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Cohomology with coefficients for operadic coalgebras

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Abstract. Corepresentations of a coalgebra over a quadratic operad are defined, and various characterizations of them are given. Cohomology of such an operadic coalgebra with coefficients in a corepresentation is then studied.

Keywords. Quadratic operad; homology; cohomology; coalgebra.

1. Introduction

Many classical cohomology theories, such as Hochschild cohomology of associative algebras [9], Harrison cohomology of commutative algebras [8], Chevalley–Eilenberg cohomology of Lie algebras [3, 21], and Loday's cohomology of Leibniz algebras [11–14], are examples of cohomology of operadic algebras. The cohomology of operadic algebras with coefficients were defined for the first time by Fox and Markl in Definition 8.3 of [5]. In [1, 2], Balavoine constructed the cohomology theory with non-trivial coefficients for an algebra A over a quadratic operad $\mathcal{P}$ and gave explicit formulas for the coboundary. When $\mathcal{P}$ is taken to be the operads for associative, commutative, Lie or Leibniz algebras, one recovers the classical cohomology theories for those types of algebras. Moreover, the cohomology cochain complex $C^*_{\mathcal{P}}(A, A)$ for a $\mathcal{P}$-algebra A with self-coefficients is the object that governs the deformations of A in the sense of Gerstenhaber [6].

The deformation complex $\bar{C}^*_{\mathcal{P}}(V, V)$ of a $\mathcal{P}$-*coalgebra* V and the dual notion to $C^*_{\mathcal{P}}(A, A)$, was studied in [15]. Thinking of $\bar{C}^*_{\mathcal{P}}(V, V)$ as the cohomology cochain complex of V with self-coefficients, there should be a cohomology theory for the $\mathcal{P}$-coalgebra V with non-trivial coefficients. This is exactly the subject of this paper.

In the first part of this paper, we study the coefficients, called *corepresentations*, for the cohomology theory of a $\mathcal{P}$-coalgebra V. Corepresentations of a $\mathcal{P}$-coalgebra are defined in a way that is very similar to the notion of left-comodules over a coassociative coalgebra. Several equivalent characterizations of V-corepresentations are then given. We show that a V-corepresentation can be described in terms of (i) a component map of a specific operad morphism, (ii) the dual $\mathcal{P}$-algebra $V^{\#}$, or (iii) an enveloping coassociative coalgebra.

In the second part of this paper, the cohomology cochain complex $\bar{C}^*_{\mathcal{P}}(M, V)$ of a $\mathcal{P}$-coalgebra V with coefficients in a corepresentation M is defined. It is constructed as a certain subcomplex of the deformation complex $\bar{C}^*_{\mathcal{P}}(C, C)$ [15], where C is a $\mathcal{P}$-coalgebra with $C = V \oplus M$ as a vector space. We will describe the differential in $\bar{C}^*_{\mathcal{P}}(M, V)$ very explicitly using the $\circ_i$ operations in the quadratic operad $\mathcal{P}$. In particular, if $\mathcal{P}$ is the operad for associative algebras, then a $\mathcal{P}$-coalgebra V is a coassociative coalgebra, and a V-corepresentation is exactly a V-bicomodule. In this case, our $\bar{C}^*_{\mathcal{P}}(M, V)$ coincides with the cochain complex for Hochschild coalgebra cohomology [4, 10, 20].

Modifying the standard arguments slightly, a description of $\bar{H}^2_{\mathcal{P}}(M, V)$, the second cohomology module of $\bar{C}^*_{\mathcal{P}}(M, V)$, in terms of extensions of $\mathcal{P}$-coalgebras will be given. Moreover, it will be shown that our $\bar{C}^*_{\mathcal{P}}(M, V)$ is canonically isomorphic to Balavoine's $C^*_{\mathcal{P}}(V^\#, M^\#)$, where $M^\#$ is the linear dual of M. Passing to cohomology, this implies that our cohomology $\bar{H}^*_{\mathcal{P}}(M, V)$ is canonically isomorphic to Balavoine's $H^*_{\mathcal{P}}(V^\#, M^\#)$. This duality isomorphism in cohomology, in the special case when $\mathcal{P}$ is the operad for associative algebras, was first observed by Parshall and Wang [20].

1.1 *Organization*

The rest of this paper is organized as follows. The following section is a preliminary section, in which definitions about operads and their (co)algebras are recalled. We also fix some notations that will be used in later sections.

The purpose of §3 is to define a *corepresentation* of a coalgebra V over a quadratic operad $\mathcal{P}$. The first characterization of a V-corepresentation is given in terms of a vanishing property of a component map of a certain operad morphism (Proposition 3.5 and Theorem 3.6).

In §4, another characterization of a V-corepresentation is given in terms of the dual $\mathcal{P}$-algebra $V^\#$. The main result of that section (Corollary 4.8) states that a module M is a V-corepresentation if and only if its linear dual $M^\#$ is a representation of the dual $\mathcal{P}$-algebra $V^\#$ in the sense of Balavoine [1].

A third characterization of a V-corepresentation is given in §5. It is shown (Theorem 5.4) that there exists an enveloping coassociative coalgebra whose left-comodules in the usual sense are exactly the V-corepresentations. This correspondence is made explicit in §5.5.

In §6, we define the cochain complex $\bar{C}^*_{\mathcal{P}}(M, V)$ of a $\mathcal{P}$-coalgebra V with coefficients in a V-corepresentation M. It is constructed using the deformation complex $\bar{C}^*_{\mathcal{P}}(C, C)$ of a $\mathcal{P}$-coalgebra C [15]. Since the differential in $\bar{C}^*_{\mathcal{P}}(C, C)$ can be described explicitly in terms of the $\circ_i$ operations in $\mathcal{P}$, the same can be done for the differential in $\bar{C}^*_{\mathcal{P}}(M, V)$ (Theorems 6.8 and 6.9).

In §7, as in the case of associative algebras, the second cohomology module $\bar{H}^2_{\mathcal{P}}(M, V)$ of $\bar{C}^*_{\mathcal{P}}(M, V)$ is identified with a certain set of equivalence classes of singular extensions of $\mathcal{P}$-coalgebras (Theorem 7.1).

In §8, our cochain complex $\bar{C}^*_{\mathcal{P}}(M, V)$ is identified with Balavoine's $C^*_{\mathcal{P}}(V^\#, M^\#)$ via a dualization isomorphism (Corollary 8.4).

2. Operads and their (co)algebras

The purpose of this section is to recall some standard definitions about (quadratic) operads and their (co)algebras that are necessary for understanding the rest of this paper.

2.1 *Conventions*

The symbol $\mathbb{N}^*$ denotes the set of positive integers. Throughout this paper, we work over a fixed field $\mathbf{k}$ of characteristic zero. Vector spaces, $\otimes$, Hom and End (endomorphisms) are all meant over $\mathbf{k}$. For any positive integer n, Σ_n will denote the group of permutations on n elements. For $\sigma \in \Sigma_n$, $\epsilon(\sigma) \in \{-1, 1\}$ will stand for the sign of σ and $\mathbf{sgn}_n$ will denote the sign representation of Σ_n.

