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Torus quotients of homogeneous spaces – minimal dimensional Schubert varieties admitting semi-stable points

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Abstract. In this paper, for any simple, simply connected algebraic group G of type B, C or D and for any maximal parabolic subgroup P of G, we describe all minimal dimensional Schubert varieties in G/P admitting semistable points for the action of a maximal torus T with respect to an ample line bundle on G/P. We also describe, for any semi-simple simply connected algebraic group G and for any Borel subgroup B of G, all Coxeter elements τ for which the Schubert variety $X(\tau)$ admits a semistable point for the action of the torus T with respect to a non-trivial line bundle on G/B.

Keywords. Semistable points; line bundle; coxeter element.

1. Introduction

Let G be a simply connected semi-simple algebraic group over an algebraic closed field k. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. In [4] and [5], the parabolic subgroups Q of G containing B for which there exists an ample line bundle $\mathcal{L}$ on G/Q such that the semistable points $(G/Q)^{ss}_T(\mathcal{L})$ are the same as the stable points $(G/Q)^{s}_T(\mathcal{L})$.

In [7], when Q is a maximal parabolic subgroup of G and $\mathcal{L} = \mathcal{L}_{\varpi}$, where ϖ is a minuscule dominant weight, it is shown that there exists unique minimal dimensional Schubert variety $X(w)$ admitting semistable points with respect to $\mathcal{L}$.

In the case of G/Q, where either G is exceptional or Q is not maximal, and $\mathcal{L}$ is an ample line bundle, the combinatorics of minimal elements $w \in W/W_Q$ for which $X(w)^{ss}_T(\mathcal{L}) \neq \emptyset$ is complicated. So, we assume that G is a simple algebraic group of type B, C or D and P is a maximal parabolic subgroup of G. Let $\mathcal{L}$ be an ample line bundle on G/P. In this paper, we describe all minimal dimensional Schubert varieties in G/P admitting semistable points with respect to $\mathcal{L}$.

For a precise statement, see Theorem 3.2.

Now, let G be a semi-simple simply connected algebraic group over an algebraic closed field k. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. A Schubert variety $X(w)$ in G/B contains a (rank G)-dimensional T-orbit if and only if $w \geq \tau$ for some Coxeter element τ.

So, it is a natural question to ask if for every Coxeter element τ, there is a non-trivial line bundle $\mathcal{L}$ on G/B such that $X(\tau)^{ss}_T(\mathcal{L}) \neq \emptyset$.

Here we describe all such Coxeter elements τ. The layout of the paper is as follows:

Section 2 consists of preliminary notation and a combinatorial lemma. Section 3 consists of minimal dimensional Schubert varieties in G/P (where G is a semi-simple algebraic group of type B, C or D and P is a maximal parabolic subgroup of G), admitting semistable points with respect to an ample line bundle on G/P. Section 4 consists of description of Coxeter elements for which the corresponding Schubert varieties admit semistable points with respect to a non-trivial line bundle on G/B.

2. Preliminary notation and a combinatorial lemma

This section consists of preliminary notation and a lemma describing a criterion for a Schubert variety to admit semistable points. Let G be a semi-simple algebraic group over an algebraically closed field k. Let T be a maximal torus of G, B a Borel subgroup of G containing T and let U be the unipotent radical of B. Let $N_G(T)$ be the normalizer of T in G. Let $W = N_G(T)/T$ be a Weyl group of G with respect to T and R denote the set of roots with respect to T, R^+ positive roots with respect to B. Let U_α denote the one-dimensional T-stable subgroup of G corresponding to the root α and let $S = \{\alpha_1, \ldots, \alpha_l\} \subseteq R^+$ denote the set of simple roots. For a subset $I \subseteq S$ denote $W^I = \{w \in W | w(\alpha) > 0,\ \alpha \in I\}$ and W_I is the subgroup of W generated by the simple reflections $s_\alpha, \alpha \in I$. Then every $w \in W$ can be uniquely expressed as $w = w^I.w_I$, with $w^I \in W^I$ and $w_I \in W_I$. Denote $R(w) = \{\alpha \in R^+ \colon w(\alpha) < 0\}$ and w_0 is the longest element of W with respect to S. Let $X(T)$ (resp. $Y(T)$) denote the set of characters of T (resp. one-parameter subgroups of T). Let $E_1 := X(T) \otimes \mathbb{R}$, $E_2 = Y(T) \otimes \mathbb{R}$. Let $\langle \cdot, \cdot \rangle \colon E_1 \times E_2 \longrightarrow \mathbb{R}$ be the canonical non-degenerate bilinear form. Choose λ_j's in E_2 such that $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ for all i. Let $\bar{C} := \{\lambda \in E_2 | \langle \lambda, \alpha \rangle \geq 0\, \forall \alpha \in R^+\}$ and for all $\alpha \in R$, there is a homomorphism $SL_2 \xrightarrow{\phi_\alpha} G$ (see page 19 of [1]). We have $\check{\alpha} \colon G_m \longrightarrow G$ defined by $\check{\alpha}(t) = \phi_\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)$. We also have $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$ for all $\alpha \in R$ and $\chi \in E_1$. Set $s_i = s_{\alpha_i}\ \forall i = 1, 2, \ldots, l$. Let $\{\omega_i \colon i = 1, 2, \ldots, l\} \subset E_1$ be the fundamental weights; i.e. $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$ for all $i, j = 1, 2, \ldots, l$.

For any character χ of B, we denote by $\mathcal{L}_\chi$, the line bundle on G/B given by the character χ. Let $X(w) = \overline{BwB/B}$ denote the Schubert variety corresponding to w. We denote by $X(w)^{\mathrm{ss}}_T(\mathcal{L}_\chi)$ the semistable points of $X(w)$ for the action of T with respect to the line bundle $\mathcal{L}_\chi$.

Lemma 2.1. *Let $\chi = \sum_{\alpha \in S} a_\alpha \varpi_\alpha$ be a dominant character of T which is in the root lattice. Let $I = \mathrm{Supp}(\chi) = \{\alpha \in S \colon a_\alpha \neq 0\}$ and let $w \in W^{I^c}$, where $I^c = S \backslash I$. Then $X(w)^{\mathrm{ss}}_T(\mathcal{L}_\chi) \neq \emptyset$ if and only if $w\chi \leq 0$.*

Proof. If $X(w)^{\mathrm{ss}}_T(\mathcal{L}_\chi) \neq \emptyset$, then, by Hilbert-Mumford criterion (Theorem 2.1 of [8] and Lemma 2.1 of [9]), we see that $w\chi \leq 0$.

Conversely, let $w\chi \leq 0$.

Step 1. We prove that if $w, \tau \in W^{I^c}$ are such that $X(w) \subseteq \bigcup_{\phi \in W} \phi X(\tau)$, then, $w \leq \tau$. Now, suppose that $X(w) \subseteq \bigcup_{\phi \in W} \phi X(\tau)$. Then, since $X(w)$ is irreducible and W is finite, we must have

$$X(w) \subseteq \phi X(\tau), \text{ for some } \phi \in W.$$

Hence, $\phi^{-1} X(w) \subseteq X(\tau)$. Now, let $P_I = BW_IB$ and consider the projection

$$\pi \colon G/B \longrightarrow G/P_I.$$

Then, $\pi^{-1}(\phi^{-1}X(w)) \subseteq \pi^{-1}(X(\tau))$. Let $w^{\max}$ (resp. $\tau^{\max}$) be the maximal representative of w (resp. τ) in W.

Hence, $\phi^{-1}X(w^{\max}) \subseteq X(\tau^{\max})$. So, we may assume that $I = S$.

