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Hypersurfaces satisfying $L_r x = Rx$ in sphere $\mathbb{S}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$

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Abstract. In this paper, using the method of moving frames, we consider hypersurfaces in Euclidean sphere $\mathbb{S}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$ whose position vector x satisfies $L_r x = Rx$, where L_r is the linearized operator of the $(r+1)$-th mean curvature of the hypersurfaces for a fixed $r = 0, \ldots, n-1$, $R \in \mathbb{R}^{(n+2)\times(n+2)}$. If the r-th mean curvature H_r is constant, we prove that the only hypersurfaces satisfying that condition are r-minimal ($H_{r+1} \equiv 0$) or isoparametric. In particular, we locally classify such hypersurfaces which are not r-minimal.

Keywords. r-Minimal; linearized operator L_r; isoparametric hypersurface.

1. Introduction

In 1992, Alías, Ferrández and Lucas [1] locally classified the surfaces M_s^2 in the 3-dimensional Lorentz–Minkowski space $\mathbb{L}^3$ satisfying the equation $\Delta x = Ax + B$. They also studied pseudo-Riemannian submanifolds M_s^n [2] in pseudo-Euclidean space $\mathbb{R}_t^{n+k}$ satisfying the condition $\Delta x = Ax + B$, where A is an endomorphism of $\mathbb{R}_t^{n+k}$ and B is a constant vector in $\mathbb{R}_t^{n+k}$. They gave a characterization theorem when A is a self-adjoint endomorphism. For hypersurfaces they obtained a classification theorem for any endomorphism A. Later, as a natural continuation of [1, 2], they studied and classified pseudo-Riemannian hypersurfaces [3] in pseudo-Riemannian space forms which satisfy the condition $\Delta x = Ax + B$. They proved that the family of such hypersurfaces consists of open pieces of minimal hypersurfaces, totally umbilical hypersurfaces, products of two non-flat totally umbilical submanifolds, and a special class of quadratic hypersurfaces. In addition, Park [9] also studied the hypersurface in a space form or in Lorentz–Minkowski space whose immersion x satisfies $\Delta x = Rx + b$, and proved it is minimal or isoparametric.

Recently, Alías and Gürbüz [4] studied hypersurfaces in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ whose position vector x satisfies $L_r x = Rx + b$. They proved the following classification result:

Theorem 1.1. *Let x: $M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface immersed into the Euclidean space and let L_r be the linearized operator of the $(r+1)$-th mean curvature of M, for some fixed $r = 0, \ldots, n-1$. Then the immersion satisfies the condition $L_r x = Rx + b$*

for some constant matrix $R \in \mathbb{R}^{(n+1)\times(n+1)}$ *and some constant vector* $b \in \mathbb{R}^{n+1}$ *if and only if it is one of the following hypersurfaces in* $\mathbb{R}^{n+1}$:

(i) *a hypersurface with zero* $(r+1)$*-th mean curvature*;
(ii) *an open piece of a round hypersphere* $\mathbb{S}^n(c)$;
(iii) *an open piece of a generalized right spherical cylinder* $\mathbb{S}^k(c)\times\mathbb{R}^{n-k}$, *with* $r+1\le k\le n-1$.

Hence, we should investigate hypersurfaces in sphere $\mathbb{S}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$ whose position vector x satisfies $L_r x = Rx$, where L_r is the linearized operator of the $(r+1)$-th mean curvature of the hypersurface for a fixed $r = 0,\dots,n-1$, $R \in \mathbb{R}^{(n+2)\times(n+2)}$ is a constant matrix. If the r-th mean curvature H_r is constant, we prove that the only hypersurfaces satisfying that condition are r-minimal or isoparametric. In fact, we prove the following result.

Theorem 1.2. *Let* $x\colon M^n \to \mathbb{S}^{n+1}$ (*or* $\mathbb{H}^{n+1}$) *be an orientable hypersurface inmmersed into sphere* $\mathbb{S}^{n+1}$ *or hyperbolic space* $\mathbb{H}^{n+1}$ *and let* L_r *be the linearized operator of the* $(r+1)$*-th mean curvature of* M, *for some fixed* $r=0,\dots,n-1$. *If* H_r *is constant and the immersion satisfies the condition* $L_r x = Rx$ *for some constant matrix* $R\in\mathbb{R}^{(n+2)\times(n+2)}$, *then* M *is* r*-minimal or isoparametric. Moreover, if* M *is isoparametric and not* r*-minimal, then* M *has distinct principal curvatures and it is an open piece of one of the following*: $\mathbb{S}^k(c_1)\times\mathbb{S}^{n-k}(c_2)$ *in* $\mathbb{S}^{n+1}$ *or* $\mathbb{S}^k(c_1)\times\mathbb{H}^{n-k}(c_2)$ *in* $\mathbb{H}^{n+1}$.

