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Hypercyclicity of the adjoint of weighted composition operators

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Abstract. In the present paper we investigate the hypercyclicity of the adjoint of weighted composition operator in special function spaces.

Keywords. Weighted composition operator; hypercyclic operator; Denjoy–Wolff point; loxodromic map; Farrell–Rubel–Shields theorem.

1. Introduction

A bounded linear operator T on a Banach space X is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit

$$\mathrm{Orb}(T, x) = \{T^n x \colon n \in \mathbb{N}\}$$

is dense in X and in this case we refer to x as a hypercyclic vector for T. The hypercyclicity of composition operators on the Hardy space H^2 have been considered by Bourdon and Shapiro in [1,2,6]. Also in [8,9] we have investigated the hypercyclicity of the weighted composition operators on some function spaces. The holomorphic self-maps of the open unit disk U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U. It is well-known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$.

The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the grand iteration theorem (p. 78 of [6]). By $\psi'(w)$ we denote the angular derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi'(w)$ has the natural meaning of derivative. Also, by ψ_n we mean the n-th iteration of the function ψ.

PROPOSITION 1.1

Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism. If ψ has a fixed point $w \in U$, then $\psi_n \to w$ uniformly on compact subsets of U and $|\psi'(w)| < 1$.

Proof. See the grand iteration theorem in p. 78 of [6]. ■

The unique attracting point w in the above proposition is called the Denjoy–Wolff point of ψ. Also, a linear fractional transformation is called loxodromic if it has a fixed point in U and a fixed point outside $\bar{U}$ (p. 5 of [6]).

Consider the weighted composition operator $M_\varphi C_\psi$ on a Banach space X of analytic functions defined by

$$M_\varphi C_\psi f = \varphi \cdot f \circ \psi \quad (f \in X).$$

We will investigate the hypercyclicity of the operator $(M\varphi C\psi)^*$.

2. Main results

From now on, let X be a Banach space of analytic functions on the open unit disk U such that for each $\lambda \in U$, the evaluation function $e_\lambda\colon X \to C$ defined by $e_\lambda(f) = f(\lambda)$ is bounded on X. Furthermore, assume that X contains the constant functions and $zf \in X$ whenever $f \in X$. It follows immediately from this assumption and the closed graph theorem that the multiplication by the independent variable z defines a bounded linear operator M_z on X. The operator M_z is called *polynomially bounded* on X if there exists a constant $c > 0$ such that

$$\|M_p\| \leq c\|p\|_U$$

for every polynomial p. Here $\|p\|_U$ denotes the supremum norm of p on U. It is well-known that any operator which is similar to a contraction is polynomially bounded. Here we also suppose that M_z is polynomially bounded. A complex-valued function φ on U for which $\varphi X \subseteq X$ is called a multiplier of X. The set of all multipliers of X is denoted by $M(X)$ and it is well-known that $M(X) \subseteq H^\infty(U)$ [7].

It is well-known that if the adjoint of a continuous operator T on a Banach space has an eigenvector, then T^* fails to be hypercyclic [4].

Theorem 2.1. *Suppose that ψ is a holomorphic self-map of U such that C_ψ acts boundedly on X and the closure of $\psi(U)$ is contained in U, and $\psi(w) = w$. If φ is a nonzero multiplier of X and $\varphi(w) \neq 0$, then $\varphi(w)$ is an eigenvalue for the operator $M_\varphi C_\psi$ acting on X. Furthermore, $(M_\varphi C_\psi)^*$ is not hypercyclic.*

Proof. First, let $w = 0$. Since the closure of $\psi(U)$ is contained in U, there exists $0 < \lambda < 1$ such that $\psi(U) \subseteq \lambda U$. Also note that $\psi(0) = 0$, so by the Schwarz's lemma we have $|\psi(z)| \leq \lambda|z|$ for all z in U. On the other hand, since φ is bounded, an application of Schwartz's lemma shows that there exists a constant $M > 0$ such that

$$|\varphi(0) - \varphi(z)| < M|z|$$

for all z in U. But $\varphi(0) \neq 0$, thus

$$\left|1 - \frac{1}{\varphi(0)}\varphi(z)\right| < \frac{M}{|\varphi(0)|}|z| \qquad (z \in U).$$