Now, since $\phi^{-1}X(w) \subseteq X(\tau)$, we have $\phi^{-1}w_1 \leq \tau$, $\forall w_1 \leq w$.

Therefore $w_1\phi \leq \tau^{-1} \forall w_1 \leq w^{-1}$. Hence, by Lemma 5.6 of [6], we have $\tau^{-}(w^{-1}, \phi^{-1})\phi \leq \tau^{-1}$.

Hence, $w^{-1} \leq \tau^{-}(w^{-1}, \phi^{-1})\phi \leq \tau^{-1}$. So $w \leq \tau$.

Now, let $w \in W^{I^c}$ be such that $w\chi \leq 0$. Then by Step 1, there exists a point $x \in X(w) \backslash W$-translates of

$$X(\tau), \tau \in W^{I^c}, \ \tau \not\geq w. \tag{1}$$

Step 2. We prove that x is semistable.

Let λ be a one-parameter subgroup of T. Choose $\phi \in W$ such that $\phi\lambda \in \bar{C}$. Let $\tau \in W^{I^c}$ be such that $\phi x \in U_\tau \tau P_I$.

By (1), $w \leq \tau$. Hence, $\tau\chi \leq w\chi \leq 0$.

Hence, by Lemma 2.1 of [9], we have $\mu^L(x, \lambda) = \mu^L(\phi x, \phi\lambda) = \langle -\tau\chi, \lambda\rangle \geq 0$. Hence, by Hilbert-Mumford criterion (Theorem 2.1 of [8]), x is semistable. □

3. Minimal dimensional Schubert variety in G/P admitting semistable points

In this section, we describe all minimal dimensional Schubert varieties $X(w)$ in G/P (where G is a simple algebraic group of type B, C or D, and P is a maximal parabolic subgroup of G) for which $X(w)$ admits a semistable point for the action of a maximal torus of G with respect to an ample line bundle on G/P.

Let $I_r = S\backslash\{\alpha_r\}$ and let $P_{I_r} = BW_{I_r}B$ be the maximal parabolic corresponding to the simple root α_r. Let $\mathcal{L}_r$ denote the line bundle associated to the weight ϖ_r. In this section we will describe all minimal elememts of W^{I_r} for which $X(w)^{\rm ss}_T(\mathcal{L}_r) \neq \emptyset$.

At this point, we recall a standard property of the fundamental weights of type A, B, C and D.

In types A_n, B_n, C_n and D_n, we have $|\langle\varpi_r, \check{\alpha}\rangle| \leq 2$ for any fundamental weight ϖ_r and any root α.

Proof. Now $\langle\varpi_r, \check{\alpha}\rangle \leq \langle\varpi_r, \check{\eta}\rangle$, where η is a highest root for the corresponding root system.

The highest root for type A_n is $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, the highest roots for type B_n are $\alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n$, the highest roots for type C_n are $2(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n$ and $\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n$ and the unique highest root for type D_n is $\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$.

In all these cases, we have $\langle\varpi_r, \check{\eta}\rangle \leq 2$. So $|\langle\varpi_r, \check{\alpha}\rangle| \leq 2$, for any root α. □

Let G be a simple simply-connected algebraic group of type B, C or D. Let T be a maximal torus of G and let S be the set of simple roots with respect to a Borel subgroup B of G containing T.

PROPOSITION 3.1

Let $I_r = S\backslash\{\alpha_r\}$ and let $w \in W^{I_r}$ be of maximal length such that $w(\varpi_r) \geq 0$. Write $w(\varpi_r) = \sum_{i=1}^{n} a_i\alpha_i$ and let $a = \max\{a_i \colon i = 1, 2, \ldots, n\}$. Then $a \in \left\{1, \frac{3}{2}\right\}$. Further, if $a = \frac{3}{2}$, then r must be odd and G must be of type D_n and $a = a_{n-1}$ or $a = a_n$.

Proof. Since $2 \leq r \leq n-2$, we have $2\varpi_r \in Z_{\geq 0}S$. Hence, if $a \in \{1, \frac{3}{2}\}$, then $a \geq 2$.

Let i_0 be the least integer such that $a_{i_0} = a$.

Clearly, $i_0 \neq 1$. We first observe that, $s_{i_0} w(\varpi_r) = w(\varpi_r) - \langle w(\varpi_r), \check{\alpha}_{i_0}\rangle \alpha_{i_0} \geq 0$, since, $\langle w(\varpi_r), \check{\alpha}_{i_0}\rangle \leq 2 \leq a = a_{i_0}$.

For all the cases except $i_0 = n$ in type B_n, $i_0 = n-1$ in type C_n and $i_0 = n-2, n-1, n$ in type D_n, we have $\langle w(\varpi_r), \check{\alpha}_{i_0}\rangle = 2a - (a_{i_0-1} + a_{i_0+1}) > 0$. Hence, $s_{\alpha_{i_0}} w(\varpi_r) < w(\varpi_r)$. So, $s_{\alpha_{i_0}} w > w$, a contradiction to the maximality of w.

Now, we treat the special cases explicitly.

Case 1. $i_0 = n$ in type B_n. In this case, $\langle w(\varpi_r), \check{\alpha}_n\rangle = -2a_{n-1} + 2a_n > 0$, since $a_n = a > a_{n-1}$. So, $s_n w(\varpi_r) < w(\varpi_r)$. Hence, $s_n w > w$, a contradiction to the maximality of w.

Case 2. $i_0 = n-1$ in type C_n. In this case $\langle w(\varpi_r), \check{\alpha}_{n-1}\rangle = -a_{n-2} + 2a_{n-1} - 2a_n$. So, we need to show that $2a_{n-1} > a_{n-2} + 2a_n$. If not, then $2a_n \geq a_{n-1} + 1$, since $a_{n-2} \leq a_{n-1} - 1$.

Now, we have $s_n w(\varpi_r) = \sum_{i \neq n} a_i \alpha_i + (a_{n-1} - a_n)\alpha_n \geq 0$, since $a_{n-1} = a \geq a_n$.

On the other hand, since $2a_n \geq a_{n-1} + 1$, we have $a_{n-1} - a_n \leq a_n - 1$. So, $s_n w(\varpi_r) < w(\varpi_r)$. Hence, $s_n w > w$, a contradiction to the maximality of w.

Case 3. $i_0 = n$ in type D_n. Here, we have $\langle w(\varpi_r), \check{\alpha}_n\rangle = 2a_n - a_{n-2} > 0$, since $a_n = a > a_{n-2}$. So, $s_n w(\varpi_r) < w(\varpi_r)$. Hence, $s_n w > w$, a contradiction to the maximality of w.

Case 4. $i_0 = n-1$ in type D_n. This case is similar to Case 3.

Case 5. $i_0 = n-2$ in type D_n. We have $\langle w(\varpi_r), \check{\alpha}_{n-2}\rangle = -a_{n-3} + 2a_{n-2} - a_{n-1} - a_n$.

In order to prove that $\langle w(\varpi_r), \check{\alpha}_{n-2}\rangle > 0$, we need to prove $a_{n-1} + a_n \leq a_{n-2}$, since $a_{n-3} < a_{n-2}$.

Suppose $a_{n-1} + a_n \geq a_{n-2} + 1$. Then, we have either $2a_{n-1} > a_{n-2}$ or $2a_n > a_{n-2}$.

Without loss of generality, we may assume that $2a_{n-1} > a_{n-2}$. Hence we have

$$s_{n-1} w(\varpi_r) = \sum_{i \neq n-1} a_i \alpha_i + (a_{n-2} - a_{n-1})\alpha_{n-1} \leq w(\varpi_r), \text{ since } a_{n-2} - a_{n-1} < a_{n-1}.$$

On the other hand, $s_{n-1} w(\varpi_r) \geq 0$, since $a_{n-2} = a \geq a_{n-1}$. So, $s_{n-1} w > w$, a contradiction to the maximality of w.