Remark 1.3. A different but related result to our Theorem 1.2 has been proved recently by Alías and Kashani [5]. In fact, instead of assuming that H_r is constant, they assume that R is self-adjoint and get the same classification result.

2. Preliminaries

In this section, we give some formulas and notions of hypersurfaces in the space forms by using the method of moving frames. Let $x\colon M \longrightarrow \bar{M}^{n+1}(c)(\subset \mathbb{R}^{n+2}$ or $\mathbb{L}^{n+2})$ be an isometric immersion from Riemannian manifold M to space forms $\bar{M}^{n+1}(c)$ with constant sectional curvature c, where $c=1$, $\bar{M}^{n+1}(1)=\mathbb{S}^{n+1}$; or $c=-1$, $\bar{M}^{n+1}(-1)=\mathbb{H}^{n+1}$.

For any $p\in M$, we can choose a local orthonormal frame fields $e_1,\dots,e_{n+2}$ in a neighborhood U of $p\in M$ such that $e_1,\dots,e_n$ are tangential to M and e_{n+1} is normal to M. In the following we shall make use of the following convention on the ranges of indices:

$$1\le A,B,C\le n+2,\quad 1\le i,j,k,l\le n,\quad n+1\le\alpha,\beta,\gamma\le n+2.$$

Let ω_A be the corresponding dual frame and the smooth connection 1-forms are denoted by ω_{AB}. Then we have structure equations of M^n in $\bar{M}^{n+1}(c)$,

$$\begin{cases} \mathrm{d}x = \sum_i \omega_i e_i, \\ \mathrm{d}e_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij}\omega_j e_{n+1} - c\omega_i x, \\ \mathrm{d}e_{n+1} = -\sum_{i,j} h_{ij}\omega_j e_i. \end{cases} \tag{2.1}$$

where h_{ij} denote the components of the second fundamental form of M^n which is given by

$$B = \sum_{i,j} h_{ij}\omega_i\otimes\omega_j, \tag{2.2}$$

and the square of the length of B is defined by $S=\|B\|^2=\sum_{i,j} h_{ij}^2$.

The Gauss equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.3}$$

$$R_{ij} = \sum_{k=1}^{n} R_{ikjk} = (n-1)c\delta_{ij} + nHh_{ij} - \sum_{k} h_{ik}h_{kj}, \tag{2.4}$$

$$n(n-1)(\rho - c) = n^2H^2 - S, \tag{2.5}$$

where $\rho = \frac{1}{n(n-1)} \sum_{i=1}^{n} R_{ii}$ is the normalized scalar curvature of M.

The Codazzi equation is

$$h_{ijk} = h_{ikj}, \tag{2.6}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_{k} h_{ijk}\omega_k = \mathrm{d}h_{ij} + \sum_{k} h_{kj}\omega_{ki} + \sum_{k} h_{ik}\omega_{kj}. \tag{2.7}$$

Associated to the shape operator A of M^n one has n invariants S_r, $1 \le r \le n$, given by the equality

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$. If $p \in M$ and $\{e_k\}_{1\le k\le n}$ is the basis of T_pM formed by eigenvectors of the shape operator A_p, with corresponding eigenvalues λ_k's, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[x_1, \dots, x_n]$ is the r-th elementary symmetric polynomial on the indeterminates $x_1, \dots, x_n$. The r-th mean curvature of M is given by

$$H_r = \frac{1}{\binom{n}{r}} S_r.$$

In particular, when $r = 1$,

$$H_1 = \frac{1}{n}\sum_{i} \lambda_i = \frac{1}{n} S_1 = H$$

is nothing but the mean curvature of M.

A hypersurface M^n in space forms $\bar{M}^{n+1}(c)$ is called r-minimal if $H_{r+1} \equiv 0$.

The classical Newton transformations $P_r\colon \mathcal{X}(M) \to \mathcal{X}(M)$ (see [10]) are defined inductively from the shape operator A by

$$P_0 = I \quad \text{and} \quad P_r = S_r I - A \circ P_{r-1},$$

for every $r = 1, \dots, n$, where I denotes the identity in $\mathcal{X}(M)$. Equivalently,

$$P_r = \sum_{j=0}^{r} (-1)^j S_{r-j} A^j,$$

$P_n = 0$ by the Cayley–Hamilton theorem. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM, diagonalizing A at $p \in M$, also diagonalize all of P_r's at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\mathrm{span}\{e_i\}^{\perp} \subset T_pM$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \le j_1 < \cdots < j_k \le n \\ j_1, \ldots, j_k \ne i}} \lambda_{j_1} \ldots \lambda_{j_k}.$$

With the above notions, it is also immediate to check that $P_r(e_i) = \sum_j T^r_{ij} e_j = S_r(A_i)e_i$, and we also have the following properties of P_r [6, 7, 10, 11].