By substituting $\psi_i(z)$ instead of z in the above inequality, we get

$$\begin{aligned}\left|1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))\right| &< \frac{M}{|\varphi(0)|}|\psi_i(z)| \\ &\leq \frac{M}{|\varphi(0)|}\lambda^i|z|.\end{aligned}$$

This implies that

$$\sum_{i=0}^{\infty} \left| 1 - \frac{1}{\varphi(0)} \varphi(\psi_i(z)) \right|$$

and consequently

$$\prod_{i=0}^{\infty} \frac{1}{\varphi(0)} \varphi(\psi_i(z))$$

converges uniformly on compact subsets of U. Set

$$g(z) = \prod_{i=0}^{\infty} \frac{1}{\varphi(0)} \varphi(\psi_i(z)),$$

then g is a nonzero holomorphic function on U. Also, note that

$$|\varphi(z)| < M|z| + |\varphi(0)|$$

for every $z \in U$. So it follows that if $z \in U$, then

$$\begin{aligned} |\varphi(\psi_n(z))| &< M|\psi_n(z)| + |\varphi(0)| \\ &\leq M\lambda^n |z| + |\varphi(0)|. \end{aligned}$$

But $\varphi(0) \neq 0$, thus

$$\begin{aligned} \frac{|\varphi(\psi_n(z))|}{|\varphi(0)|} &< M\lambda^n \frac{|z|}{|\varphi(0)|} + 1 \\ &\leq \exp\left(M\lambda^n \frac{|z|}{|\varphi(0)|} \right) \\ &\leq \exp\left(\frac{M}{|\varphi(0)|} \lambda^n \right). \end{aligned}$$

Therefore we get

$$\begin{aligned} \prod_{n=0}^{\infty} \left| \frac{1}{\varphi(0)} \varphi(\psi_n(z)) \right| &\leq \exp\left(\sum_{n=0}^{\infty} \frac{M}{|\varphi(0)|} \lambda^n \right) \\ &= \exp\left(\frac{M}{|\varphi(0)|} \frac{1}{1-\lambda} \right) \end{aligned}$$

for every $z \in U$. Hence indeed g belongs to $H^{\infty}(U)$. Now by the Farrell–Rubel–Shields theorem (Theorem 5.1, p. 151 of [3]) there is a sequence $\{p_n\}_n$ of polynomials converging to g such that for all n, $\|p_n\|_U \leq c_0$ for some $c_0 > 0$. So we obtain

$$\|M_{p_n}\| \leq c\|p_n\|_U \leq cc_0$$

for all n. But ball $B(X)$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(X)$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_n} \longrightarrow A^*$ in the weak operator topology and acting these operators on e_λ we obtain that

$$p_n(\lambda)e_\lambda = M^*_{p_n} e_\lambda \longrightarrow A^* e_\lambda$$

weakly. Since $p_n(\lambda) \longrightarrow g(\lambda)$ we see that $A^*e_\lambda = g(\lambda)e_\lambda$. Because the closed linear span of $\{e_\lambda\colon \lambda \in U\}$ is dense in X^*, we conclude that $A = M_g$ and this implies that $g \in M(X)$. Hence indeed $g \in X$, since X contains the constant functions. But easily one can see that $\varphi \cdot g \circ \psi = \varphi(0)g$ and so in the case $w = 0$, $\varphi(0)$ is an eigenvalue for the operator $M_\varphi C_\psi$. If $w \neq 0$, consider the self-maps $\Psi = \alpha_w \circ \psi \circ \alpha_w$ and $\Phi = \varphi \circ \alpha_w$ where

$$\alpha_w(z) = \frac{w - z}{1 - \bar{w}z}.$$

Clearly we can see that $\Psi(0) = 0$ and $\Phi(0) \neq 0$. Also, note that by the invariant Schwarz's lemma (p. 60 of [6]), $\alpha_w(\lambda U) \subseteq \lambda U$ which implies that the closure of $\Psi(U)$ is contained in U. So the first step of the proof shows that there exists a nonzero bounded analytic function G on U such that $\Phi \cdot G \circ \Psi = \Phi(0)G$. Thus we obtain

$$\varphi \circ \alpha_w \cdot G \circ (\alpha_w \circ \psi \circ \alpha_w)(z) = (\varphi \circ \alpha_w(0))G(z) = \varphi(w)G(z).$$