Thus, we conclude that $a \in \{1, \frac{3}{2}\}$.

Now, if $a = \frac{3}{2}$, then clearly r is odd and G is not of type B_n. We now prove that G can not be of type C_n.

Suppose on the contrary, let t be the least positive integer such that $\sum_{i=t}^{n-1} \alpha_i + \frac{3}{2}\alpha_n \leq w(\varpi_r)$.

Since $\langle w(\varpi_r), \check{\alpha}_n\rangle = 3 - a_{n-1} \leq 2$, we have $a_{n-1} = 1$.

If $t \leq n-2$, then $0 \leq s_t w(\varpi_r) = \sum_{i \neq t} a_i \alpha_i < w(\varpi)$. So, $s_t w > w$, a contradiction to the maximality of w. Hence, $a_{n-2} = 0$.

We now claim that $a_i = 0 \, \forall i \leq n-3$. For otherwise, let $m \leq n-3$ be the largest integer such that $a_m = 1$.

Now, $\langle w(\varpi_r), \check{\alpha}_{m+1} + \alpha_{m+2} + \ldots \alpha_{n-1}\rangle = -3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \leq 2$ for all root β.

Thus, $a_i = 0\,\forall i \le n-2$. Hence, $w(\varpi) = \alpha_{n-1} + \frac{3}{2}\alpha_n$. But $\langle w(\varpi_r), \check{\alpha}_{n-1} + \alpha_n\rangle = 3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \le 2$ for all root β.

Thus, G can not be of type C_n.

If G is of type D_n, then $a_i \le \frac{3}{2}$, $\forall i = 1, 2, \dots, n$. We now claim that $a_{n-1} + a_n \le 2$. Suppose on the contrary, let $a_{n-1} = a_n = \frac{3}{2}$.

We claim that $a_m = 0$, $\forall m \le n-3$. Otherwise, let t be the least positive integer such that $\sum_{i=t}^{n-2} \alpha_i + \frac{3}{2}\alpha_{n-1} + \frac{3}{2}\alpha_n \le w(\varpi_r)$. Then, $a_{t-1} = 0$ and $t \le n-3$.

Hence, $\langle w(\varpi_r), \check{\alpha}_t + \alpha_{t+1} + \dots \alpha_{n-1} + \alpha_n\rangle = 3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \le 2$ for all root β.

Thus, $a_m = 0$, $\forall m \le n-3$. Hence, $w(\varpi) = \alpha_{n-2} + \frac{3}{2}(\alpha_{n-1} + \alpha_n)$.

So, $\langle w(\varpi_r), \check{\alpha}_{n-2} + \alpha_{n-1} + \alpha_n\rangle = 3$, a contradiction to the fact that $|\langle w(\varpi_r), \check{\beta}\rangle| \le 2$ for all root β.

Thus, in type D_n, both a_{n-1} and a_n cannot be $\frac{3}{2}$. □

Notation. $J_{p,q} = \{(i_1, i_2, \dots, i_p) \colon i_k \in \{1, 2, \dots, q\}, \forall k \text{ and } i_{k+1} - i_k \ge 2\}$.

Now, we will describe the set of all elements $w \in W^{I_r}$ of minimal length such that $w\varpi_r \le 0$ for types B_n, C_n and D_n.

Theorem 3.2. *Let $W^{I_r}_{\min}$ = minimal elements of the set of all $\tau \in W^{I_r}$ such that $X(\tau)^{ss}_T(\mathcal{L}_{\varpi_r}) \neq \emptyset$.*

(1) *Type B_n.*

- (i) *Let $r = 1$. Then $w = s_n s_{n-1} \dots s_1$.*
- (ii) *Let r be an even integer in $\{2, 3, \dots, n\}$. For any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$, there exists unique $w_{\underline{i}} \in W^{I_r}_{\min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Further, $W^{I_r}_{\min} = \{w_{\underline{i}} \colon \underline{i} \in J_{\frac{r}{2}, n-1}\}$.*
- (iii) *Let r be an odd integer in $\{2, 3, \dots, n\}$. For any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-2}$, there exists an unique $w_{\underline{i}} \in W^{I_r}_{\min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \alpha_n)$. Further, $W^{I_r}_{\min} = \{w_{\underline{i}} \colon \underline{i} \in J_{\frac{r-1}{2}, n-2}\}$.*
- (iv) *Let $r = n$. If n is even, then, $w = w_{\frac{n}{2}} \dots w_1$, where, $w_i = s_{2i-1} \dots s_n$, $i = 1, 2, \dots, \frac{n}{2}$ and if n is odd, then, $w = w_{[\frac{n}{2}]+1} \dots w_1$, where, $w_i = s_{2i-1} \dots s_n$, $i = 1, 2, \dots \left[\frac{n}{2}\right] + 1$.*

(2) *Type C_n.*

- (i) *Let $r = 1$. Then $w = s_n s_{n-1} \dots s_1$.*
- (ii) *Let r be an even integer in $\{2, 3, \dots, n-1\}$. For any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$, there exists an unique $w_{\underline{i}} \in W^{I_r}_{\min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Further, $W^{I_r}_{\min} = \{w_{\underline{i}} \colon \underline{i} \in J_{\frac{r}{2}, n-1}\}$.*
- (iii) *Let r be an odd integer in $\{2, 3, \dots, n-1\}$. For any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-2}$, there exists an unique $w_{\underline{i}} \in W^{I_r}_{\min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_n)$. Further, $W^{I_r}_{\min} = \{w_{\underline{i}} \colon \underline{i} \in J_{\frac{r-1}{2}, n-2}\}$.*

(3) *Type D_n.*

(i) *Let $r=1$. Then $w = s_n s_{n-1}\dots s_1$.*

(ii) *Let r be an even integer in $\{2,3,\dots,n-2\}$. For any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n}\backslash Z$, there exists an unique $w_{\underline{i}} \in W^{I_r}_{\min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$, where $Z = \{(i_1, i_2, \dots, i_{\frac{r}{2}-2}, n-2, n) \colon i_k \in \{1,2,\dots,n-4\}$ and $i_{k+1} - i_k \ge 2,\ \forall k\}$. Further, $W^{I_r}_{\min} = \{w_{\underline{i}} \colon \underline{i} \in J_{\frac{r}{2},n}\backslash Z\}$.*

(iii) *Let r be an odd integer in $\{2,3,\dots,n-2\}$. For any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2},n-3}$, there exists an unique $w_{\underline{i}} \in W^{I_r}_{\min}$ such that $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n)$. Also, for any $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2},n-2}$, there exists an unique $w_{\underline{i},1} \in W^{I_r}_{\min}$ such that $w_{\underline{i},1}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_{n-1} + \frac{3}{2}\alpha_n)$ and there exists an unique $w_{\underline{i},2} \in W^{I_r}_{\min}$ such that $w_{\underline{i},2}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{3}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n)$. Further, $W^{I_r}_{\min} = \{w_{\underline{i}} \colon \underline{i} \in J_{\frac{r-1}{2},n-3}\} \bigcup \{w_{\underline{i},j} \colon \underline{i} \in J_{\frac{r-1}{2},n-2}$ and $j = 1,2\}$.*

(iv) *Let $r = n-1$ or n. Then, $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$, where,*

$$w_i = \begin{cases} \tau_i s_n, & \text{if } i \text{ is odd.} \\ \tau_i s_{n-1}, & \text{if } i \text{ is even.} \end{cases}$$

with, $\tau_i = s_{2i-1}\dots s_{n-2}$, $i = 1,2,\dots,\left[\frac{n-1}{2}\right]$.