2.2 *Operads*

An *operad* [16–19] $\mathcal{P}$ consists of a right $\mathbf{k}[\Sigma_n]$-module $\mathcal{P}(n)$, one for each $n \in \mathbb{N}^*$. For positive integers $n, j_1, \ldots, j_n$, there is a structure map

$$\gamma \colon \mathcal{P}(n) \otimes \mathcal{P}(j_1) \otimes \cdots \otimes \mathcal{P}(j_n) \to \mathcal{P}(j_1 + \cdots + j_n).$$

These structure maps satisfy some associativity, equivariance, and unity conditions, which can be found in [18]. Using the operad structure maps, one defines the $\circ_i$ operations as

$$f \circ_i g = \gamma(f; 1, \ldots, 1, g, 1, \ldots, 1) \in \mathcal{P}(n + m - 1) \tag{2.2.1}$$

for $f \in \mathcal{P}(n)$ and $g \in \mathcal{P}(m)$, where there are $(i-1)$ copies of 1's in front of g. Conversely, the structure maps γ can be recovered from the $\circ_i$ operations as

$$\gamma(f; g_1, \ldots, g_n) = (\cdots(((f \circ_1 g_1) \circ_{j_1+1} g_2) \circ_{j_1+j_2+1} g_3) \cdots) \tag{2.2.2}$$

for $f \in \mathcal{P}(n)$ and $g_i \in \mathcal{P}(j_i)$ $(1 \leq i \leq n)$. In the presence of the unit $1 \in \mathcal{P}(1)$, the operad structure maps γ are completely determined by the $\circ_i$ operations ([16] or §1.7.1, p. 66 of [17]). In what follows, by using (2.2.1) and (2.2.2), we will use these two equivalent definitions of an operad interchangeably.

For example, let V be a vector space over $\mathbf{k}$ and for every $n \in \mathbb{N}^*$ let $\mathrm{End}(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$. Then $\mathrm{End}(V) = \{\mathrm{End}(V)(n),\ n \in \mathbb{N}^*\}$ is naturally an operad, called the *endomorphism operad of* V.

From the definition of an operad, if we omit the parts concerning the symmetric groups Σ_n $(n \geq 1)$, then we obtain the definition of a *non-*Σ *operad*.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. A *morphism* of operads from $\mathcal{P}$ to $\mathcal{Q}$ is a sequence $a = \{a(n), n \in \mathbb{N}^*\}$ of $\mathbf{k}[\Sigma_n]$-linear maps $a(n) \colon \mathcal{P}(n) \to \mathcal{Q}(n)$ satisfying the conditions, $a(1)(1) = 1$ and

$$a(n + m - 1)(\mu \circ_i \nu) = a(n)(\mu) \circ_i a(m)(\nu)$$

for $n, m \in \mathbb{N}^*$, $1 \leq i \leq n$, $\mu \in \mathcal{P}(n)$ and $\nu \in \mathcal{P}(m)$.

Let $\mathcal{P}$ be an operad. A $\mathcal{P}$*-algebra* or an algebra over $\mathcal{P}$ is a vector space V over $\mathbf{k}$ along with a morphism of operads $a \colon \mathcal{P} \to \mathrm{End}(V)$.

Let V be a vector space. Let $\mathrm{Coend}(V) = \{\mathrm{Hom}(V, V^{\otimes n})\}$ be the *coendomorphism operad* of V with the obvious structure maps, dual to those in $\mathrm{End}(V)$. For an operad $\mathcal{P}$, a $\mathcal{P}$*-coalgebra structure* on V is a morphism $\mathcal{P} \to \mathrm{Coend}(V)$ of operads.

For example, a coassociative coalgebra structure is equivalent to an As-coalgebra structure, where 'As' is the associative algebra operad.

2.3 *Free graded* $\mathcal{P}$*-algebras*

For an operad $\mathcal{P}$ and a vector space V, define the *free graded* $\mathcal{P}$*-algebra generated by* V as

$$\mathcal{F}^{\mathrm{gr}}_{\mathcal{P}}(V) = \oplus_{n \geq 1} \mathcal{P}(n) \otimes_{\Sigma_n} (V^{\otimes n} \otimes \mathrm{sgn}_n),$$

where $\sigma(v_1 \otimes \cdots \otimes v_n) = \epsilon(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for $\sigma \in \Sigma_n$ and $v_i \in V$. The homogeneous degree n component of $\mathcal{F}^{\text{gr}}_{\mathcal{P}}(V)$ is denoted by $\mathcal{F}^{\text{gr}_n}_{\mathcal{P}}(V)$. The $\mathcal{P}$-algebra structure on $\mathcal{F}^{\text{gr}}_{\mathcal{P}}(V)$ is the natural one defined by the operad structure on $\mathcal{P}$ and concatenation on $V^{\otimes *}$ (§§1.6 of [2]).

2.4 *Quadratic operads*

PROPOSITION 2.5 (§2 of [7])

Let E be a right $\mathbf{k}[\Sigma_2]$-module. Then there exists an operad $\mathcal{F}(E)$ with $\mathcal{F}(E)(1) = \mathbf{k}$ and $\mathcal{F}(E)(2) = E$ such that the following property holds: For any operad $\mathcal{Q}$ and for any morphism of right $\mathbf{k}[\Sigma_2]$-modules $a\colon E \to \mathcal{Q}(2)$, there exists a unique morphism of operads, $\hat{a}\colon \mathcal{F}(E) \to \mathcal{Q}$, such that $\hat{a}(2) = a$.

The operad $\mathcal{F}(E)$ is called the *free operad generated by E*. By the usual arguments, the free operad $\mathcal{F}(E)$ is unique up to operad isomorphisms.

Let E be a right $\mathbf{k}[\Sigma_2]$-module and R be a right $\mathbf{k}[\Sigma_3]$-submodule of $\mathcal{F}(E)(3)$. Let (R) be the ideal generated by R. Then the quotient operad $\mathcal{F}(E)/(R)$ is called the *quadratic operad generated by E with relations R*, denoted by $\mathcal{P}(\mathbf{k}, E, R)$ [7]. A quadratic operad $\mathcal{P}(\mathbf{k}, E, R)$ is said to be *finitely generated* if E is a finite dimensional vector space.

2.6 *Quadratic duality*

Let F be a right $\mathbf{k}[\Sigma_n]$-module. By $F^{\#}$ we mean the right $\mathbf{k}[\Sigma_n]$-module $F^{\#} = \text{Hom}(F, \mathbf{k}) \otimes \text{sgn}_n$, where the right Σ_n-action is given by $(\phi \cdot \sigma)(x) = \epsilon(\sigma)\phi(x \cdot \sigma^{-1})$ for $\phi \in \text{Hom}(F, \mathbf{k})$ and $x \in F$.

