
(3) $\sup_{z\in\Omega} \int_{\Omega} |k_z(w)|^{\frac{2\alpha q}{p}} |\psi(w)|^q \mathrm{d}v_{\beta}(w) < \infty$.

Theorem 3.2. *Suppose* $0 < q < p < \infty$. *Then* $\psi \in M(L_a^p(\Omega, \mathrm{d}v_{\alpha}), L_a^q(\Omega, \mathrm{d}v_{\beta}))$ *if and only if*

$$\frac{\int_{E(z,r)} |\psi(w)|^q \mathrm{d}v_{\beta}(w)}{|E(z,r)|^{\alpha}} \in L^s(\Omega, \mathrm{d}v_{\alpha}),$$

where $\frac{1}{s} + \frac{q}{p} = 1$.

Set $\psi \equiv 1$. Then $W_{\varphi,\psi}$ is just the composition operator C_φ, and we can easily obtain the main theorems in [7] and [8] by the results in §2. Furthermore, we can obtain the results as follows.

Theorem 3.3. *Let* $0 < p, t < \infty$, *and let* C_φ *be bounded on* $L_a^p(\Omega, \mathrm{d}v_\alpha)$. *Then* C_φ *is bounded on* $L_a^t(\Omega, \mathrm{d}v_\alpha)$.

Theorem 3.4. *Let* $0 < p, t < \infty$, *and let* C_φ *be compact on* $L_a^p(\Omega, \mathrm{d}v_\alpha)$. *Then* C_φ *is compact on* $L_a^t(\Omega, \mathrm{d}v_\alpha)$.

Acknowledgement

The authors would like to express their thanks to Professor Zhangjian Hu and the referees, for the useful help. This project is supported by the National Natural Science Foundation of China (10771064), the National Natural Science Foundation of Zhejiang province (Y7080197, Y606197, D7080080) and the Foundation of Department of Education of Zhejiang province (20070482).

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