Lemma 2.1. *For each* $1 \le r \le n-2$,

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;
(b) $\mathrm{tr}(P_r) = (n-r)S_r$;
(c) $\mathrm{tr}(A \circ P_r) = (r+1)S_{r+1}$;
(d) $\mathrm{tr}(A^2 \circ P_r) = S_1 S_{r+1} - (r+2)S_{r+2}$;
(e) $\mathrm{tr}(P_r \circ \nabla_X A) = \langle \nabla S_{r+1}, X \rangle$, *for any* $X \in \mathcal{X}(M)$.

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r \colon C^\infty(M) \to C^\infty(M)$ given by

$$L_r(f) = \mathrm{tr}(P_r \circ \nabla^2 f),$$

where $\nabla^2 f \colon \mathcal{X}(M) \to \mathcal{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(M).$$

Using that P_r is a symmetrical operator, we have

$$L_r(fg) = fL_r g + gL_r f + 2\langle P_r(\nabla f), \nabla g \rangle \tag{2.8}$$

for every $f, g \in C^\infty(M)$.

Now we can let

$$f = \langle x, a \rangle, \quad g = \langle e_{n+1}, a \rangle,$$

where a is a fixed vector in $\mathbb{R}^{n+2}$ if $c = 1$; or a fixed vector in $\mathbb{L}^{n+2}$ if $c = -1$. Then it is easy to see that

$$\nabla f = a^\top, \quad \nabla g = -A(a^\top), \tag{2.9}$$

where $a^\top \in \mathcal{X}(M)$ denotes the tangential component of a.

Thus, we have the following lemma.

Lemma 2.2. *For each* $1 \leq r \leq n-2$,

$$L_r x = (r+1)S_{r+1}e_{n+1} - c(n-r)S_r x, \tag{2.10}$$

$$L_r e_{n+1} = -\nabla S_{r+1} - (S_1 S_{r+1} - (r+2)S_{r+2})e_{n+1} + c(r+1)S_{r+1}x. \tag{2.11}$$

Proof. Since

$$f_i = e_i\langle x, a\rangle = \langle e_i, a\rangle,$$

then

$$\begin{aligned}
\sum_j f_{ij}\omega_j &= \mathrm{d}f_i + \sum_j f_j\omega_{ji} \\
&= \langle \mathrm{d}e_i, a\rangle - \sum_j \langle e_j, a\rangle\omega_{ij} \\
&= \left\langle \sum_j \omega_{ij}e_j + \sum_j h_{ij}\omega_j e_{n+1} - c\omega_i x, a\right\rangle - \sum_j \langle e_j, a\rangle\omega_{ij} \\
&= \left\langle \sum_j h_{ij}\omega_j e_{n+1} - c\omega_i x, a\right\rangle \\
&= \sum_j h_{ij}\langle e_{n+1}, a\rangle\omega_j - c\sum_j \delta_{ij}\langle x, a\rangle\omega_j.
\end{aligned}$$

Thus

$$\begin{aligned}
f_{ij} &= h_{ij}\langle e_{n+1}, a\rangle - c\delta_{ij}\langle x, a\rangle, \\
x_{ij} &= h_{ij}e_{n+1} - c\delta_{ij}x.
\end{aligned}$$

With that in mind we calculate

$$\begin{aligned}
L_r f &= \sum_{i,j} T^r_{ij} f_{ij} \\
&= \sum_{i,j} T^r_{ij} h_{ij}\langle e_{n+1}, a\rangle - c\sum_{i,j} T^r_{ij}\delta_{ij}\langle x, a\rangle \\
&= \mathrm{tr}(A \circ P_r)\langle e_{n+1}, a\rangle - c\,\mathrm{tr}(P_r)\langle x, a\rangle \\
&= (r+1)S_{r+1}\langle e_{n+1}, a\rangle - c(n-r)S_r\langle x, a\rangle \\
&= (r+1)S_{r+1}g - c(n-r)S_r f.
\end{aligned}$$