Let E be a right $\mathbf{k}[\Sigma_2]$-module. Then as right $\mathbf{k}[\Sigma_3]$-modules, one has that [7] $\mathcal{F}(E^{\#})(3) \cong (\mathcal{F}(E)(3))^{\#}$. Let $R \subset \mathcal{F}(E)(3)$ be a right $\mathbf{k}[\Sigma_3]$-submodule, and let $R^{\perp} \subset \mathcal{F}(E^{\#})(3)$ be the annihilator of R in $(\mathcal{F}(E)(3))^{\#} \cong \mathcal{F}(E^{\#})(3)$. The *Koszul dual* of the quadratic operad $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$ is defined as the quadratic operad $\mathcal{P}^! = \mathcal{P}(\mathbf{k}, E^{\#}, R^{\perp})$.

2.7 *Algebras and coalgebras over a quadratic operad*

PROPOSITION 2.8 (Proposition 1.5.5 of [2])

Let $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$ be a quadratic operad. Then a $\mathcal{P}$-algebra structure on a vector space V is determined by a morphism of right $\mathbf{k}[\Sigma_2]$-modules $\pi\colon \mathcal{P}(2) = E \to \text{End}(V)(2)$ such that $\hat{\pi}(3)(R) = 0$.

In this case, the morphism $\pi\colon \mathcal{P}(2) \to \text{End}(V)(2)$, or equivalently its adjoint $\pi\colon \mathcal{P}(2) \otimes_{\Sigma_2} V^{\otimes 2} \to V$, is called the *structural morphism* of the $\mathcal{P}$-algebra V.

PROPOSITION 2.9 (Theorem 3.2 in [15])

Let $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$ be a finitely generated quadratic operad, and let V be a finite dimensional vector space. Then a $\mathcal{P}$-coalgebra structure on V is determined by a $\mathbf{k}[\Sigma_2]$-equivariant morphism

$$\pi\colon E = \mathcal{P}(2) \to \text{Coend}(V)(2)$$

such that

$$\hat{\pi}(3)(R) = 0,$$

where $\hat{\pi}\colon \mathcal{F}(E) \to \mathrm{Coend}(V)$ *is the unique operad morphism associated to* π.

2.10 *Standing assumptions*

For the rest of this paper, unless otherwise specified, $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$ will denote a finitely generated quadratic operad, and $V = (V, \pi)$ will denote a finite dimensional $\mathcal{P}$-coalgebra with structure map

$$\pi \in \bar{C}^2_{\mathcal{P}}(V) = \mathrm{Hom}(V, \mathcal{P}^{!}(2) \otimes_{\Sigma_2} (V^{\otimes 2} \otimes \mathrm{sgn}_n)).$$

Equivalently, using adjunctions and the finite dimensionality of $\mathcal{P}^{!}(2)$, the element π can also be considered as a $\mathbf{k}[\Sigma_2]$-linear map

$$\pi\colon \mathcal{P}(2) \to \mathrm{Coend}(V)(2).$$

In what follows, we will identify these two descriptions of an element in $\bar{C}^2_{\mathcal{P}}(V)$ for any finite dimensional vector space V.

3. Corepresentations of a $\mathcal{P}$-coalgebra

The main purpose of this section is to give a proper definition of a corepresentation of a $\mathcal{P}$-coalgebra (V, π) (Definition 3.3). We then provide a way to check whether a given module is a V-corepresentation in terms of a component map of an operad morphism (Proposition 3.5 and Theorem 3.6).

3.1 *Corepresentations of* (V, π)

Consider a finite dimensional vector space M together with a linear map

$$\eta\colon \mathcal{P}(2) \to \mathrm{Hom}(M, V \otimes M).$$

Define another linear map

$$\eta^{\tau}\colon \mathcal{P}(2) \to \mathrm{Hom}(M, M \otimes V) \tag{3.1.1}$$

as the composition of the maps

$$\mathcal{P}(2) \xrightarrow{\tau} \mathcal{P}(2) \xrightarrow{\eta} \mathrm{Hom}(M, V \otimes M) \xrightarrow{\tau} \mathrm{Hom}(M, M \otimes V),$$

where τ is the permutation (1 2) in Σ_2, which acts on $\mathcal{P}(2)$ from the right and permutes $V \otimes M$ to $M \otimes V$. Think of η (or η^{τ}) as the left (or right) V-coaction on M.

Set $C = V \oplus M$. Consider the linear map

$$\eta_C(2)\colon \mathcal{P}(2) \to \mathrm{Hom}(C, C^{\otimes 2}) = \mathrm{Coend}(C)(2)$$

defined by the conditions

$$\eta_C(2)(\mu)|_V = \pi(\mu)\colon V \to V^{\otimes 2},$$

$$\eta_C(2)(\mu)|_M = (\eta(\mu), \eta^{\tau}(\mu))\colon M \to (V \otimes M) \oplus (M \otimes V) \tag{3.1.2}$$

for $\mu \in \mathcal{P}(2)$.

PROPOSITION 3.2

The map $\eta_C(2)$ is Σ_2-equivariant.

Proof. This follows from the facts that π is Σ_2-equivariant and that

$$\eta(\mu\tau) = (\eta^\tau(\mu))\tau,$$
$$\eta^\tau(\mu\tau) = (\eta(\mu))\tau$$

for $\mu \in \mathcal{P}(2)$. ■

DEFINITION 3.3

We say that (M, η) is a (V, π)-*corepresentation*, or a V-*corepresentation* for short, if the element $\eta_C(2) \in \bar{C}^2_{\mathcal{P}}(C)$ defines a $\mathcal{P}$-coalgebra structure on $C = V \oplus M$.

Note that we only consider finite dimensional corepresentations.

DEFINITION 3.4

A *morphism* $f\colon (M, \eta) \to (M', \eta')$ of (V, π)-corepresentations is a linear map $f\colon M \to M'$ such that the diagram

$$\begin{array}{ccc}
\mathcal{P}(2) & \xrightarrow{\eta} & \mathrm{Hom}(M, V \otimes M) \\
\downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \mathrm{Hom}(M, V\otimes f)} \\
\mathrm{Hom}(M', V \otimes M') & \xrightarrow[\mathrm{Hom}(f, V\otimes M')]{} & \mathrm{Hom}(M, V \otimes M')
\end{array}$$

commutes.

PROPOSITION 3.5

The pair (M, η) is a (V, π)-corepresentation if and only if

$$\eta_C(3)(R) = 0,$$

where

$$\eta_C(3)\colon \mathcal{F}(E)(3) \to \mathrm{Coend}(C)(3)$$

is the unique map that extends $\eta_C(2)$.

Proof. This is just Theorem 3.2 in [15]. ■

Since $R \subseteq \mathcal{F}(E)(3)$, in order to check the condition $\eta_C(3)(R) = 0$, it suffices to know what the elements

$$\eta_C(3)(\mu \circ_i \nu) \in \mathrm{Coend}(C)(3)$$

are for $\mu, \nu \in \mathcal{P}(2)$ and $i = 1, 2$.