Proof of (1).

(i) $\varpi_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Take $w = s_n s_{n-1}\dots s_1$. Then $w(\varpi_1) = -\alpha_n \le 0$.

(ii) Let r be an even integer in $\{2,3,\dots,n-2\}$. We have,

$$\varpi_r = \sum_{i=1}^{r-1} i\alpha_i + r(\alpha_r + \cdots + \alpha_n),\ \ 4 \le r \le (n-1).$$

Now, $J_{\frac{r}{2},n-1} = \{(i_1, i_2, \dots, i_{\frac{r}{2}}) \colon i_k \in \{1,2,\dots,n-1\}$ and $i_{k+1} - i_k \ge 2,\ \forall k\}$. Consider the partial order on $J_{\frac{r}{2},n-1}$, given by $(i_1, i_2, \dots, i_{\frac{r}{2}}) \le (j_1, j_2, \dots, j_{\frac{r}{2}})$ if $i_k \le j_k$, $\forall k$ and $(i_1, i_2, \dots, i_{\frac{r}{2}}) < (j_1, j_2, \dots, j_{\frac{r}{2}})$ if $i_k < j_k$ for some k. We will prove the theorem by induction on this order.

For $(j_1, j_2, \dots, j_{\frac{r}{2}}) = (n-r+1, n-r+3, \dots, n-1)$, we have

$$(s_{n-r+1}\dots s_1)(s_{n-r+3}\dots s_2)\dots(s_{n-1}\dots s_{\frac{r}{2}})(s_n s_{n-1}\dots s_{\frac{r}{2}+1}) \times (s_n s_{n-1}\dots s_{\frac{r}{2}+2})\dots(s_n s_{n-1}\dots s_r)(\varpi_r) = -\left(\sum_{t=1}^{\frac{r}{2}} \alpha_{n-r+2t-1}\right).$$

Now, if $(i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$ is not maximal, then there exists t maximal such that $i_t < n-r+2t-1$.

Now, $(i_1, i_2, \dots, i_{t-1}, 1+i_t, i_{t+1}, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$ and $(i_1, i_2, \dots, i_{t-1}, 1+i_t, i_{t+1}, \dots, i_{\frac{r}{2}}) > (i_1, i_2, \dots, i_{\frac{r}{2}})$. So, by induction, there exists $w_1 \in W^{I_r}$ such that $w_1\varpi_r = -(\sum_{k\ne t}\alpha_{i_k} + \alpha_{1+i_t})$. Taking $w = s_{1+i_t} s_{i_t} w_1$, we have $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Hence, for any $(i_1, i_2, \ldots, i_{\frac{r}{2}}) \in J_{\frac{r}{2},n-1}$, there exists $w \in W^{I_r}$ of minimal length such that $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Now, we will prove that the w's in W^{I_r} having this property are minimal.

Let $w \in W^{I_r}$ such that $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Suppose w is not minimal. Then there exist $\beta \in R^+$ such that $s_\beta w(\varpi_r) \leq 0$ and $l(s_\beta w) = l(w) - 1$. Since $s_\beta w(\varpi_r) \leq 0$ and $i_{k+1} - i_k \geq 2$, $\forall k$, $\beta = \alpha_{i,k}$ for some $k = 1, 2, \ldots \frac{r}{2}$.

Since $l(s_\beta w) = l(w) - 1$, $\beta = \alpha_{i_t}$ for some t. Hence, $s_\beta w(\varpi_r) = -(\sum_{k\neq t} \alpha_{i_k} +)\alpha_{i_t} \nleq 0$, a contradiction. Thus, all the w's are minimal.

Now, it remains to prove that for all elements of the type $-(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ in the weight lattice such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$, for some k, there does not exist $w \in W^{I_r}$, of minimal length such that $w\varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$.

Let $\mu = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ be such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$ for some k. Choose k minimal such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$.

If $i_k = n - 1$, then $i_{k+1} = 1$ and $s_n w(\varpi_n) = -(\sum_{i_j \neq n} \alpha_{i_j}) > -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Hence, $s_n w < w$, a contradiction to the minimality of w.

Otherwise, $s_{i_k} w(\varpi_r) = -(\sum_{j\neq k} \alpha_{i_j}) > -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$. Hence, $s_{i_k} w < w$, a contradiction to the minimality of w.

$W_{\min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2},n-1}\}$ follows from Lemma 2.1.

(iii) Let r be an odd integer in $\{2, 3, \ldots, n-1\}$. The proof is similar to the case when r is even.

(iv) We have, $\varpi_n = \frac{1}{2} \sum_{i=1}^{n} i\alpha_i$. Then, $2\varpi_n = \sum_{i=1}^{n} i\alpha_i$.

Case 1. *n is even. Take $w_i = s_{2i-1} \ldots s_n$, $i = 1, 2, \ldots \frac{n}{2}$. Let $w = w_{\frac{n}{2}} \ldots w_1$. Then* $w(2\varpi_n) = -\sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1} \leq 0$.

Case 2. *n is odd. Take $w_i = s_{2i-1} \ldots s_n$, $i = 1, 2, \ldots, \frac{n+1}{2}$. Let $w = w_{\frac{n+1}{2}} \ldots w_1$. Then* $w(2\varpi_n) = -\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} \leq 0$.

Proof of (2).

(i) We have, $\varpi_1 = \alpha_1 + \alpha_2 + \cdots + \frac{1}{2}\alpha_n$. Then, $2\varpi_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n$. Take $w = s_n s_{n-1} \ldots s_1$. Then $w(2\varpi_1) = -\alpha_n \leq 0$.

Proof of (ii) and (iii) are similar to Cases (ii) and (iii) of type B_n.

Proof of (3).

(i) We have, $\varpi_1 = \sum_{i=1}^{n-2} \alpha_i + \frac{1}{2}(\alpha_{n-1} + \alpha_n)$. Then, $2\varpi_1 = 2(\sum_{i=1}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_n$. Take $w = s_n s_{n-1} \ldots s_1$. Then $w(2\varpi_1) = -(\alpha_{n-1} + \alpha_n) \leq 0$.

Proof of (ii) and (iii) are similar to Cases (ii) and (iii) of type B_n.

(iv) We have, $\varpi_{n-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2}) + \frac{1}{4}(n\alpha_{n-1} + (n-2)\alpha_n)$. Then, $4\varpi_{n-1} = 2(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2}) + n\alpha_{n-1} + (n-2)\alpha_n$. Take

$$w_i = \begin{cases} \tau_i s_{n-1}, & \text{if } i \text{ is odd,} \\ \tau_i s_n, & \text{if } i \text{ is even,} \end{cases}$$

where $\tau_i = s_{2i-1} \ldots s_{n-2}$, $i = 1, 2, \ldots [\frac{n-1}{2}]$.

Let $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$. Then,

$$w(4\varpi_{n-1}) = \begin{cases} \mu - 2\alpha_n, & \text{if } n \equiv 0 \pmod 4, \\ \mu - 2\alpha_{n-1}, & \text{if } n \equiv 2 \pmod 4, \\ \mu - 2\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n, & \text{if } n \equiv 1 \pmod 4, \\ \mu - 2\alpha_{n-2} - \alpha_{n-1} - 3\alpha_n, & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where $\mu = -2(\sum_{i=1}^{[\frac{n-1}{2}]} \alpha_{2i-1})$.