This shows that

$$L_r x = (r+1)S_{r+1}e_{n+1} - c(n-r)S_r x.$$

Note that

$$g_i = e_i\langle e_{n+1}, a\rangle = \langle \mathrm{d}e_{n+1}(e_i), a\rangle = -\sum_j h_{ij}\langle e_j, a\rangle,$$

so by a direct calculation we can get

$$\begin{aligned}\sum_j g_{ij}\omega_j &= \mathrm{d}g_i + \sum_j g_j\omega_{ji}\\ &= -\sum_j \mathrm{d}h_{ij}\langle e_j, a\rangle - \sum_j h_{ij}\langle de_j, a\rangle - \sum_j g_j\omega_{ij}\\ &= -\sum_{j,k}(h_{ijk}\omega_k - h_{kj}\omega_{ki} - h_{ik}\omega_{kj})\langle e_j, a\rangle\\ &\quad - \left\langle \sum_{j,k} h_{ij}\omega_{jk}e_k + \sum_{j,k} h_{ij}h_{jk}\omega_k e_{n+1} - c\sum_j h_{ij}\omega_j x, a\right\rangle\\ &\quad + \sum_{j,k} h_{jk}\langle e_k, a\rangle\omega_{ij}\\ &= -\sum_{j,k} h_{ijk}\omega_j\langle e_k, a\rangle + c\sum_j h_{ij}\omega_j\langle x, a\rangle - \sum_{j,k} h_{ik}h_{jk}\omega_j\langle e_{n+1}, a\rangle.\end{aligned}$$

This shows that

$$g_{ij} = -h_{ijk}\langle e_k, a\rangle + ch_{ij}\langle x, a\rangle - \sum_k h_{ik}h_{jk}\langle e_{n+1}, a\rangle.$$

With this we have

$$\begin{aligned}L_r g &= \sum_{i,j} T^r_{ij} g_{ij}\\ &= -\sum_{i,j,k} T^r_{ij}h_{ijk}\langle e_k, a\rangle + c\sum_{i,j} T^r_{ij}h_{ij}\langle x, a\rangle - \sum_{i,j,k} T^r_{ij}h_{ik}h_{jk}\langle e_{n+1}, a\rangle\\ &= -\sum_{i,j,k} T^r_{ij}h_{ijk}\langle e_k, a\rangle - \mathrm{tr}(A^2\circ P_r)\langle e_{n+1}, a\rangle + c\,\mathrm{tr}(A\circ P_r)\langle x, a\rangle\\ &= -\langle\nabla S_{r+1}, a\rangle - (S_1S_{r+1} - (r+2)S_{r+2})\langle e_{n+1}, a\rangle\\ &\quad + c(r+1)S_{r+1}\langle x, a\rangle\\ &= -\langle\nabla S_{r+1}, a\rangle - (S_1S_{r+1} - (r+2)S_{r+2})g + c(r+1)S_{r+1}f.\end{aligned}$$

This means

$$L_r e_{n+1} = -\nabla S_{r+1} - (S_1S_{r+1} - (r+2)S_{r+2})e_{n+1} + c(r+1)S_{r+1}x. \qquad \square$$

We can now define $r_{AB} = \epsilon_B\langle Re_A, e_B\rangle$, where $e_{n+2} = x$, and $\epsilon_A = 1$ $(1 \le A \le n+1)$, $\epsilon_{n+2} = \langle x, x\rangle = c$, then

$$R = \sum_{A,B}\epsilon_B r_{AB}\omega_A\otimes e_B.$$

Recall that

$$L_r x = (r+1)S_{r+1}e_{n+1} - c(n-r)S_r x = Rx, \tag{2.12}$$

can imply that

$$\begin{cases} r_{n+2i} = 0, \\ r_{n+2,n+1} = (r+1)S_{r+1}, \\ r_{n+2,n+2} = -c(n-r)S_r, \end{cases} \tag{2.13}$$

and taking exterior differential of (2.12), we can obtain

$$\begin{aligned} R\mathrm{d}x &= R\left(\sum_i \omega_i e_i\right) \\ &= \sum_i Re_i\omega_i \\ &= (r+1)\mathrm{d}S_{r+1}e_{n+1} + (r+1)S_{r+1}\mathrm{d}e_{n+1} \\ &\quad - c(n-r)\mathrm{d}S_r x - c(n-r)S_r\mathrm{d}x \\ &= \sum_i (r+1)S_{r+1;i}e_{n+1}\omega_i - \sum_{i,j}(r+1)S_{r+1}h_{ij}e_j\omega_i \\ &\quad - \sum_i c(n-r)S_{r;i}x\omega_i - \sum_i c(n-r)S_r e_i\omega_i. \end{aligned}$$

Thus we have

$$\begin{cases} r_{ij} = -(r+1)S_{r+1}h_{ij} - c(n-r)S_r\delta_{ij}, \\ r_{in+1} = (r+1)S_{r+1;i}, \\ r_{in+2} = -c(n-r)S_{r;i}. \end{cases} \tag{2.14}$$

