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On Artinian generalized local cohomology modules

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Abstract. Let R be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ be a maximal ideal of R. An R-module M is called *minimax* if there is a finitely generated submodule N of M such that M/N is Artinian. Over a Gorenstein local ring R of finite Krull dimension, we proved that the Socle of $H^n_{\mathfrak{a}}(R)$ is a minimax R-module for each $n \geq 0$.

Keywords. Generalized local cohomology modules; Artinian modules; minimax modules.

1. Introduction

Throughout this paper, we will assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R and M, N are two R-modules.

For each $i \geq 0$, the i-th local cohomology module of N with respect to an ideal $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(N) = \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, N).$$

For an R-module N, one may identify $\operatorname{Hom}_R(R/\mathfrak{a}, N)$ with the submodule $\{x \in N | \mathfrak{a}x = 0\}$, which is an $R/\mathfrak{a}$-vector space called the Socle of N. In general, it is well-known that the local cohomology modules are not finitely generated. So we have the famous problem concerning local cohomology theory (cf. [6]):

(i) When is the Socle of $H^n_{\mathfrak{a}}(N)$ finitely generated?

It is known that if R is an unramified regular local ring, then the local cohomology modules $H^i_{\mathfrak{a}}(R)$ have finite dimensional Socles for all $i \geq 0$ and all ideals I of R. The first example of a local cohomology module with an infinite dimensional Socle was given by Hartshorne [4].

Let M be an R-module. According to Zöschinger [9], M is said to be a *minimax module* if there is a finitely generated submodule N of M such that M/N is Artinian. Clearly, all finitely generated modules and all Artinian modules are minimax. Lorestani, Sahandi and Yassemi proved the following in Theorem 2.2 of [7]. Let s be a non-negative integer and M be an R-module such that $\operatorname{Ext}^s_R(R/_{\mathfrak{a}}, M)$ is a minimax R-module. If $H^i_{\mathfrak{a}}(M)$ is minimax for all $i < s$, then $\operatorname{Hom}_R(R/_{\mathfrak{a}}, H^s_{\mathfrak{a}}(M))$ is a minimax module.

Since the concept of minimax module is a natural generalization of the concept of finitely generated modules, we have the following natural question.

(ii) When is the Socle of $H^n_{\mathfrak{a}}(N)$ minimax?

In this paper, we provide some partial answer to question (ii).

For each $i \geq 0$, the *i-th generalized local cohomology module* of (M, N) with respect to an ideal $\mathfrak{a}$ is

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

Clearly, $H^i_{\mathfrak{a}}(R, N) \cong H^i_{\mathfrak{a}}(N)$ for all $i \geq 0$.

The *generalized transform factor* with respect to an ideal $\mathfrak{a}$ of R is

$$D_{\mathfrak{a}}(M, -) \cong \varinjlim_{n \in \mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n M, -).$$

Throughout the paper, we freely use the conventions of the notions for commutative algebra from the book [8]. And we use the well-known theorems concerning ordinary (generalized) local cohomology without citing any references for which the reader should refer to [2], [5] and [3].

2. Results

We begin with the following useful lemmas.

Lemma 1. *Let* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of* R*-modules. Then* M *is minimax if and only if* L *and* N *are minimax modules. Hence, if* $L \longrightarrow M \longrightarrow N$ *is an exact sequence such that both end terms are minimax* R*-modules, then* M *is also minimax* [1].

Lemma 2. *Let* $s \geq 0$ *and* $\mathfrak{a} \subseteq \mathfrak{b}$ *be ideals of a commutative Noetherian ring* R. *Let* M *and* N *be two* R*-modules (not necessarily minimax). Assume that* $\operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N))$ *and* $\operatorname{Ext}^2_R(R/\mathfrak{b}, H^o_{\mathfrak{a}}(M, N))$ *are minimax* R*-modules and* $\operatorname{Ext}^1_R(M, N) = 0$. *Then* $\operatorname{Hom}_R(R/J, H^1_{\mathfrak{a}}(M, N))$ *is minimax.*

Proof. Assume that $\operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N))$ and $\operatorname{Ext}^2_R(R/\mathfrak{b}, H^o_{\mathfrak{a}}(M, N))$ are minimax modules. Consider the exact sequence

$$0 \to H^0_{\mathfrak{a}}(M, N) \subseteq \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N)/H^0_{\mathfrak{a}}(M, N) \to 0.$$

Then we have the long exact sequence

$$\begin{aligned} \cdots &\to \operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N)) \\ &\to \operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N)/H^0_{\mathfrak{a}}(M, N)) \\ &\to \operatorname{Ext}^2_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M, N)) \to \cdots . \end{aligned}$$