Theorem 3.6. *With the notations above, we have*

$$\eta_C(3)(\mu \circ_i \nu)|_V = \pi(\mu) \circ_i \pi(\nu) = \begin{cases} (\pi(\nu) \otimes 1_V) \circ \pi(\mu), & \textit{if } i = 1, \\ (1_V \otimes \pi(\nu)) \circ \pi(\mu), & \textit{if } i = 2, \end{cases}$$

and

$$\eta_C(3)(\mu \circ_i \nu)|_M$$
$$= \begin{cases} ((\pi(\nu) \otimes 1_M) \circ \eta(\mu), (\eta(\nu) \otimes 1_V) \circ \eta^\tau(\mu), (\eta^\tau(\nu) \otimes 1_V) \circ \eta^\tau(\mu)), & \textit{if } i = 1, \\ ((1_V \otimes \eta(\nu)) \circ \eta(\mu), (1_V \otimes \eta^\tau(\nu)) \circ \eta(\mu), (1_M \otimes \pi(\nu)) \circ \eta^\tau(\mu)), & \textit{if } i = 2. \end{cases}$$

Proof. For $i \in \{1, 2\}$ and $\nu \in \mathcal{P}(2)$, define the map

$$\eta_C(2)(\nu)^i \colon C^{\otimes 2} \to C^{\otimes 3}$$

by setting

$$\eta_C(2)(\nu)^i = \begin{cases} \eta_C(2)(\nu) \otimes 1_C, & \text{if } i = 1, \\ 1_C \otimes \eta_C(2)(\nu), & \text{if } i = 2. \end{cases}$$

Then we have that

$$\begin{aligned} \eta_C(3)(\mu \circ_i \nu)|_M &= (\eta_C(2)(\mu) \circ_i \eta_C(2)(\nu))|_M \\ &= \eta_C(2)(\nu)^i|_{(V\otimes M)\oplus(M\oplus V)} \circ (\eta_C(2)(\mu)|_M) \\ &= (\eta_C(2)(\nu)^i|_{V\otimes M} \circ \eta(\mu), \eta_C(2)(\nu)^i|_{M\otimes V} \circ \eta^\tau(\mu)). \end{aligned}$$

In the last entry, the first component is given by

$$\begin{aligned} &\eta_C(2)(\nu)^i|_{V\otimes M} \circ \eta(\mu) \\ &\quad = \begin{cases} (\eta_C(2)(\nu)|_V \otimes 1_M) \circ \eta(\mu), & \text{if } i = 1, \\ (1_V \otimes \eta_C(2)(\nu)|_M) \circ \eta(\mu), & \text{if } i = 2, \end{cases} \\ &\quad = \begin{cases} (\pi(\nu) \otimes 1_M) \circ \eta(\mu), & \text{if } i = 1, \\ ((1_V \otimes \eta(\nu)) \circ \eta(\mu), (1_V \otimes \eta^\tau(\nu)) \circ \eta(\mu)), & \text{if } i = 2. \end{cases} \end{aligned}$$

Likewise, we have that

$$\begin{aligned} &\eta_C(2)(\nu)^i|_{M\otimes V} \circ \eta^\tau(\mu) \\ &\quad = \begin{cases} ((\eta(\nu) \otimes 1_V) \circ \eta^\tau(\mu), (\eta^\tau(\nu) \otimes 1_V) \circ \eta^\tau(\mu)), & \text{if } i = 1, \\ (1_M \otimes \pi(\nu)) \circ \eta^\tau(\mu), & \text{if } i = 2. \end{cases} \end{aligned}$$

This proves the assertion about $\eta_C(3)(\mu \circ_i \nu)|_M$. The assertion about $\eta_C(3)(\mu \circ_i \nu)|_V$ is proved by a similar argument. ■

Example 3.7.

(1) If $\mathcal{P} = \mathrm{As}$ is the associative algebras operad, then a $\mathcal{P}$-coalgebra is a coassociative coalgebra. In this case, a V-corepresentation is the same thing as a V-bicomodule in the usual sense.
(2) If $\mathcal{P} = \mathrm{Com}$ is the associative commutative algebras operad, then a $\mathcal{P}$-coalgebra is a cocommutative coassociative coalgebra. In this case, a V-corepresentation is the same thing as a symmetric V-bicomodule in the usual sense.

4. V-corepresentations as $V^{\#}$-representations

The purpose of this section is to establish a correspondence (Corollary 4.8) between V-corepresentations and representations of its dual $\mathcal{P}$-algebra $V^{\#}$ in the sense of Balavoine [1]. This gives another characterization of a V-corepresentation.

For the rest of this paper, unless otherwise specified, (M, η) will denote a V-corepresentation (Definition 3.3), and $C = V \oplus M$ will denote the $\mathcal{P}$-coalgebra with structural morphism $\theta = \eta_C(2)$ (eq. (3.1.2)).

We begin by recalling Balavoine's notation of representations of a $\mathcal{P}$-algebra.

4.1 *Representations of a $\mathcal{P}$-algebra*

Let $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$ be a finite dimensional quadratic operad such that $\mathcal{P}(1) = \mathbf{k}$. Let (A, π) be a $\mathcal{P}$-algebra, where π is the structural map of A. Let M be a vector space over $\mathbf{k}$ and $\psi\colon \mathcal{P}(2) \otimes A \otimes M \to M$ be a linear map. Set $B = A \oplus M$. Define a linear map $\bar{\psi} : \mathcal{P}(2) \otimes_{\Sigma_2} B^{\otimes 2} \to B$ by

$$\begin{aligned}\bar{\psi}(\mu, a_1 + m_1, a_2 + m_2) = \pi(\mu)(a_1, a_2)\\ + \psi(\mu, a_1, a_2) + \psi(\mu(12), a_2, m_1).\end{aligned}$$

It is easy to verify that the $\bar{\psi}$ so defined satisfy the following conditions:

(i) $\bar{\psi}|_{\mathcal{P}(2)\otimes_{\Sigma_2} A^{\otimes 2}} = \pi$;
(ii) $\bar{\psi}(\mu, a, m) = \psi(\mu, a, m)$ for all $\mu \in \mathcal{P}(2)$, $(a, m) \in A \times M$;
(iii) $\bar{\psi}(\mu, m_1, m_2) = 0$ for all $\mu \in \mathcal{P}(2)$, $m_1, m_2 \in M$.

DEFINITION 4.2

A representation of A is a vector space M equipped with a linear map $\psi\colon \mathcal{P}(2)\otimes A\otimes M \to M$, such that the extended map $\bar{\psi}$ defined above makes $A \oplus M$ a $\mathcal{P}$-algebra.

Let $\psi(3)$ denote the extended map of $\bar{\psi}$ to $\mathcal{F}(E)(3) \otimes_{\Sigma_3} B^{\otimes 3}$.