We have

$$\varpi_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2}) + \frac{1}{4}((n-2)\alpha_{n-1} + n\alpha_n).$$

Then

$$4\varpi_n = 2(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2}) + (n-2)\alpha_{n-1} + n\alpha_n.$$

Take

$$w_i = \begin{cases} \tau_i s_n, & \text{if } i \text{ is odd}, \\ \tau_i s_{n-1}, & \text{if } i \text{ is even}, \end{cases}$$

where $\tau_i = s_{2i-1} \ldots s_{n-2}$, $i = 1, 2, \ldots [\frac{n-1}{2}]$.

Let $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$. Then,

$$w(4\varpi_n) = \begin{cases} \mu - 2\alpha_{n-1}, & \text{if } n \equiv 0 \pmod 4, \\ \mu - 2\alpha_n, & \text{if } n \equiv 2 \pmod 4, \\ \mu - 2\alpha_{n-2} - \alpha_{n-1} - 3\alpha_n, & \text{if } n \equiv 1 \pmod 4, \\ \mu - 2\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n, & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where $\mu = -2(\sum_{i=1}^{[\frac{n-1}{2}]} \alpha_{2i-1})$. □

4. Coxeter elements admitting semistable points

In this section, we describe all Coxeter elements $w \in W$ for which the corresponding Schubert variety $X(w)$ admit a semistable point for the action of a maximal torus with respect to a non-trivial line bundle on G/B.

We now assume that the root system R is irreducible (see page 52 of [2]).

Coxeter elements of Weyl group

An element $w \in W$ is said to be a Coxeter element if it is of the form $w = s_{i_1} s_{i_2} \ldots s_{i_n}$, with $s_{i_j} \neq s_{i_k}$ unless $j = k$ (see page 74 of [3]).

Let $\chi = \sum_{\alpha \in S} a_\alpha \alpha$ be a non-zero dominant weight and let w be a Coxeter element of W.

Lemma 4.1. *If* $w\chi \leq 0$ *and* $\alpha \in S$ *is such that* $l(ws_\alpha) = l(w) - 1$, *then*

(1) $|\{\beta \in S\backslash\{\alpha\}\colon \langle\beta, \check{\alpha}\rangle \neq 0\}| = 1$ *or* 2.

(2) *Further, if* $|\{\beta \in S\backslash\{\alpha\}\colon \langle\beta, \check{\alpha}\rangle \neq 0\}| = 2$, *then* R *must be of type* A_3 *and* χ *is of the form* $a(2\alpha + \beta + \gamma)$ *for some* $a \in \mathbb{Z}_{\geq 0}$, *where* α, β *and* γ *are labelled as* $\circ_\beta$—$\circ_\alpha$—$\circ_\gamma$.

Proof of (1). Since S is irreducible and χ is non-zero dominant weight, a_β is a positive rational number for each $\beta \in S$. Further since $w\chi \leq 0$, χ must be in the root lattice and so a_β is a positive integer for every β in S.

Since w is a Coxeter element and $l(ws_\alpha) = l(w) - 1$,

$$\text{the coefficient of } \alpha \text{ in } w\chi = \text{coefficient of } \alpha \text{ in } s_\alpha\chi. \tag{1}$$

We have

$$\begin{aligned} s_\alpha\chi &= \chi - \langle\chi, \check{\alpha}\rangle\alpha \\ &= \chi - \left\langle \sum_{\beta\in S} a_\beta\beta, \check{\alpha} \right\rangle \alpha \\ &= \sum_{\beta\in S} a_\beta\beta - \sum_{\beta\in S} a_\beta\langle\beta, \check{\alpha}\rangle\alpha. \end{aligned}$$

The coefficient of α in $s_\alpha\chi$ is

$$-\left(\sum_{\beta\in S\backslash\{\alpha\}} \langle\beta, \check{\alpha}\rangle a_\beta + a_\alpha \right). \tag{2}$$

Since $w\chi \leq 0$, from (1) and (2) we have

$$-\left(\sum_{\beta\in S\backslash\{\alpha\}} \langle\beta, \check{\alpha}\rangle a_\beta + a_\alpha \right) \leq 0.$$

Hence,

$$-\left(\sum_{\beta\in S\backslash\{\alpha\}} \langle\beta, \check{\alpha}\rangle a_\beta \right) \leq a_\alpha$$

Thus, we have

$$-2\left(\sum_{\beta\in S\backslash\{\alpha\}} \langle\beta, \check{\alpha}\rangle a_\beta \right) \leq 2a_\alpha. \tag{3}$$

Since χ is dominant, we have

$$\begin{aligned} &\langle\chi, \check{\beta}\rangle \geq 0,\ \forall\beta \in S \\ &\qquad \Rightarrow \left\langle \sum_{\gamma\in S} a_\gamma\gamma, \check{\beta} \right\rangle \geq 0 \\ &\qquad \Rightarrow \sum_{\gamma\in S} a_\gamma\langle\gamma, \check{\beta}\rangle \geq 0. \end{aligned}$$

Now if $\langle\beta, \check{\alpha}\rangle \neq 0$, the left-hand side of the inequality is $2a_\beta - a_\alpha$ (a non-negative integer).

Thus, we have

$$2a_\beta \geq a_\alpha \text{ if } \langle \beta, \check{\alpha} \rangle \neq 0 \tag{4}$$

Now if $|\{\beta \in S\setminus\{\alpha\}: \langle \beta, \check{\alpha} \rangle \neq 0\}| \geq 3$, from (3) and (4) we have

$$3a_\alpha \leq -\left(2 \sum_{\beta \in S\setminus\{\alpha\}} \langle \beta, \check{\alpha} \rangle a_\beta\right) \leq 2a_\alpha.$$

This is a contradiction to the fact that a_α is a positive integer. So

$$|\{\beta \in S\setminus\{\alpha\}: \langle \beta, \check{\alpha} \rangle \neq 0\}| \leq 2.$$

Proof of (2). Suppose $|\{\beta \in S\setminus\{\alpha\}: \langle \beta, \check{\alpha} \rangle \neq 0\}| = 2$. Let β, γ be the two distinct elements of this set.

Using (3) and the facts $\langle \beta, \check{\alpha} \rangle \leq -1$, $\langle \gamma, \check{\alpha} \rangle \leq -1$, we have

$$2(a_\beta + a_\gamma) \leq -2(\langle \beta, \check{\alpha} \rangle a_\beta + \langle \gamma, \check{\alpha} \rangle a_\gamma) \leq 2a_\alpha. \tag{5}$$

Since $\langle \chi, \check{\beta} \rangle \geq 0$ and $\langle \chi, \check{\gamma} \rangle \geq 0$, we have

$$2a_\beta \geq -\sum_{\delta \neq \beta,\alpha} \langle \delta, \check{\beta} \rangle a_\delta + a_\alpha \text{ and } 2a_\gamma \geq -\sum_{\delta \neq \gamma,\alpha} \langle \delta, \check{\gamma} \rangle a_\delta + a_\alpha.$$

Hence

$$-\sum_{\delta \neq \beta,\alpha} \langle \delta, \check{\beta} \rangle a_\delta - \sum_{\delta \neq \gamma,\alpha} \langle \delta, \check{\gamma} \rangle a_\delta + 2a_\alpha \leq 2(a_\beta + a_\gamma).$$

Using (5), we get

$$-\sum_{\delta \neq \beta,\alpha} \langle \delta, \check{\beta} \rangle a_\delta - \sum_{\delta \neq \gamma,\alpha} \langle \delta, \check{\gamma} \rangle a_\delta + 2a_\alpha \leq 2a_\alpha.$$

$$\Rightarrow \sum_{\delta \neq \gamma,\beta,\alpha} \langle -\delta, \check{\beta} \rangle a_\delta + \sum_{\delta \neq \gamma,\beta,\alpha} \langle -\delta, \check{\gamma} \rangle a_\delta \leq 0, \text{ since } \langle \beta, \check{\gamma} \rangle = \langle \gamma, \check{\beta} \rangle = 0.$$

Since each a_δ is positive and $\langle -\delta, \check{\beta} \rangle$, $\langle -\delta, \check{\gamma} \rangle$ are non-negative integers, we have

$$\langle -\delta, \check{\beta} \rangle = 0 \text{ and } \langle -\delta, \check{\gamma} \rangle = 0,\ \forall \delta \neq \alpha, \beta, \gamma.$$

Since R is irreducible, we have $S = \{\alpha, \beta, \gamma\}$. So, from the classification theorem (see pages 57 and 58 of [2]) of irreducible root systems, we have $\langle \beta, \check{\alpha} \rangle \in \{-1, -2\}$.