This implies that $\operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N)/H^0_{\mathfrak{a}}(M, N))$ is minimax by Lemma 1. Then the sequence

$$\begin{aligned} \cdots &\to \operatorname{Hom}_R(R/\mathfrak{b}, D_{\mathfrak{a}}(M, N)) \\ &\to \operatorname{Hom}_R(R/\mathfrak{b}, H^1_{\mathfrak{a}}(M, N)) \\ &\to \operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N)/H^0_{\mathfrak{a}}(M, N)) \to \cdots \end{aligned}$$

is exact. Since $D_{\mathfrak{a}}(M, N)$ contains no non-zero element annihilated by $\mathfrak{b}$, the module $\operatorname{Hom}_R(R/\mathfrak{b}, D_{\mathfrak{a}}(M, N))$ is zero. Since $\operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N)/H^0_{\mathfrak{a}}(M, N))$ is minimax, so is $\operatorname{Hom}_R(R/\mathfrak{b}, H^1_{\mathfrak{a}}(M, N))$ by Lemma 1. $\square$

The following theorem is inspired by a result of Lorestani, Sahandi and Yassemi for local cohomology modules (cf. Theorem 2.2 of [7]), but it concerns the setting of general local cohomology modules. Let

$$E^\bullet \colon 0 \longrightarrow N \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots,$$

be a (fix) minimal injective coresolution of N. Also, suppose that $\Omega_i(N)$ is the cosyzygy of N in $E^\bullet$. Moreover, we set $E_0 := \Omega_0(N)(= N)$.

Theorem 3. *Let $s \geq 0$, $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of a commutative Noetherian ring R. Let M and N be two R-modules (not necessarily minimax).*

(1) *For $s = 0$, assume that $\operatorname{Hom}_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, N))$ is minimax.*
For $s \geq 1$, assume that $\operatorname{Ext}^s_R(M, N) = 0$, and that $\operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, E_{s-1}))$ and $\operatorname{Ext}^2_R(R/\mathfrak{b}, H^0_{\mathfrak{a}}(M, \Omega_{s-1}(N)))$ are minimax R-modules.
Then $\operatorname{Hom}_R(R/\mathfrak{b}, H^s_{\mathfrak{a}}(M, N))$ is a minimax R-module. Furthermore $\operatorname{Supp}_R(F) \subseteq V(\mathfrak{b})$, then $\operatorname{Hom}_R(F, H^s_{\mathfrak{a}}(M, N))$ is a minimax R-module.

(2) *The following statements are equivalent.*

  (i) $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M, N))$ *is a minimax module for all $s \geq 0$.*
  (ii) $\operatorname{Hom}_R(R/\mathfrak{b}, H^s_{\mathfrak{a}}(M, N))$ *is a minimax module for all $s \geq 0$.*
  (iii) $\operatorname{Hom}_R(R/P, H^s_{\mathfrak{a}}(M, N))$ *is a minimax module for all $s \geq 0$ and all primes P minimal over $\mathfrak{a}$.*

Proof.
(1) We prove the claim by induction on s. For $s = 0$, proof is clear. If $s = 1$, then, by Lemma 2, $\operatorname{Hom}_R(R/\mathfrak{b}, H^1_{\mathfrak{a}}(M, N))$ is a minimax module.

Now suppose that $s > 1$ and assume that the claim is true for $t := s - 1$. We will prove it for s. Note that

$$\begin{aligned} H^s_{\mathfrak{a}}(M, N) &= \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^s_R(M/\mathfrak{a}^n M, N) \\ &\cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^t_R(M/\mathfrak{a}^n M, \Omega_1(N)) = H^t_{\mathfrak{a}}(M, \Omega_1(N)). \end{aligned}$$

So it suffices to prove that $\operatorname{Hom}_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M, \Omega_1(N)))$ is minimax.

Clearly, $\operatorname{Ext}^t_R(M, \Omega_1(N)) = 0$ and $\operatorname{Ext}^1_R(R/\mathfrak{b}, \operatorname{Hom}_R(M, E_{1+t-1}))$ is a minimax R-module; but this is just the second assumption for (M, N). The third assumptions for $(M, \Omega_1(N))$ and (M, N) coincide because it is easy to see that $\Omega_{t-1}\Omega_1(N) = \Omega_t(N)$.

For the last assertion, we first recall a result of Gruson that says that F contains a finite chain $0 = F_0 \subseteq \cdots \subseteq F_n = F$ such that F_i/F_{i-1} is a homomorphic image of a finite direct sum of copies of $R/\mathfrak{a}$ for each $0 < i \leq n$. Without loss of generality, we can assume that $n = 1$, so there is an epimorphism $(R/\mathfrak{b})^m \to F$ for some $m > 0$. Then $\operatorname{Hom}_R(F, H^s_{\mathfrak{a}}(M, N)) \subseteq \operatorname{Hom}_R((R/\mathfrak{b})^m, H^s_{\mathfrak{a}}(M, N))$ and the latter module is minimax by the above and Lemma 2.