PROPOSITION 4.3 (Proposition 1.3 of [1])

With the above notations, M is a representation of A iff for all $a_1, a_2 \in A$, $m \in M$ and $r \in R$,

$$\psi(3)(\mu, a_1, a_2, m) = 0.$$

4.4 *Dual $\mathcal{P}$-algebra of a $\mathcal{P}$-coalgebra*

Let V be a finite dimensional vector space. Denote by $V^{\#}$ its linear dual $\mathrm{Hom}(V, \mathbf{k})$. Then for each $n \geq 1$, there is a linear isomorphism (Proposition 2.8 in [20])

$$\zeta^n \colon \mathrm{Coend}(V)(n) \xrightarrow{\cong} \mathrm{End}(V^{\#})(n) \tag{4.4.1}$$

given by

$$\zeta^n(f) = f^{\#},$$

where

$$f^{\#}(\alpha_1 \otimes \cdots \otimes \alpha_n)(a) = \sum \left(\prod_{i=1}^{n} \alpha_i(f(a)_i) \right)$$

for $\alpha_i \in V^{\#}$ and $a \in V$. The notations on the right-hand side of the previous line is given by

$$f(a) = \sum f(a)_1 \otimes \cdots \otimes f(a)_n \in V^{\otimes n}.$$

Theorem 4.5 (Theorem 3.1 of [15]). *Let V be a finite dimensional vector space. Then the maps ζ^n $(n \geq 1)$ assemble to form an isomorphism*

$$\zeta \colon \mathrm{Coend}(V) \xrightarrow{\cong} \mathrm{End}(V^{\#})$$

of operads.

In view of the above theorem, we infer that if $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$ is a finitely generated quadratic operad, and V be a finite dimensional $\mathcal{P}$-coalgebra, then the linear dual $V^{\#}$ is a $\mathcal{P}$-algebra.

4.6 *Dualizing a V-corepresentation*

Let V be a $\mathcal{P}$-coalgebra and M be a V corepresentation. Let $C = V \oplus M$. Define the map

$$\eta_C(2)^{\#} = \zeta^2 \circ \eta_C(2) \colon \mathcal{P}(2) \to \mathrm{End}(C^{\#})(2),$$

where ζ^2 is part of the dualization isomorphism

$$\zeta \colon \mathrm{Coend}(C) \cong \mathrm{End}(C^{\#})$$

of operads (Theorem 3.4 of [15]). From the definition of $\eta_C(2)$ in eq. (3.1.2), we have that

$$\eta_C(2)^{\#}(\mu) = \begin{cases} \pi(\mu)^{\#} = \zeta^2(\pi(\mu)), & \text{on } V^{\#} \otimes V^{\#}, \\ \eta(\mu)^{\#}, & \text{on } V^{\#} \otimes M^{\#}, \\ \eta^{\tau}(\mu)^{\#}, & \text{on } M^{\#} \otimes V^{\#}, \\ 0, & \text{on } M^{\#} \otimes M^{\#}, \end{cases} \tag{4.6.1}$$

for $\mu \in \mathcal{P}(2)$. Here

$$\eta(\mu)^{\#} \colon V^{\#} \otimes M^{\#} \to M^{\#}$$

is defined as the image of $\eta(\mu) \in \operatorname{Hom}(M, V \otimes M)$ under the dualization isomorphism

$$\operatorname{Hom}(M, V \otimes M) \xrightarrow{\cong} \operatorname{Hom}(V^{\#} \otimes M^{\#}, M^{\#}).$$

The element $\eta^{\tau}(\mu)^{\#}$ is defined similarly.

Denote by

$$\widehat{\eta_C(2)^{\#}}(3): \mathcal{F}(E)(3) \to \operatorname{End}(C^{\#})(3)$$

the unique extension of $\eta_C(2)^{\#}$ to $\mathcal{F}(E)(3)$. Set

$$\eta_C(3)^{\#} = \zeta^3 \circ \eta_C(3): \mathcal{F}(E)(3) \to \operatorname{End}(C^{\#})(3).$$

Theorem 4.7. *One has that* $\eta_C(3)^{\#} = \widehat{\eta_C(2)^{\#}}(3)$.

In other words, dualization commutes with extension to $\mathcal{F}(E)(3)$.

Proof. This follows from the fact that ζ is an operad isomorphism, which in particular implies that ζ commutes with the $\circ_i$ operations. Indeed, given any generator $\mu \circ_i \nu \in \mathcal{F}(E)(3)$ with $\mu, \nu \in \mathcal{P}(2)$ and $i \in \{1, 2\}$, we have

$$\begin{aligned}
\eta_C(3)^{\#}(\mu \circ_i \nu) &= \zeta^3(\eta_C(3)(\mu \circ_i \nu)) \\
&= \zeta^3(\eta_C(2)(\mu) \circ_i \eta_C(2)(\nu)) \\
&= \zeta^2(\eta_C(2)(\mu)) \circ_i \zeta^2(\eta_C(2)(\nu)) \\
&= \eta_C(2)^{\#}(\mu) \circ_i \eta_C(2)^{\#}(\nu) \\
&= \widehat{\eta_C(2)^{\#}}(3)(\mu \circ_i \nu),
\end{aligned}$$

as desired. ∎

Let

$$\eta^{\#}: \mathcal{P}(2) \to \operatorname{Hom}(V^{\#} \otimes M^{\#}, M^{\#})$$

be the composition of η followed by dualization. We also consider $\eta^{\#}$ as a linear map

$$\eta^{\#}: \mathcal{P}(2) \otimes V^{\#} \otimes M^{\#} \to M^{\#}.$$

Note that

$$\begin{aligned}
\eta(\mu)^{\#} &= \eta^{\#}(\mu), \\
\eta^{\tau}(\mu)^{\#} &= (\eta^{\tau})^{\#}(\mu) = (\eta^{\#})^{\tau}(\mu),
\end{aligned}$$

where $(\eta^{\tau})^{\#}$ is η^{τ} followed by dualization and $(\eta^{\#})^{\tau}$ is the composition

$$\mathcal{P}(2) \xrightarrow{\tau} \mathcal{P}(2) \xrightarrow{\eta^{\#}} \operatorname{Hom}(V^{\#} \otimes M^{\#}, M^{\#}) \xrightarrow{\tau} \operatorname{Hom}(M^{\#} \otimes V^{\#}, M^{\#}).$$

COROLLARY 4.8

The pair (M, η) *is a* (V, π)*-corepresentation if and only if* $(M^{\#}, \eta^{\#})$ *is a* $(V^{\#}, \pi^{\#})$*-representation in the sense of Definition* 1.2 *of* [1], *where* $(V^{\#}, \pi^{\#})$ *is the dual* $\mathcal{P}$*-algebra of* (V, π) (*Corollary* 5.3 *of* [15]).