If $\langle \beta, \check{\alpha} \rangle = -2$, then $\langle \gamma, \check{\alpha} \rangle = -1$.

Hence, from (3) we get

$$4a_\beta + 2a_\gamma \leq 2a_\alpha \tag{6}$$

Again, from (4) we have $2a_\beta \geq a_\alpha$ and $2a_\gamma \geq a_\alpha$. So using (6), we get $3a_\alpha \leq 4a_\beta + 2a_\alpha \leq 2a_\alpha$, a contradiction to the fact that a_α is a positive integer. Thus $\langle \beta, \check{\alpha} \rangle = -1$.

Using a similar argument, we see that $\langle \gamma, \check{\alpha} \rangle = -1$.

Now, let us assume that $\langle \alpha, \check{\beta} \rangle = -2$.

Then,

$$0 \le \langle \chi, \check{\beta} \rangle = a_\gamma \langle \gamma, \check{\beta} \rangle - 2a_\alpha + 2a_\beta$$
$$= -2a_\alpha + 2a_\beta, \text{ since } \langle \gamma, \check{\beta} \rangle = 0$$
$$\Rightarrow 2a_\alpha \le 2a_\beta.$$

From (3), we have

$$2a_\beta + 2a_\gamma \le 2a_\alpha \le 2a_\beta.$$

Hence, $2a_\gamma \le 0$, a contradiction. So $\langle \alpha, \check{\beta} \rangle = -1$. Similarly $\langle \alpha, \check{\gamma} \rangle = -1$.
Hence R is of the type A_3.

$$\circ_\beta \text{———} \circ_\alpha \text{———} \circ_\gamma \ .$$

We now show that $\chi = a(\beta + 2\alpha + \gamma)$, for some $a \in \mathbb{Z}_{\ge 0}$. Let $\chi = a_\alpha \alpha + a_\beta \beta + a_\gamma \gamma$. By assumption, we have $s_\gamma s_\beta s_\alpha(\chi) \le 0$. So $(a_\beta + a_\gamma - a_\alpha)\alpha + (a_\beta - a_\alpha)\gamma + (a_\gamma - a_\alpha)\beta \le 0$. Hence, we have

$$a_\beta + a_\gamma \le a_\alpha \tag{7}$$

Since χ is dominant, we have $\langle \chi, \check{\beta} \rangle \ge 0$ and $\langle \chi, \check{\gamma} \rangle \ge 0$. So we have

$$a_\alpha \le 2a_\beta \text{ and } a_\alpha \le 2a_\gamma \tag{8}$$

Using (7) and (8), $2a_\alpha \ge 2(a_\beta + a_\gamma) \ge 2a_\alpha$. This is possible only if $2a_\beta = a_\alpha = 2a_\gamma$. Then, χ must be of the form $a(\beta + 2\alpha + \gamma)$, for some $a \in \mathbb{Z}_{\ge 0}$. □

Let G be a simple simply connected algebraic group. We now describe all the Coxeter elements $w \in W$ for which $X(w)^{ss}_T(\mathcal{L}_\chi) \ne \emptyset$. For the Dynkin diagrams and labelling of simple roots, we refer to page 58 of [2].

Theorem 4.2.

(A) *Type* A_n.

1. A_3. *For any Coxeter element* w, $X(w)^{ss}_T(\mathcal{L}_\chi) \ne \emptyset$ *for some non-zero dominant weight* χ.
2. $A_n, n \ge 4$. *If* $X(w)^{ss}_T(\mathcal{L}_\chi) \ne \emptyset$ *for some non-zero dominant weight* χ *and* w *is a Coxeter element, then* w *must be either* $s_n s_{n-1} \ldots s_1$ *or* $s_i \ldots s_1 s_{i+1} \ldots s_n$ *for some* $1 \le i \le n-1$.

(B) *Type* B_n.

1. B_2. *For any Coxeter element* w, $X(w)^{ss}_T(\mathcal{L}_\chi) \ne \emptyset$ *for some non-zero dominant weight* χ.
2. $B_n, n \ge 3$. *If* $X(w)^{ss}_T(\mathcal{L}_\chi) \ne \emptyset$ *for some non-zero dominant weight* χ *and* w *is a Coxeter element, then* $w = s_n s_{n-1} \ldots s_1$.

(C) *Type* C_n.

If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ *for some non-zero dominant weight* χ *and* w *is a Coxeter element, then* $w = s_n s_{n-1} \ldots s_1$.

(D) *Type* D_n.

(1) D_4. $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ *for some non-zero dominant weight* χ *and* w *is a Coxeter element if and only if* $l(ws_2) = l(w) + 1$.
(2) $D_n, n \geq 5$. *If* $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ *for some non-zero dominant weight* χ *and* w *is a Coxeter element, then* $w = s_n s_{n-1} \ldots s_1$.

(E) E_6, E_7, E_8.

There is no Coxeter element w *for which there exists a non-zero dominant weight* χ *such that* $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

(F) F_4.

There is no Coxeter element w *for which there exists a non-zero dominant weight* χ *such that* $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

(G) G_2.

There is no Coxeter element w *for which there exist a non-zero dominant weight* χ *such that* $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.

Proof. By Lemma 2.1, $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for a non-zero dominant weight χ if and only if $w\chi \leq 0$. So, using this lemma we investigate all the cases.

Proof of (A).
(1) The Coxeter elements of A_3 are precisely $s_1s_2s_3, s_1s_3s_2, s_2s_1s_3, s_3s_2s_1$. For $w = s_1s_3s_2$, take $\chi = \alpha_1 + 2\alpha_2 + \alpha_3$. Otherwise, take $\chi = \alpha_1 + \alpha_2 + \alpha_3$. Then $w\chi \leq 0$.
(2) Let $n \geq 4$, and let $w\chi \leq 0$ for some dominant weight χ. By Lemma 4.1, if $l(ws_i) = l(w) - 1$, then $i = 1$ or $i = n$.

If $l(ws_n) \neq l(w) - 1$, then using the fact that s_i commute with s_j for $j \neq i-1, i+1$, it is easy to see that $w = s_n s_{n-1} \ldots s_2 s_1$.

If $l(ws_n) = l(w) - 1$, then, let i be the least integer in $\{1, 2, \ldots, n-1\}$ such that $w = \phi s_{i+1} \ldots s_n$, for some $\phi \in W$ with $l(w) = l(\phi) + (n - i)$. Then, we have to show that $\phi = s_i s_{i-1} \ldots s_1$.

If $\phi = \phi_1 s_j$ for some $j \in \{2, 3, \ldots, i-1\}$, then w is of the form

$$\begin{aligned} w &= \phi_1 s_j (s_{i+1} \ldots s_{n-1} s_n) \\ &= \phi_1 (s_{i+1} \ldots s_{n-1} s_n s_j). \end{aligned}$$

This contradicts Lemma 4.1. So $j \in \{1, i\}$. Again $j = i$ is not possible unless $i = 1$ by the minimality of i.