(2) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Let $P_1, P_2, \ldots, P_n$ be the minimal primes over $\mathfrak{a}$. Let $M = R/P_1 \oplus R/P_2 \oplus \cdots \oplus R/P_n$. Then $\operatorname{Hom}_R(R/\mathfrak{b}, H^s_{\mathfrak{a}}(M, N))$ is a minimax module. Since $\operatorname{Supp}_R(R/\mathfrak{a}) = \operatorname{Supp}_R(M)$, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M, N))$ is a minimax module by (1). □

We are now in a position to prove our main result.

Theorem 4. *Let R be a local Gorenstein ring of Krull dimension k. Then* $\operatorname{Soc}_R(G_i/S_i)$ *is finitely generated (minimax) for each $1 \leq i < k$ if and only if $H^n_{\mathfrak{a}}(R)$ is a minimax R-module for each $n \geq 0$.*

Proof.
($:\Rightarrow$) In order to apply Theorem 3(1), we put $s = n$, $M = N = R$, $\mathfrak{b} = \mathfrak{m}$ and $F = R/\mathfrak{m}$ where $\mathfrak{m}$ denotes the maximal ideal of R.

Step I. Note that $\operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Hom}_R(R, R)) \cong \operatorname{Hom}_R(R/\mathfrak{m}, R) \cong \operatorname{Soc}(R)$ is finitely generated and $\operatorname{Ext}^n_R(R, R) = 0$ since R is projective.

For $n = 1$, clearly $\operatorname{Ext}^1_R(R/\mathfrak{m}, \operatorname{Hom}_R(R, E_0)) = \operatorname{Ext}^1_R(R/\mathfrak{m}, R)$ is minimax.

If $n > 1$, then $\operatorname{Ext}^1_R(R/\mathfrak{m}, \operatorname{Hom}_R(R, E_{n-1})) = \operatorname{Ext}^1_R(R/\mathfrak{m}, E_{n-1}) = 0$ because E_{n-1} is injective.

Step II. Now we claim that $\operatorname{Ext}^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}R))$ is minimax for each $n \geq 1$. We recall that $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$ for a module M, where

$$\Gamma_{\mathfrak{a}}(M) = \{x \in M \mid \exists k \geq 0\colon \mathfrak{a}^k x = 0\}.$$

In particular,

$$\Gamma_{\mathfrak{a}}(E(R/p)) = \begin{cases} E(R/p), & \text{if } p \in V(\mathfrak{a}) \\ 0, & \text{if } p \notin V(\mathfrak{a}). \end{cases}$$

For $n = 1$, we have $H^0_{\mathfrak{a}}(\Omega_0(R)) = H^0_{\mathfrak{a}}(R) \subseteq \operatorname{Hom}_R(R, R) \cong R$, so $H^0_{\mathfrak{a}}(\Omega_0(R))$ is minimax. This implies that $\operatorname{Ext}^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}R))$ is minimax.

If $n > k$, then $\Gamma_{\mathfrak{a}}(\Omega_{n-1}(R)) = \Omega_{n-1}(R)$ is injective. So we have

$$\operatorname{Ext}^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}(R))) = 0.$$

Assume $1 < n \leq k$. Then $\Omega_{n-1}(R)$ is an essential submodule of K_{n-1}. So $\Gamma_{\mathfrak{a}}(\Omega_{n-1}(R)) = \Omega_{n-1}(R) \cap K_{n-1}$ and K_{n-1} is the injective envelope of $\Gamma_{\mathfrak{a}}(\Omega_{n-1}(R))$. Hence

$$\begin{aligned} K_{n-1}/H^0_{\mathfrak{a}}(\Omega_{n-1}(R)) &= K_{n-1}/\Gamma_{\mathfrak{a}}(\Omega_{n-1}(R)) \\ &\cong S_{n-1}/\Omega_{n-1}(R) \\ &\subseteq G_{n-1}/\Omega_{n-1}(R) \\ &= \Omega_n(R) \subseteq G_n. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{Ext}^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}(R))) &= \operatorname{Ext}^2_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(\Omega_{n-1}(R))) \\ &\cong \operatorname{Ext}^1_R(R/\mathfrak{m}, S_{n-1}/\Omega_{n-1}(R)). \end{aligned}$$

Now we have four subcases on k.