Proof. Indeed, (M, η) is a (V, π)-corepresentation if and only if

$$\eta_C(3)(R) = 0,$$

by Proposition 3.5. Since ζ is an isomorphism, it follows from Theorem 4.7 that $\eta_C(3)(R) = 0$ if and only if

$$\widehat{\eta_C(2)^{\#}}(3)(R) = 0.$$

This last condition is equivalent to saying that $\eta_C(2)^{\#}$ defines a $\mathcal{P}$-algebra structure on $C^{\#}$. Using the description (4.6.1) of $\eta_C(2)^{\#}$, one observes that this is equivalent to $(M^{\#}, \eta^{\#})$ being a $(V^{\#}, \pi^{\#})$-representation in the sense of Definition 1.2 of [1]. ■

COROLLARY 4.9

The correspondence between (M, η) *and* $(M^{\#}, \eta^{\#})$ *above gives an equivalence from the category of finite dimensional* (V, π)*-corepresentations to the category of finite dimensional* $(V^{\#}, \pi^{\#})$*-representations.*

5. Corepresentations as comodules

The purpose of this section is to construct the *enveloping coalgebra of* V. It is a unital graded coassociative coalgebra, with the property that its left-comodules are exactly the V-corepresentations (Theorem 5.4).

As in any abelian category, the category of $\mathcal{P}$-algebra representations can be described in terms of left modules over a certain associative algebra. Such an algebra indeed exists and is called the universal enveloping algebra of a $\mathcal{P}$-algebra A, denoted by $\mathcal{U}_{\mathcal{P}}(A)$. The construction in such generality for the first time was given by Ginzburg and Kapranov in [7]. We recall Balavoine's enveloping algebra for a $\mathcal{P}$-algebra, which we will dualize to obtain the desired enveloping coalgebra.

5.1 *Enveloping algebra for a* $\mathcal{P}$*-algebra*

Recall from §1 of [1] that for a $\mathcal{P}$-algebra A, its *enveloping algebra* is defined as the quotient

$$\mathcal{U}_{\mathcal{P}}(A) = T(\mathcal{P}(2) \otimes A)/I_R,$$

where $T(\mathcal{P}(2) \otimes A)$ is the unital tensor algebra on the vector space $\mathcal{P}(2) \otimes A$ and I_R is a certain two-sided ideal in it. The enveloping algebra inherits a non-negative grading from the unital tensor algebra $T(\mathcal{P}(2) \otimes A)$, and we denote by $\mathcal{U}_{\mathcal{P}}(A)_n$ the homogeneous degree n component in $\mathcal{U}_{\mathcal{P}}(A)$.

Consider the graded vector space

$$\begin{aligned}\mathcal{U}_{\mathcal{P}}(A)^{\#} &\stackrel{\text{def}}{=} \oplus_{n\geq 0}\mathcal{U}_{\mathcal{P}}(A)_n^{\#}\\ &= \oplus_{n\geq 0}\operatorname{Hom}(\mathcal{U}_{\mathcal{P}}(A)_n, \mathbf{k}).\end{aligned}$$

Theorem 5.2. *If A is finite dimensional, then the multiplication on $\mathcal{U}_{\mathcal{P}}(A)$ induces a comultiplication on $\mathcal{U}_{\mathcal{P}}(A)^{\#}$, making it into a unital graded coassociative coalgebra.*

Proof. Since A is finite dimensional, so is $\mathcal{P}(2) \otimes A$. Therefore, each $(\mathcal{P}(2) \otimes A)^{\otimes n}$ is also finite dimensional. Since $\mathcal{U}_{\mathcal{P}}(A)_n$ is a quotient of $(\mathcal{P}(2) \otimes A)^{\otimes n}$, it is finite dimensional. The comultiplication on $\mathcal{U}_{\mathcal{P}}(A)^{\#}$ is defined on its homogeneous degree n component as the following composition:

$$\begin{aligned}\mathcal{U}_{\mathcal{P}}(A)_n^{\#} \to (\oplus \mathcal{U}_{\mathcal{P}}(A)_i \otimes \mathcal{U}_{\mathcal{P}}(A)_j)^{\#} &\xrightarrow{\cong} \oplus (\mathcal{U}_{\mathcal{P}}(A)_i \otimes \mathcal{U}_{\mathcal{P}}(A)_j)^{\#} \\ &\xrightarrow{\cong} \oplus \mathcal{U}_{\mathcal{P}}(A)_i^{\#} \otimes \mathcal{U}_{\mathcal{P}}(A)_j^{\#},\end{aligned}$$

in which $\oplus = \oplus_{i+j=n}$. The first map is the linear dual of the multiplication on $\mathcal{U}_{\mathcal{P}}(A)$. The first isomorphism holds, since the direct sum is finite. The second isomorphism uses the finite dimensionality of each $\mathcal{U}_{\mathcal{P}}(A)_i$. ∎

5.3 *Enveloping coalgebra for a $\mathcal{P}$-coalgebra*

Now we go back to the setting of the previous section. Since $(V^{\#}, \pi^{\#})$ is a finite dimensional $\mathcal{P}$-algebra, it follows from Theorem 5.2 that $\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}$ is a unital graded coassociative coalgebra, which we call the *enveloping coalgebra of V*. In particular, we can consider left (or right) comodules over $\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}$.

Theorem 5.4. *There is an equivalence between the categories of finite dimensional left $\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}$-comodules and of finite dimensional (V, π)-corepresentations.*

Proof. There is an equivalence between the category of finite dimensional left $\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}$-comodules and the category of finite dimensional left $\mathcal{U}_{\mathcal{P}}(V^{\#})$-modules. It associates to a left $\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}$-coaction map

$$M \xrightarrow{\lambda^{\#}} \mathcal{U}_{\mathcal{P}}(V^{\#})^{\#} \otimes M \tag{5.4.1}$$

its dual left $\mathcal{U}_{\mathcal{P}}(V^{\#})$-action map

$$\mathcal{U}_{\mathcal{P}}(V^{\#}) \otimes M^{\#} \xrightarrow{\lambda} M^{\#}.$$

The latter category is equivalent to the category of finite dimensional $(V^{\#}, \pi^{\#})$-representations (Theorem 1.7.2 of [1]), which in turn is equivalent to the category of finite dimensional (V, π)-corepresentations (Corollary 4.9). ∎

5.5 *Explicit description of the equivalence*

The equivalence in Theorem 5.4 can be described more explicitly as follows. Suppose that M is a finite dimensional left $\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}$-comodule with structure map $\lambda^{\#}$ as in (5.4.1). Then its associated (V, π)-corepresentation is (M, λ), where

$$\lambda \colon M \to \operatorname{Hom}(\mathcal{P}(2), V \otimes M)$$

is the composition of the following maps:

$$M \xrightarrow{\lambda^{\#}} \mathcal{U}_{\mathcal{P}}(V^{\#})^{\#} \otimes M \cong \oplus_{n\geq 0}(\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}_{n} \otimes M) \twoheadrightarrow \mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}_{1} \otimes M$$

$$\overset{j}{\hookrightarrow} \mathcal{P}(2)^{\#} \otimes V \otimes M \cong \mathrm{Hom}(\mathcal{P}(2), V \otimes M).$$

The first isomorphism comes from the fact that direct sum commutes with tensor product. The last isomorphism uses the finite dimensionality of $\mathcal{P}(2)$. The map j arises as follows. Consider the projection map

$$\mathcal{P}(2) \otimes V^{\#} \twoheadrightarrow \mathcal{U}_{\mathcal{P}}(V^{\#})_{1}.$$

Its linear dual is an injection

$$\mathcal{U}_{\mathcal{P}}(V^{\#})^{\#}_{1} \hookrightarrow \mathcal{P}(2)^{\#} \otimes V.$$

The map j is obtained from this injection by tensoring with M, which stays injective because $\mathbf{k}$ is a field.