On the one hand, from (2.8), (2.9) and using Lemma 2.2, we have

$$\begin{aligned} L_r(L_r f) &= L_r((r+1)S_{r+1}g - c(n-r)S_r f) \\ &= (r+1)\{L_r(S_{r+1})g + S_{r+1}L_r g + 2\langle P_r(\nabla S_{r+1}), \nabla g\rangle\} \\ &\quad - c(n-r)\{L_r(S_r)f + S_r L_r f + 2\langle P_r(\nabla S_r), \nabla f\rangle\}\} \\ &= (r+1)\{L_r(S_{r+1})g + 2\langle P_r(\nabla S_{r+1}), -A(a^\top)\rangle \\ &\quad + S_{r+1}[-\langle \nabla S_{r+1}, a\rangle - (S_1S_{r+1} - (r+2)S_{r+2})g + c(r+1)S_{r+1}f]\} \\ &\quad - c(n-r)\{L_r(S_r)f + S_r((r+1)S_{r+1}g - c(n-r)S_r f) + 2\langle P_r(\nabla S_r), a^\top\rangle\} \\ &= (r+1)[L_r(S_{r+1}) - S_{r+1}(S_1S_{r+1} - (r+2)S_{r+2}) - c(n-r)S_rS_{r+1}]g \\ &\quad + [c(r+1)^2S_{r+1}^2 - c(n-r)L_r(S_r) + c^2(n-r)^2S_r^2]f \\ &\quad - \langle (r+1)S_{r+1}\nabla S_{r+1} + 2(r+1)A\circ P_r(\nabla S_{r+1}) + 2c(n-r)P_r(\nabla S_r), a\rangle. \end{aligned}$$

Hence

$$L_r(L_r x) = (r+1)\{L_r(S_{r+1}) - S_{r+1}[S_1 S_{r+1} - (r+2)S_{r+2} + c(n-r)S_r]\}e_{n+1}$$
$$+ [c(r+1)^2 S_{r+1}^2 - c(n-r)L_r(S_r) + c^2(n-r)^2 S_r^2]x$$
$$- [(r+1)S_{r+1}\nabla S_{r+1} + 2(r+1)A \circ P_r(\nabla S_{r+1}) + 2c(n-r)P_r(\nabla S_r)].$$

On the other hand,

$$L_r(L_r x) = L_r(Rx) = R(L_r x) = (r+1)S_{r+1}Re_{n+1} - c(n-r)S_r Rx$$
$$= (r+1)S_{r+1}Re_{n+1} - c(n-r)(r+1)S_r S_{r+1}e_{n+1}$$
$$+ c^2(n-r)^2 S_r^2 x.$$

Consequently, we have

$$\begin{cases} (r+1)S_{r+1}r_{n+1,n+1} = (r+1)L_r(S_{r+1}) - (r+1)S_{r+1}(S_1 S_{r+1} - (r+2)S_{r+2}) \\ (r+1)S_{r+1}r_{n+1,n+2} = c(r+1)^2 S_{r+1}^2 - c(n-r)L_r(S_r), \\ (r+1)S_{r+1}r_{n+1i} = -(r+1)S_{r+1}S_{r+1;i} - 2(r+1)\lambda_i S_{r+1;i}S_r(A_i) \\ \qquad -2c(n-r)S_{r;i}S_r(A_i). \end{cases} \tag{2.15}$$

Note that

$$dr_{ij} = \mathrm{d}\langle Re_i, e_j\rangle = \langle R\mathrm{d}e_i, e_j\rangle + \langle Re_i, \mathrm{d}e_j\rangle$$
$$= \left\langle R\left(\sum_k \omega_{ik}e_k + \sum_k h_{ik}\omega_k e_{n+1} - c\omega_i x\right), e_j\right\rangle$$
$$+ \left\langle Re_i, \sum_k \omega_{jk}e_k + \sum_k h_{jk}\omega_k e_{n+1} - c\omega_j x\right\rangle$$
$$= \sum_k r_{kj}\omega_{ik} + \sum_k r_{ik}\omega_{jk} + \sum_k r_{n+1j}h_{ik}\omega_k$$
$$+ \sum_k r_{in+1}h_{jk}\omega_k - cr_{n+2j}\omega_i - r_{in+2}\omega_j.$$

By (2.14), we know $r_{ij} = r_{ji}$, so we obtain

$$(r_{n+1j} - r_{jn+1})h_{ik} + (r_{jn+2} - cr_{n+2j})\delta_{ik}$$
$$= (r_{n+1i} - r_{in+1})h_{jk} + (r_{in+2} - cr_{n+2i})\delta_{jk}. \tag{2.16}$$

3. Proof of Theorem 1.2

In this section, using the similar method in [4] we will prove Proposition 3.1 which will be essential to Theorem 1.2. Its proof is presented here to make this work self-contained.

PROPOSITION 3.1

If $x\colon M \to \mathbb{S}^{n+1}$ (or $\mathbb{H}^{n+1}$) satisfies the condition $L_r x = Rx$, for some constant matrix $R \in \mathbb{R}^{(n+2)\times(n+2)}$ and if H_r is constant, then the $(r+1)$-th mean curvature H_{r+1} is also constant on M.