Thus, we have $\phi = s_i \ldots s_1$.

Proof of (B).
(1) For $w = s_1s_2$, take $\chi = \alpha_1 + 2\alpha_2$. For $w = s_2s_1$, take $\chi = \alpha_1 + \alpha_2$.
(2) For $w = s_n s_{n-1} \ldots s_1$, take $\chi = \alpha_1 + \alpha_2 + \ldots \alpha_n$. Then $w\chi = -\alpha_n \leq 0$.

Conversely, let w be a Coxeter element and let χ be a non-zero dominant weight such that $w\chi \leq 0$. By Lemma 4.1, if $l(ws_i) = l(w) - 1$, then either $i = 1$ or $i = n$.

If $l(ws_n) \neq l(w) - 1$, then using the fact that s_i commute with s_j for $j \neq i-1, i+1$, it is easy to see that $w = s_n s_{n-1} \ldots s_2 s_1$.

We now claim that $l(ws_n) = l(w) + 1$. If not, then, the coefficient of α_n in $w\chi$ = coefficient of α_n in $s_n\chi$.

Now, the coefficient of α_n in $s_n\chi$ is $2a_{n-1} - a_n$. Since $w\chi \leq 0$, we have $2a_{n-1} - a_n \leq 0$.

$$\Rightarrow 2a_{n-1} \leq a_n. \tag{1}$$

Since χ is dominant, we have $\langle \chi, \check{\alpha}_{n-1} \rangle \geq 0$. Thus, we get

$$-a_{n-2} + 2a_{n-1} - a_n \geq 0.$$

$$\Rightarrow a_{n-2} \leq 2a_{n-1} - a_n \leq 0, \text{ by (1)}.$$

So $a_{n-2} = 0$, a contradiction to the assumption that $n \geq 3$ and χ is a non-zero dominant weight. Thus $l(ws_n) = l(w) + 1$.

So the only possibility for w is $s_n s_{n-1} \ldots s_1$.

Proof of (C). For $w = s_n s_{n-1} \ldots s_1$, take $\chi = 2(\sum_{i \neq n} \alpha_i) + \alpha_n$. Then, χ is dominant and $w\chi = -\alpha_n$.

Conversely, let w be a Coxeter element and let χ be a non-zero dominant weight such that $w\chi \leq 0$. By Lemma 4.1, if $l(ws_i) = l(w) - 1$, then $i \in \{1, n\}$.

If $l(ws_n) \neq l(w) - 1$, then using the fact s_i commute with s_j for $j \neq i-1, i+1$, it is easy to see that $w = s_n s_{n-1} \ldots s_2 s_1$.

Claim. $l(ws_n) = l(w) + 1$. If not, then, the coefficient of α_n in $w\chi$ = coefficient of α_n in $s_n\chi$.

Now, the coefficient of α_n in $s_n\chi$ is $a_{n-1} - a_n$. Since $w\chi \leq 0$, we have $a_{n-1} - a_n \leq 0$. Hence, we have

$$a_{n-1} \leq a_n. \tag{2}$$

Since χ is dominant, we have $\langle \chi, \check{\alpha}_{n-1} \rangle \geq 0$. Thus, we get

$$-a_{n-2} + 2a_{n-1} - 2a_n \geq 0$$

$$\Rightarrow a_{n-2} \leq 2a_{n-1} - 2a_n \leq 0, \text{ by (2)}.$$

So $a_{n-2} = 0$, a contradiction to the assumption that χ is a non-zero dominant weight.

Thus $l(ws_n) = l(w) + 1$. So the only possibility for w is $s_n s_{n-1} \ldots s_1$.

Proof of (D).

(1) For $w = s_4 s_3 s_2 s_1$, take $\chi = 2(\alpha_1 + \alpha_2) + \alpha_3 + \alpha_4$, for $w = s_4 s_1 s_2 s_3$, take $\chi = 2(\alpha_3 + \alpha_2) + \alpha_1 + \alpha_4$ and for $w = s_3 s_1 s_2 s_4$, take $\chi = 2(\alpha_4 + \alpha_2) + \alpha_1 + \alpha_3$.

The converse follows from Lemma 4.1.

(2) For $w = s_n s_{n-1} \ldots s_1$, take $\chi = 2(\sum_{i=1}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_n$. Then $w\chi \leq 0$.

Conversely, let w be a Coxeter element and let χ be a non-zero dominant weight such that $w\chi \leq 0$. By Lemma 4.1, if $l(ws_i) = l(w) - 1$ then $i \in \{1, n-1, n\}$.

Now, if $l(ws_1) = l(w) - 1$, then, it is easy to see that $w = s_n s_{n-1} \ldots s_2 s_1$.

So, it is sufficient to prove that $l(ws_n) = l(w) + 1$ and $l(ws_{n-1}) = l(w) + 1$.

If $l(ws_n) = l(w) - 1$, then, the coefficient of α_n in $w\chi$ = coefficient of α_n in $s_n\chi = a_{n-2} - a_n$.

Since $w\chi \leq 0$, we have

$$a_{n-2} - a_n \leq 0. \tag{4}$$

Since χ is dominant we have $\langle \chi, \check{\alpha}_{n-2} \rangle \geq 0$. Therefore, we have

$$2a_{n-2} \geq a_{n-1} + a_{n-3} + a_n. \tag{5}$$

Also, since $\langle \chi, \check{\alpha}_{n-1} \rangle \geq 0$ and $\langle \chi, \check{\alpha}_{n-3} \rangle \geq 0$, we have

$$2a_{n-1} - a_{n-2} \geq 0 \tag{6}$$

and

$$2a_{n-3} - a_{n-4} - a_{n-2} \geq 0. \tag{7}$$

From (5), we get

$$\begin{aligned} 4a_{n-2} &\geq 2a_{n-1} + 2a_{n-3} + 2a_n \\ &\geq a_{n-2} + (a_{n-4} + a_{n-2}) + 2a_n, \text{ from (6) and (7)} \\ &\geq 2a_{n-2} + 2a_{n-2} + a_{n-4}, \text{ by (4)} \\ &= 4a_{n-2} + a_{n-4}. \end{aligned}$$

So $a_{n-4} = 0$, a contradiction to the assumption that χ is a non-zero dominant weight. So $l(ws_n) = l(w) + 1$.

Using a similar argument, we can show that $l(ws_{n-1}) = l(w) + 1$.

Proof of (E).

Type E_8. Let w be a Coxeter element and let χ be a non-zero dominant weight χ such that $w\chi \leq 0$. Further, if $l(ws_i) = l(w) - 1$, then by Lemma 4.1, $i \in \{1, 2, 8\}$.

Case 1. $i = 8$. Co-efficient of α_8 in $w\chi$ = co-efficient of α_8 in $s_8(\chi) = a_7 - a_8 \leq 0$.

Since χ is dominant, $\langle \chi, \check{\alpha}_i \rangle \geq 0,\ \forall i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$\langle \chi, \check{\alpha}_7 \rangle \geq 0 \Rightarrow 2a_7 \geq a_6 + a_8 \geq a_6 + a_7.$$

Hence, we have $a_7 \geq a_6$.

$$\begin{aligned} \langle \chi, \check{\alpha}_6 \rangle \geq 0 \Rightarrow 2a_6 \geq a_5 + a_7 &\geq a_5 + a_6 \\ &\Rightarrow a_6 \geq a_5. \\ \langle \chi, \check{\alpha}_5 \rangle \geq 0 \Rightarrow 2a_5 \geq a_4 + a_6 &\geq a_4 + a_5. \\ &\Rightarrow a_5 \geq a_4 \\ \langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 &\geq a_1 + a_4. \\ \langle \chi, \check{\alpha}_2 \rangle \geq 0 \Rightarrow 2a_2 &\geq a_4. \end{aligned}$$

Now,

$$\begin{aligned}\langle \chi, \check{\alpha}_4 \rangle \geq 0 \Rightarrow 2a_4 &\geq a_2 + a_3 + a_5 \\ \Rightarrow 4a_4 &\geq 2a_2 + 2a_3 + 2a_5. \\ &\geq a_4 + a_1 + a_4 + 2a_4, \text{ since } a_5 \geq a_4.\end{aligned}$$

So, $a_1 = 0$. Thus in this case, there is no Coxeter element w for which there is a non-zero dominant weight such that $w\chi \leq 0$.