6. Cohomology of $\mathcal{P}$-coalgebras

The purpose of this section is to define the cochain complex $\bar{C}^{*}_{\mathcal{P}}(M, V)$ that gives rise to the cohomology $\bar{H}^{*}_{\mathcal{P}}(M, V)$ of V with coefficients in M. It is constructed as a subcomplex of $\bar{C}^{*}_{\mathcal{P}}(C, C)$, the deformation complex of the $\mathcal{P}$-coalgebra C (§4 of [15]). As a side benefit of our construction, we obtain an explicit formula (see eqs (6.8.1) and (6.9.1)) of the differential in $\bar{C}^{*}_{\mathcal{P}}(M, V)$ in terms of the $\circ_i$ operations in $\mathcal{P}$.

In the case of the associative algebras operad $\mathcal{P} = \mathrm{As}$, our cohomology $\bar{H}^{*}_{\mathcal{P}}(M, V)$ coincides with the Hochschild coalgebra cohomology [10] $H^{*}_{c}(M, V)$ of V with coefficients in the V-bicomodule M.

First we recall the deformation complex of a $\mathcal{P}$-coalgebra that was constructed in §4 of [15].

6.1 *Deformation complex of a $\mathcal{P}$-coalgebra*

Define the vector spaces

$$\bar{L}^{n}_{\mathcal{P}}(V) = \mathcal{P}^{!}(n+1) \otimes_{\Sigma_{n+1}} \overline{\mathrm{Coend}}(V)(n+1) \quad (n \geq 0),$$

$$\bar{C}^{n}_{\mathcal{P}}(V) = \mathrm{Hom}(V, \mathcal{P}^{!}(n) \otimes_{\Sigma_n} (V^{\otimes n} \otimes \mathrm{sgn}_n)) \quad (n \geq 1)$$

$$= \mathrm{Hom}(V, \mathcal{F}^{\mathrm{gr}_n}_{\mathcal{P}^{!}}(V)).$$

Here

$$\overline{\mathrm{Coend}}(V)(n) = \mathrm{Hom}(V, V^{\otimes n}) \otimes \mathrm{sgn}_n,$$

which is the same as $\mathrm{Hom}(V, V^{\otimes n})$ as a vector space and has the natural left Σ_n-action.

Theorem 6.2 (Theorem 4.1 of [15]). *For each $n \geq 1$, there is an isomorphism*

$$\bar{\Gamma}\colon \bar{L}^{n-1}_{\mathcal{P}}(V) \xrightarrow{\cong} \bar{C}^{n}_{\mathcal{P}}(V)$$

of vector spaces.

PROPOSITION 6.3 (Proposition 4.3 of [15])

$(\bar{L}^{*}_{\mathcal{P}}(V), [-,-])$ *is a graded Lie algebra.*

6.4 *The graded Lie algebra* $\bar{C}^*_{\mathcal{P}}(V)$

Define the operation $[-,-]$ on $\bar{C}^*_{\mathcal{P}}(V)$ via $\bar{\Gamma}$. Namely, define

$$[f, g] = \bar{\Gamma}([\bar{\Gamma}^{-1}(f), \bar{\Gamma}^{-1}(g)]) \tag{6.4.1}$$

for $f, g \in \bar{C}^*_{\mathcal{P}}(V)$. The following result is an immediate consequence of Theorem 6.2 and Proposition 6.3.

COROLLARY 6.5

$(\bar{C}^*_{\mathcal{P}}(V), [-,-])$ *is a graded Lie algebra of degree* -1.

6.6 *Coboundary in* $\bar{C}^*_{\mathcal{P}}(V)$

Now let V be a finite dimensional $\mathcal{P}$-coalgebra with structural morphism $\pi \in \bar{C}^2_{\mathcal{P}}(V)$, i.e., $[\pi, \pi] = 0$. Following Balavoine [2], define a map

$$\bar{\delta}^n_\pi \colon \bar{C}^n_{\mathcal{P}}(V) \to \bar{C}^{n+1}_{\mathcal{P}}(V)$$

by setting

$$\bar{\delta}^n_\pi(f) = -\frac{n+1}{2}[f, \pi] \tag{6.6.1}$$

for $f \in \bar{C}^n_{\mathcal{P}}(V)$. The map $\bar{\delta}_\pi$ is a differential on $\bar{C}^*_{\mathcal{P}}(V)$ [15]. In particular, $(\bar{C}^*_{\mathcal{P}}(V), \bar{\delta}_\pi, [-,-])$ is a differential graded Lie algebra.

The cohomology of the cochain complex $(\bar{C}^*_{\mathcal{P}}(V), \bar{\delta}_\pi)$ is denoted by $\bar{H}^n_{\mathcal{P}}(V)$ or $\bar{H}^n_{\mathcal{P}}(V, \pi)$ and is called the *cohomology of V with coefficients in itself.*

Essentially the same discussion as in §4 of [2] also applies here, showing that the differential graded Lie algebra $(\bar{C}^*_{\mathcal{P}}(V), \bar{\delta}_\pi, [-,-])$ controls the deformations of the $\mathcal{P}$-coalgebra (V, π).

6.7 *The cochain complex* $\bar{C}^*_{\mathcal{P}}(M, V)$

For $n \geq 1$, define the vector space

$$\bar{C}^n_{\mathcal{P}}(M, V) \stackrel{\text{def}}{=} \mathrm{Hom}(M, \mathcal{P}^!(n) \otimes_{\Sigma_n} (V^{\otimes n} \otimes \mathrm{sgn}_n)).$$

Recall that $C = V \oplus M$ is the $\mathcal{P}$-coalgebra with structural morphism $\theta = \eta_C(2)$ (eq. (3.1.2)). The module $\bar{C}^n_{\mathcal{P}}(M, V)$ is identified as a submodule of $\bar{C}^n_{\mathcal{P}}(C) = \bar{C}^n_{\mathcal{P}}(C, C)$ via the injection

$$\iota \colon \bar{C}^n_{\mathcal{P}}(M, V) \hookrightarrow \bar{C}^n_{\mathcal{P}}(C) \tag{6.7.1}$$

defined by

$$(\iota f)(v + m) = f(m)$$

for $f \in \bar{C}^n_{\mathcal{P}}(M, V)$, $v \in V$ and $m \in M$. Define a map

$$\bar{d}^n_\eta \colon \bar{C}^n_{\mathcal{P}}(M, V) \to \bar{C}^{n+1}_{\mathcal{P}}(C)$$

by setting

$$\bar{d}^n_\eta(f) = \bar{\delta}^n_\theta(\iota f),$$

where $\bar{\delta}^*_\theta = \bar{\delta}^{1,*}_\theta + \bar{\delta}^{2,*}_\theta$ is the differential in the deformation complex $\bar{C}^*_{\mathcal{P}}(C)$ (§4.4 of [15]). In order to make $\bar{d}_\eta$ into a differential on $\bar{C}^*_{\mathcal{P}}(M, V)$, we need to make sure that its image lies in $\bar{C}^*_{\mathcal{P}}(M, V)$.