Proof. The case $r = 0$, naturally $S_0 = 1$, was already solved by Park (Proposition 4.3 of [9]) (see also Proposition 3.2 of [3]) for the general case), so we may consider the case $r \geq 1$. Since S_r is a constant, by (2.14), $r_{in+2} = 0$. From (2.13) and (2.16), we obtain

$$(r_{n+1j} - r_{jn+1})h_{ik} = (r_{n+1i} - r_{in+1})h_{jk}. \tag{3.1}$$

Consider the open set

$$\mathcal{U} = \{p \in M | S_{r+1}(p)\nabla S_{r+1}(p) \neq 0\}.$$

We will show that $\mathcal{U}$ is empty, using the method of reduction to absurdity. Otherwise, we assume that $\mathcal{U}$ is non-empty. In the following, we will discuss on $\mathcal{U}$ if without special statement.

From (2.15), we can obtain

$$r_{n+1i} = -S_{r+1;i} - \frac{2}{S_{r+1}}\lambda_i S_{r+1;i} S_r(A_i). \tag{3.2}$$

Putting (3.2) and (2.14) into (3.1), we obtain

$$\begin{aligned}&\left((n+2)S_{r+1;i} + \frac{2}{S_{r+1}}\lambda_i S_{r+1;i} S_r(A_i)\right)\lambda_j\\ &\quad = \left((n+2)S_{r+1;j} + \frac{2}{S_{r+1}}\lambda_j S_{r+1;j} S_r(A_i)\right)\lambda_i,\end{aligned} \tag{3.3}$$

i.e.

$$\langle T_r(\nabla S_{r+1}), e_i\rangle Ae_j = \langle T_r(\nabla S_{r+1}), e_j\rangle Ae_i, \tag{3.4}$$

where T_r is the operator given by

$$T_r = \frac{2}{S_{r+1}} A \circ P_r + (r+2)I.$$

Therefore

$$\langle T_r(\nabla S_{r+1}), X\rangle AY = \langle T_r(\nabla S_{r+1}), Y\rangle AX \tag{3.5}$$

for every $X, Y \in \mathcal{X}(M)$.

We claim that

$$T_r(\nabla S_{r+1}) = 0. \tag{3.6}$$

Actually, if $T_r(\nabla S_{r+1})(p_0) \neq 0$ at some point $p_0 \in \mathcal{U}$, then we can choose, on a neighborhood of p_0 where $T_r(\nabla S_{r+1}) \neq 0$, and a local orthonormal frame $e_1, \ldots, e_n$ with e_1 in the direction of $T_r(\nabla S_{r+1})$. Thus it yields to $Ae_i = 0$ for every $i = 2, \ldots, n$ from (3.4), which implies that rank $A = 1$ on $\mathcal{U}$. But this implies that $S_{r+1} = 0$ on $\mathcal{U}$ for every $r \geq 1$, which is not possible. Therefore, $T_r(\nabla S_{r+1}) = 0$ on $\mathcal{U}$, or equivalently

$$2A \circ P_r(\nabla S_{r+1}) = -(r+2)S_{r+1}\nabla S_{r+1} \tag{3.7}$$

on $\mathcal{U}$. Therefore

$$S_{r+1;i} = 0 \ \text{ or } \ \lambda_i S_r(A_i) = -\frac{r+2}{2}S_{r+1}. \tag{3.8}$$

This implies that $S_{r+1;i} = 0$ necessarily for some i. Otherwise, we would have

$$(r+1)S_{r+1} = \operatorname{tr}(A \circ P_r) = \sum_i \lambda_i S_r(A_i) = -\frac{n(r+2)}{2}S_{r+1},$$

and thus $S_{r+1} = 0$ on $\mathcal{U}$, which is a contraction. Therefore, rearranging the local orthonormal frame if necessary, we may assume $S_{r+1;i} = 0$ for $1 \leq i \leq m < n$, $S_{r+1;j} \neq 0$ for $m+1 \leq j \leq n$.

In the case $r = n-1$, since $P_n = 0$ and from the inductive definition of P_r and (3.7), it can reduce to

$$P_n(\nabla S_n) = \frac{n+3}{2}S_n\nabla S_n = 0, \tag{3.9}$$

which is a contradiction. So in the following we consider the case $1 \leq r \leq n-2$.