Case 2. $i = 1$. Co-efficient of α_1 in $w\chi$ = co-efficient of α_1 in $s_1\chi = a_3 - a_1 \leq 0$.

Since χ is dominant, we have $\langle \chi, \check{\alpha}_3 \rangle \geq 0$. Therefore, $2a_3 \geq a_1 + a_4 \geq a_3 + a_4$

Hence, we have $a_3 \geq a_4$. Since $\langle \chi, \check{\alpha}_4 \rangle \geq 0$, we have $2a_4 \geq a_3 + a_2 + a_5$. Since, $\langle \chi, \check{\alpha}_2 \rangle \geq 0$ and $\langle \chi, \check{\alpha}_5 \rangle \geq 0$ we have $2a_2 \geq a_4$ and $2a_5 \geq a_4 + a_6$. Then, $4a_4 \geq 2a_3 + 2a_2 + 2a_5 \geq 2a_4 + a_4 + a_4 + a_6$, from the above inequalities.

So, $a_6 = 0$. Hence we have $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exists a non-zero dominant weight χ such that $w\chi \leq 0$.

Case 3. $i = 2$. Co-efficient of α_2 in $w\chi$ = co-efficient of α_2 in $s_2\chi = a_4 - a_2 \leq 0$.

Since χ is dominant, $\langle \chi, \check{\alpha}_i \rangle \geq 0$, $\forall i \in \{1, 2, 3, 4, 5, 6\}$.

$$\langle \chi, \check{\alpha}_5 \rangle \geq 0 \Rightarrow 2a_5 \geq a_4 + a_6.$$

$$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq a_1 + a_4.$$

$$\langle \chi, \check{\alpha}_4 \rangle \geq 0 \Rightarrow 2a_4 \geq a_3 + a_2 + a_5.$$

Hence, we have $4a_4 \geq 2a_3 + 2a_2 + 2a_5$.

$$\geq (a_1 + a_4) + 2a_4 + (a_4 + a_6) = a_1 + a_6 + 4a_4.$$

$\Rightarrow a_1 + a_6 = 0$. So, $a_1 = a_6 = 0$.

Hence, we have $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exists a non-zero dominant weight χ such that $w\chi \leq 0$.

Type E_6, E_7. Proof is similar to the case of E_8.

Proof of (F). Let w be a Coxeter element. Let χ be a non-zero dominant weight such that $w\chi \leq 0$. If $l(ws_i) = l(w) - 1$, then $i \in \{1, 4\}$, by Lemma 4.1.

Case 1. $i = 1$. Co-efficient of α_1 in $w\chi$ = co-efficient of α_1 in $s_1\chi = a_2 - a_1 \leq 0$. Since χ is dominant, we have $\langle \chi, \check{\alpha}_3 \rangle \geq 0$ and $\langle \chi, \check{\alpha}_2 \rangle \geq 0$.

$$\langle \chi, \check{\alpha}_2 \rangle \geq 0 \Rightarrow 2a_2 \geq a_1 + a_3 \geq a_2 + a_3, \text{ since } a_2 \leq a_1.$$

Hence, we have $a_2 \geq a_3$.

$$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq 2a_2 + a_4 \geq 2a_3 + a_4.$$

So, we have $a_4 = 0$. Hence, $\chi = 0$. Thus, in this case there is no Coxeter element w for which there exists a non-zero dominant weight χ such that $w\chi \leq 0$.

Case 2. $i = 4$. Co-efficient of α_4 in $w\chi$ = co-efficient of α_4 in $s_4\chi = a_3 - a_4 \leq 0$.
Since χ is dominant, we have $\langle \chi, \check{\alpha}_3 \rangle \geq 0$ and $\langle \chi, \check{\alpha}_2 \rangle \geq 0$.

$$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq 2a_2 + a_4 \geq 2a_2 + a_3, \text{ since } a_3 \leq a_4.$$

Hence, we have $a_3 \geq 2a_2$.

$$\langle \chi, \check{\alpha}_2 \rangle \geq 0 \Rightarrow 2a_2 \geq a_1 + a_3 \geq a_1 + 2a_2.$$

So, we have $a_1 = 0$. Hence, $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exists a non-zero dominant weight χ such that $w\chi \leq 0$.

Proof of (G). Let w be a Coxeter element and $\chi = a_1\alpha_1 + a_2\alpha_2$, be a dominant weight such that $w\chi \leq 0$

Case 1. $l(ws_1) = l(w) - 1$. Co-efficient of α_1 in $w\chi$ = co-efficient of α_1 in $s_1\chi = a_2 - a_1 \leq 0$.
Since χ is dominant, we have $\langle \chi, \check{\alpha}_2 \rangle \geq 0$.

$$\Rightarrow 2a_2 \geq 3a_1 \geq 3a_2.$$

So, we have $a_2 = 0$. Hence, $\chi = 0$. Thus, in this case, there is no Coxeter element w for which there exist a non-zero dominant weight χ such that $w\chi \leq 0$.

Case 2. $l(ws_2) = l(w) - 1$. Co-efficient of α_2 in $w\chi$ = co-efficient of α_2 in $s_2\chi = 3a_1 - a_2 \leq 0$. Since χ is dominant, we have $\langle \chi, \check{\alpha}_1 \rangle \geq 0$.

$$\Rightarrow 2a_1 \geq a_2 \geq 3a_1.$$

So, we have $a_1 = 0$. Hence, $\chi = 0$. Thus, in this case also, there is no Coxeter element w for which there exists a non-zero dominant weight χ such that $w\chi \leq 0$. □

We now turn to the general case. Let G be a semisimple simply connected algebraic group. Then G is of the form $G = \prod_{i=1}^{r} G_i$, for some simple simply connected algebraic groups $G_1, \ldots, G_r$. So, a maximal torus T (resp. a Borel subgroup B containing T) is of the form $\prod_{i=1}^{r} T_i$ (resp. $\prod_{i=1}^{r} B_i$), where each T_i is a maximal torus of G_i, and each B_i is a Borel subgroup of G_i containing T_i. Also the Weyl group of G with respect to T is of the form $\prod_{i=1}^{r} W_i$, where each W_i is the Weyl group of G_i with respect to T_i.

Now, let $\chi = (\chi_1, \ldots \chi_r) \in \oplus_{i=1}^{r} X(T_i)$ be a dominant weight, where $X(T_i)$ denote the group of characters of T_i. Then, clearly each χ_i is dominant. Let $w = (w_1, w_2, \ldots, w_r) \in \prod_{i=1}^{r} W_i$ be a coxeter element of W. Then, each w_i is a coxeter element.

Then, we have

Theorem 4.3. *$X(w)_T^{\mathrm{ss}}(\mathcal{L}_\chi) \neq \emptyset$ if and only if w_i must be as in Theorem* 4.2 *for all i such that χ_i is nonzero.*

Proof. Follows from Theorem 4.2 and the fact that $w\chi \leq 0$ if and only if $w_i\chi_i \leq 0$ for all $i = 1, 2, \ldots, r$. □

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