Note that $\alpha_w \circ \alpha_w(z) = z$. Substitute $\alpha_w(z)$ instead of z in the above equality. So we get $\varphi \cdot g \circ \psi = \varphi(w)g$ where $g = G \circ \alpha_w$ is a nonzero function in $H^\infty(U)$ and by the same proof of the first part, indeed $g \in M(X) \subseteq X$. Now since the operator $M_\varphi C_\psi$ has a non-zero eigenvalue, then $(M_\varphi C_\psi)^*$ fails to be hypercyclic. This completes the proof. ■

In Theorem 2.1 by some assumptions on ψ we investigated the hypercyclicity of the operator $(M_\varphi C_\psi)^*$ and in the following theorem by some assumptions on φ we investigate the hypercyclicity of the operator $(M_\varphi C_\psi)^*$.

Theorem 2.2. *Suppose φ is a nonzero multiplier of X and ψ is a loxodromic map of U that is not an elliptic authomorphism and C_ψ acts boundedly on X. Let $w \in U$ be a fixed point of ψ and $\varphi(w) \neq 0$. If φ can be extended to a bounded holomorphic map on $B(0, 1 + 2|w|)$, then $\varphi(w)$ is an eigenvalue for the operator $M_\varphi C_\psi$ acting on X, and so $(M_\varphi C_\psi)^*$ fails to be hypercyclic.*

Proof. Let w_0 be the another fixed point of ψ and first let $w_0 \neq \infty$. Note that w_0 is outside of $\bar{U}$ and the map

$$S(z) = (z - w)/N(z - w_0),$$

where $N(|w_0| - 1) > 2$, takes w to 0 and w_0 to ∞. Let $V = S(U)$, so the map S takes the unit disk U into itself and for all $z \in V$ we have $S \circ \psi \circ S^{-1}(z) = \lambda z$ for some complex number λ with $0 < |\lambda| < 1$ (see [6]). Note that by Proposition 1.1 indeed $|\lambda| = |\psi'(w)| < 1$. Define the holomorphic map f by the relation

$$f(z) = (z - w_0 + w)(\varphi(z + w) - \varphi(w)) \qquad (|z| < 1 + |w|).$$

Because $f(0) = 0$ and f is bounded, an application of the generalization of the Schwarz's lemma (p. 177 of [5]) shows that there exist some constant $M > 0$ such that

$$|z - w_0 + w||\varphi(z + w) - \varphi(w)| < M|z| \qquad (|z| < 1 + |w|).$$

Now suppose that $z \in U$. By substituting $z - w$ instead of z in the above relation, we get

$$|z - w_0||\varphi(z) - \varphi(w)| < M|z - w| \qquad (z \in U)$$

and consequently

$$|\varphi(z) - \varphi(w)| < M\frac{|z - w|}{|z - w_0|} = MN|S(z)| \qquad (z \in U).$$

Define $\Psi(z) = S \circ \psi \circ S^{-1}(z)$ and $\Phi(z) = \varphi \circ S^{-1}(z)$ for all $z \in V$. Note that V is an open subset of U and $0 = S(w) \in S(U) = V$, and

$$|\Phi(z) - \Phi(0)| < MN|z| \qquad (z \in V).$$

But $\Phi(0) = \varphi(w) \neq 0$ and $\Psi(V) \subseteq V$, because

$$\Psi(V) = S \circ \psi \circ S^{-1}(S(U)) \subseteq S(U)),$$

thus for each $z \in V$ we have

$$\left|1 - \frac{1}{\Phi(0)}\Phi(z)\right| < \frac{M}{|\Phi(0)|}|z| \qquad (z \in V).$$

By substituting $\Psi_i(z)$ instead of z in the above inequality, we get

$$\left|1 - \frac{1}{\Phi(0)}\Phi(\Psi_i(z))\right| < \frac{M}{|\Phi(0)|}|\Psi_i(z)|.$$