By (3.2) and (3.8), we have

$$r_{n+1i} = -S_{r+1;i} - \frac{2}{S_{r+1}}\lambda_i S_r(A_i)S_{r+1;i} = (r+1)S_{r+1;i}.$$

Using (2.13)–(2.15), we can see that the matrix R with respect to e_A is of the form

$$\begin{pmatrix} \mu_1 & \cdots & 0 & (r+1)S_{r+1;1} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \mu_n & (r+1)S_{r+1;n} & 0 \\ (r+1)S_{r+1;1} & \cdots & (r+1)S_{r+1;n} & \mu_{n+1} & c(r+1)S_{r+1} \\ 0 & \cdots & 0 & (r+1)S_{r+1} & -c(n-r)S_r \end{pmatrix}, \tag{3.10}$$

where $\mu_i = -(r+1)S_{r+1}\lambda_i - c(n-r)S_r$ for $1 \leq i \leq n$, $\mu_{n+1} = \frac{1}{S_{r+1}}L_r(S_{r+1}) - S_1S_{r+1} + (r+2)S_{r+2}$.

Thus, from (3.10) it is easy to see that the characteristic polynomial of the matrix R is given by

$$\begin{aligned} \det(tI - R) = (t + c(n-r)S_r) & \left\{ \prod_{i=1}^{n+1}(t-\mu_i) \right. \\ & \left. - \sum_{i=1}^{n}(r+1)^2S_{r+1;i}^2(t-\mu_1)\cdots\widehat{(t-\mu_i)}\cdots(t-\mu_n) \right\} \\ & - c(r+1)^2S_{r+1}^2\prod_{i=1}^{n}(t-\mu_i). \end{aligned} \tag{3.11}$$

where $\widehat{\ }$ denotes deletion.

Since $S_{r+1;i} = 0$ for $1 \le i \le m$, it is easy to see that μ_i $(1 \le i \le m)$ are eigenvalues of R, and thus they are constant. If $\lambda_j = \lambda_k$ for $m+1 \le j < k \le n$, then $\mu_j = \mu_k$, and by (3.11) we know $\mu_j = \mu_k$ are also eigenvalues of R, and thus they are constant. Therefore, without generality, we can assume $\lambda_{m+1} < \cdots < \lambda_n$. By (3.8) we know that

$$\lambda_j S_r(A_j) = -\frac{r+2}{2} S_{r+1} \neq 0, \tag{3.12}$$

for $m+1 \le j \le n$ on $\mathcal{U}$. Taking notice of

$$\lambda_j S_r(A_j) = \sum_{s=0}^{r} (-1)^s \lambda_j^{s+1} S_{r-s} = -\frac{r+2}{2} S_{r+1},$$

hence λ_j $(m+1 \le j \le n)$ are distinct real roots of the following equation of degree $r+1$,

$$\sum_{s=0}^{r} (-1)^s t^{s+1} S_{r-s} + \frac{r+2}{2} S_{r+1} = 0.$$

And thus $n - m \le r + 1$. On the other hand, each λ_j is also the distinct roots of the characteristic polynomial of A,

$$\sum_{s=0}^{n} (-1)^s t^s S_{n-s}.$$

Therefore λ_j $(m+1 \le j \le n)$ are distinct real roots of the following equation of degree $n - r - 1$,

$$\sum_{s=0}^{n-r-1} (-1)^s t^s S_{n-s} + \frac{r+2}{2} (-1)^{n-r} S_{r+1} t^{n-r-1} = 0.$$

In particular, $n - m \le n - r - 1$, that is $m \ge r + 1$.

Now we claim that

$$S_{r+1}(A_{m+1}) = \cdots = S_{r+1}(A_n) = S_{r+1}(A_{m+1\ldots n}). \tag{3.13}$$

In fact, by $m+1 \le j < k \le n$,

$$S_{r+1}(A_j) = \lambda_k S_r(A_{jk}) + S_{r+1}(A_{jk}) = \lambda_j S_r(A_{jk}) + S_{r+1}(A_{jk}) = S_{r+1}(A_k),$$

this means $S_r(A_{jk}) = 0$. Therefore

$$S_{r+1}(A_j) = S_{r+1}(A_{jk}).$$

Thereby, we can obtain (3.13) by induction. From (3.13) we get that

$$\begin{aligned}\frac{r+4}{2} S_{r+1} &= S_{r+1}(A_{m+1}) = S_{r+1}(A_{m+1\ldots n}) \\ &= \sum_{1 \le i_1 < \cdots < i_{r+1} \le m} \lambda_{i_1} \cdots \lambda_{i_{r+1}} \\ &= \frac{(-1)^{r+1}}{(r+1)^{r+1} S_{r+1}^{r+1}} \cdot \sum_{1 \le i_1 < \cdots < i_{r+1} \le m} \prod_{k=1}^{r+1} (\mu_{i_k} + c(n-r) S_r)\end{aligned}$$

on $\mathcal{U}$. But this mean that S_{r+1} is locally constant on $\mathcal{U}$, which is a contradiction with definition of $\mathcal{U}$. Hence S_{r+1} is constant. □