If $n = k$, then $G_n = E(R/\mathfrak{m}) = \Omega_n(R)$ is an injective module containing $S_{n-1}/\Omega_{n-1}(R)$. So in order to prove that $\operatorname{Ext}^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}(R)))$ is minimax, it is enough to show that $\operatorname{Hom}_R(R/\mathfrak{m}, G_n/(S_{n-1}/\Omega_{n-1}(R))$ is minimax. This is clear because the module G_n, and hence also $G_n/(S_{n-1}/\Omega_{n-1}(R))$ is Artinian.

Now, assume $1 < n < k$. In order to prove that $\operatorname{Ext}^1_R(R/\mathfrak{m}, S_{n-1}/\Omega_{n-1}(R))$ is minimax, it suffices to show that $\operatorname{Hom}_R(R/\mathfrak{m}, G_n/(S_{n-1}/\Omega_{n-1}(R)))$ is such.

We have the exact sequence

$$0 \to (G_{n-1}/\Omega_{n-1}(R))/(S_{n-1}/\Omega_{n-1}(R)) \subseteq G_n/(S_{n-1}/\Omega_{n-1}(R))$$
$$\to \Omega_{n+1}(R) = G_n/(G_{n-1}/\Omega_{n-1}(R)) \to 0.$$

Note that if $n < k-1$, then $\operatorname{Hom}_R(R/\mathfrak{m}, G_{n+1}) = 0$ by Theorem 3.3.8(5) of [3] and if $n = k-1$, then $\operatorname{Hom}_R(R/\mathfrak{m}, G_{n+1}) = \operatorname{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}) \cong \operatorname{Soc}_R(E(R/\mathfrak{m})) \cong R/\mathfrak{m}$. So in either case $\operatorname{Hom}_R(R/\mathfrak{m}, G_n/(G_{n-1}/\Omega_{n-1}(R)))$ is minimax. Now, by assumption, $\operatorname{Hom}_R(R/\mathfrak{m}, G_{n-1}/S_{n-1})$ is minimax.
($\Leftarrow$:) First, note that $\operatorname{Hom}_R(R/\mathfrak{m}, H^1_I(\Omega_i(R)))$ is a minimax R-module for each $i \geq 1$, because $H^n_I(N) \cong H^{n-1}_I(\Omega_1(N))$ for each module N and each $n > 1$. By Remark 2.2.7 of [2] and the injectivity of G_i, we have the exact sequence

$$0 \to G_i/\Gamma_{\mathfrak{a}}(G_i) \to D_{\mathfrak{a}}(G_i) \to H^1_{\mathfrak{a}}(G_i) = 0,$$

where $G_i/\Gamma_{\mathfrak{a}}(G_i)$ is injective by Corollary 2.1.5 of [2], so $\operatorname{Ext}^1_R(R/\mathfrak{m}, D_{\mathfrak{a}}(G_i)) = 0$. So

$$0 = \operatorname{Hom}_R(R/\mathfrak{m}, T_i) \to \operatorname{Ext}^1_R(R/\mathfrak{m}, D_{\mathfrak{a}}(\Omega_i(R)))$$
$$\to \operatorname{Ext}^1_R(R/\mathfrak{m}, D_{\mathfrak{a}}(G_i)) = 0.$$

Now we have the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{m}, H^1_{\mathfrak{a}}(\Omega_i(R))) \to \operatorname{Ext}^1_R(R/\mathfrak{m}, \Omega_i(R)/\Gamma_{\mathfrak{a}}(\Omega_i(R)))$$
$$\to \operatorname{Ext}^1_R(R/\mathfrak{m}, D_{\mathfrak{a}}(\Omega_i(R))) = 0$$

by Remark 2.2.7 of [2]. Hence the module $\operatorname{Ext}^1_R(R/\mathfrak{m}, \Omega_i(R)/\Gamma_{\mathfrak{a}}(\Omega_i(R)))$ is minimax. Therefore,

$$\Omega_i(R)/\Gamma_{\mathfrak{a}}(\Omega_i(R)) \cong S_i/K_i \subseteq G_i/K_i,$$

where $G_i/K_i \cong \bigoplus_{p \in (P_i \setminus Q_i)} E(R/p)$ is injective. Since $i < k$, $\operatorname{Hom}_R(R/\mathfrak{m}, G_i/K_i) = 0$, so $\operatorname{Hom}_R(R/\mathfrak{m}, G_i/S_i) \cong \operatorname{Ext}^1_R(R/\mathfrak{m}, S_i/K_i)$ is minimax. □

Remark. Let R be a local Gorenstein ring of Krull dimension ≤ 2. As a result of Theorem 4, we can obtain that the Socle of $H^n_I(R))$ is a minimax R-module for each $n \geq 0$. Because $G_2/(S_1/\Omega_1(R))$ is Artinian for $k = 2$ (see the proof (:$\Rightarrow$) in Theorem 4), so the module $\operatorname{Soc}_R(G_1/S_1)$ is minimax.

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