This implies that

$$\sum_{i=0}^{\infty}\left|1 - \frac{1}{\Phi(0)}\Phi(\Psi_i(z))\right|$$

and consequently

$$\prod_{i=0}^{\infty}\frac{1}{\Phi(0)}\varphi(\Psi_i(z))$$

converges uniformly on compact subsets of V. Also note that

$$\begin{aligned}|\Phi(\Psi_n(z))| &< MN|\Psi_n(z)| + |\Phi(0)| \\ &< MN|\lambda|^n|z| + |\Phi(0)|\end{aligned}$$

and consequently we get

$$\begin{aligned}\frac{|\Phi(\Psi_n(z))|}{|\Phi(0)|} &< MN|\lambda|^n\frac{|z|}{|\Phi(0)|} + 1 \\ &\leq \exp\left(MN|\lambda|^n\frac{|z|}{|\Phi(0)|}\right) \\ &\leq \exp\left(\frac{MN}{|\Phi(0)|}|\lambda|^n\right).\end{aligned}$$

Thus for every $z \in V$ we have

$$\prod_{n=0}^{\infty}\left|\frac{1}{\Phi(0)}\Phi(\Psi_n(z))\right| \leq \exp\left(\sum_{n=0}^{\infty}\frac{MN}{|\Phi(0)|}\lambda^n\right) = \exp\left(\frac{MN}{|\Phi(0)|}\frac{1}{1-|\lambda|}\right).$$

Now set

$$G(z) = \prod_{n=0}^{\infty}\frac{1}{\Phi(0)}\Phi(\Psi_n(z)).$$

Then G is a bounded nonzero holomorphic map on V and consequently $g = G \circ S$ is a bounded nonzero holomorphic map on U. Also, note that $\Phi \cdot G \circ \Psi = \Phi(0)G$. Thus

$$\varphi \circ S^{-1}(z) \cdot G \circ (S \circ \psi \circ S^{-1})(z) = (\varphi \circ S^{-1}(0))G(z) = \varphi(w)G(z)$$

for every $z \in V$. If $z \in U$ substitute $S(z) \in V$ instead of z in the above equality. So we get $\varphi \cdot g \circ \psi = \varphi(w)g$ where $g = G \circ S \in H^{\infty}(U)$. Now since M_z is polynomially bounded by the same method used in the proof of Theorem 2.1 we can see that $g \in M(X) \subseteq X$. If $w_0 = \infty$, then the map $S(z) = (z - w)/2$ takes w to 0 and w_0 to ∞. Then S maps the unit disk U into itself and for all $z \in V = S(U)$ we have $S \circ \psi \circ S^{-1}(z) = \lambda z$ for some complex number λ with $0 < |\lambda| < 1$ (pages 5, 6 of [6]). Now define the holomorphic map f by the relation

$$f(z) = 2(\varphi(z+w) - \varphi(w)) \qquad (|z| < 1 + |w|).$$

Because $f(0) = 0$ and f is bounded, an application of the Schwarz's lemma shows that there exists a constant $M > 0$ ($M = \|\varphi\|_U$) such that

$$2|\varphi(z+w) - \varphi(w)| < M|z| \qquad (|z| < 1 + |w|)$$

and consequently

$$|\varphi(z) - \varphi(w)| < M\frac{|z-w|}{2} = M|S(z)| \qquad (z \in U).$$

Define $\Psi(z) = S \circ \psi \circ S^{-1}(z)$ and $\Phi(z) = \varphi \circ S^{-1}(z)$ for all $z \in V$. Now similar to the case $w_0 \neq \infty$ we can see that there exists $G \in H^{\infty}(V)$ such that $\Phi \cdot G \circ \Psi = \Phi(0)G$, hence $\varphi \cdot g \circ \psi = \varphi(w)g$ where $g = G \circ S \in X$. Therefore in each case $\varphi(w)$ is indeed an eigenvalue for $M_{\varphi}C\psi$ as an operator on X and so $(M_{\varphi}C_{\psi})^*$ is not hypercyclic on X^*. ■

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