Now, we begin to prove Theorem 1.2. By Proposition 3.1, we already know S_{r+1} is constant on M. If $S_{r+1}=0$, then M is r-minimal, and there is nothing to prove. Hence, we may suppose that S_{r+1} is a non-zero constant. Since S_r and S_{r+1} all are constant, using (3.10), we can see that the matrix R with respect to e_A is of the form

$$\begin{pmatrix} \mu_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \mu_n & 0 & 0 \\ 0 & \cdots & 0 & \mu_{n+1} & c(r+1)S_{r+1} \\ 0 & \cdots & 0 & (r+1)S_{r+1} & -c(n-r)S_r \end{pmatrix}, \tag{3.14}$$

where $\mu_i=-(r+1)S_{r+1}\lambda_i-c(n-r)S_r$ for $1\le i\le n$, $\mu_{n+1}=\frac{1}{S_{r+1}}L_r(S_{r+1})-S_1S_{r+1}+(r+2)S_{r+2}$. It is easy to see that μ_i's are eigenvalues of R, thus they are constant, i.e. $\lambda_i=-\frac{\mu_i+c(n-r)S_r}{(r+1)S_{r+1}}$ $(1\le i\le n)$ are also constant. Hence M is isoparametric. Meanwhile, since we know that S_r and S_{r+1} are all constants, by (2.14) and (2.15) we have

$$Re_i=-[(r+1)S_{r+1}\lambda_i+c(n-r)S_r]e_i, \tag{3.15}$$

$$Re_{n+1}=-[S_1S_{r+1}-(r+2)S_{r+2}]e_{n+1}+c(r+1)S_{r+1}x. \tag{3.16}$$

Then, on the one hand, taking covariant derivative in (3.16) we obtain

$$\begin{aligned}\bar\nabla_{e_i}(Re_{n+1})&=-[S_1S_{r+1}-(r+2)S_{r+2}]\bar\nabla_{e_i}e_{n+1}+c(r+1)S_{r+1}\bar\nabla_{e_i}x\\&=[S_1S_{r+1}-(r+2)S_{r+2}]Ae_i+c(r+1)S_{r+1}e_i\\&=[S_1S_{r+1}-(r+2)S_{r+2}]\lambda_ie_i+c(r+1)S_{r+1}e_i.\end{aligned} \tag{3.17}$$

On the other hand,

$$\begin{aligned}\bar\nabla_{e_i}(Re_{n+1})&=R(\bar\nabla_{e_i}e_{n+1})\\&=R(-Ae_i)\\&=-\lambda_iR(e_i)\\&=[(r+1)S_{r+1}\lambda_i+c(n-r)S_r]\lambda_ie_i.\end{aligned} \tag{3.18}$$

Consequently, using (3.17) and (3.18), we can obtain

$$\begin{aligned}&[(r+1)S_{r+1}\lambda_i+c(n-r)S_r]\lambda_i\\&\quad=[S_1S_{r+1}-(r+2)S_{r+2}]\lambda_i+c(r+1)S_{r+1},\end{aligned}$$

i.e.

$$(r+1)S_{r+1}\lambda_i^2+[-S_1S_{r+1}+(r+2)S_{r+2}+c(n-r)S_r]\lambda_i-c(r+1)S_{r+1}=0. \tag{3.19}$$

Hence λ_i's are the roots of the following equation

$$(r+1)S_{r+1}t^2+[-S_1S_{r+1}+(r+2)S_{r+2}+c(n-r)S_r]t-c(r+1)S_{r+1}=0. \tag{3.20}$$

This means that there are at most two constant distinct principal curvatures among λ_i ($1 \leq i \leq n$) and M is an isoparametric hypersurface of $\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$ with at most two distinct principal curvatures. Consequently, either M is totally umbilical in $\bar{M}^{n+1}(c)$ (but not totally geodesic, because of $H_{r+1} \neq 0$) or M an isoparametric hypersurface of $\bar{M}^{n+1}(c)$ with two constant principal curvatures. The former cannot occur, because all the totally umbilical hypersurfaces in $\bar{M}^{n+1}(c)$ which are not totally geodesic do not satisfy $L_r x = Rx$ (see [12]). In the latter, from well-known results by Lawson (Lemma 2 of [8]) and Ryan (Theorem 2.5 of [12]), we conclude that M is an open piece of a standard Riemannian product. Hence, if M is not r-minimal, M is an open piece of one of the following: $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-k}(c_2)$ in $\mathbb{S}^{n+1}$ or $\mathbb{S}^k(c_1) \times \mathbb{H}^{n-k}(c_2)$ in $\mathbb{H}^{n+1}$.

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