Write

$$\bar{\Gamma}^{-1}(\theta) = \sum_\alpha \rho_\alpha \otimes \Theta_\alpha \in \bar{L}^1_{\mathcal{P}}(C),$$

$$\bar{\Gamma}^{-1}(\pi) = \sum_\alpha \mu_\alpha \otimes \Pi_\alpha \in \bar{L}^1_{\mathcal{P}}(V).$$

Then it follows from the definition of $\theta = \eta_C(2)$ and the formula for $\bar{\Gamma}$ that

$$\Theta_\alpha|_V = \Pi_\alpha,$$

$$im(\Theta_\alpha|_M) \subseteq (V \otimes M) \oplus (M \otimes V).$$

Theorem 6.8. *One has that*

$$\bar{\delta}^{1,n}_\theta(\iota f) = \iota(\bar{d}^{1,n}_\eta f),$$

where

$$(\bar{d}^{1,n}_\eta f)(m) = \frac{1}{2n!} \sum{}' (z_{(1)} \circ_j \rho^*_\alpha)\sigma_j \otimes \epsilon(\sigma_j)\sigma_j^{-1} \Pi^j_\alpha(z_{(2)}) \qquad (6.8.1)$$

with the notations being the same as in Theorem 6.3 *of* [15], *except that* $f(m) = \sum_{(z)} z_{(1)} \otimes z_{(2)}$ *here.*

Proof. Applying Theorem 6.3 of [15] to $\bar{\delta}^{1,n}_\theta(\iota f)$, we note that the computation of $\bar{\delta}^{1,n}_\theta(\iota f)(v + m)$ begins with

$$(\iota f)(v + m) = f(m) \in \mathcal{P}^!(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

In particular, it is independent of v. Since

$$\Theta_\alpha|_V = \Pi_\alpha,$$

it follows that

$$\Theta^j_\alpha|_{V^{\otimes n}} = \Pi^j_\alpha.$$

Therefore, $\bar{\delta}^{1,n}_\theta(\iota f)(v + m)$ is given by the right-hand side of (6.8.1), which finishes the proof. ∎

There is a similar description for $\bar{\delta}_\theta^{2,n}(\iota f)$. Given an element $m \in M$, we know that $\Theta_\alpha(m) \in (V \otimes M) \oplus (M \otimes V)$. Write

$$\Theta_\alpha(m) = \sum\nolimits_1 w^1_{(1)} \otimes w^1_{(2)} + \sum\nolimits_2 w^2_{(1)} \otimes w^2_{(2)}$$

with the first sum in $M \otimes V$ and the second sum in $V \otimes M$. For $i \in \{1, 2\}$, we extend the notation in Theorem 6.5 of [15] as follows:

$$f(w^i_{(i)}) = \sum_{(f(w^i_{(i)}))} f(w^i_{(i)})_{(1)} \otimes f(w^i_{(i)})_{(2)} \in \mathcal{P}^!(n) \otimes_{\Sigma_n} V^{\otimes n},$$

$$\sum\nolimits^i = \sum_\alpha \sum\nolimits_i \sum_{(f(w^i_{(i)}))} \sum_{\sigma \in \Sigma_{n+1}}.$$

Using the fact that

$$(\iota f)|_V = 0,$$
$$(\iota f)|_M = f,$$

and Theorem 6.5 of [15], a similar argument as in Theorem 6.8 establishes the following result.

Theorem 6.9. *One has that*

$$\bar{\delta}_\theta^{2,n}(\iota f) = \iota(\bar{d}_\eta^{2,n} f),$$

where

$$(\bar{d}_\eta^{2,n} f)(m) = \frac{1}{2n!} \sum\nolimits^2 \epsilon(\sigma)(\rho^*_\alpha \circ_2 f(w^2_{(2)})_{(1)})\sigma \otimes \sigma^{-1}(w^2_{(1)} \otimes f(w^2_{(2)})_{(2)})$$
$$+ (-1)^{n+1} \frac{1}{2n!} \sum\nolimits^1 \epsilon(\sigma)(\rho^*_\alpha \circ_1 f(w^1_{(1)})_{(1)})$$
$$\times \sigma \otimes \sigma^{-1}(f(w^1_{(1)})_{(2)} \otimes w^1_{(2)}). \tag{6.9.1}$$

COROLLARY 6.10

The image of the map

$$\bar{d}^n_\eta = \bar{d}^{1,n}_\eta + \bar{d}^{2,n}_\eta$$

lies in $\bar{C}^{n+1}_{\mathcal{P}}(M, V)$.

DEFINITION 6.11

From Corollary 6.10, $(\bar{C}^*_{\mathcal{P}}(M, V), \bar{d}^*_\eta)$ is a subcomplex of $(\bar{C}^*_{\mathcal{P}}(C), \bar{\delta}^*_\theta)$. Define $\bar{H}^n_{\mathcal{P}}(M, V)$ as the n-th cohomology module of $(\bar{C}^*_{\mathcal{P}}(M, V), \bar{d}^*_\eta)$.

Example 6.12. If $\mathcal{P} = \mathrm{As}$, then

$$\bar{H}^n_{\mathcal{P}}(M, V) = H^n_c(M, V),$$

the Hochschild coalgebra cohomology of the coassociative coalgebra V with coefficients in the V-bicomodule M [4, 10, 20].

Example 6.13. If $(M, \eta) = (V, \pi)$ (i.e., V coacting on itself via its structural morphism), then

$$(\bar{C}^*_{\mathcal{P}}(V, V), \bar{d}^*_\pi) = (\bar{C}^*_{\mathcal{P}}(V), \bar{\delta}^*_\pi)$$

and

$$\bar{H}^*_{\mathcal{P}}(V, V) = \bar{H}^*_{\mathcal{P}}(V).$$

7. Interpreting $\bar{H}^2_{\mathcal{P}}(M, V)$ as extensions

The purpose of this section is to give an interpretation of $\bar{H}^2_{\mathcal{P}}(M, V)$ in terms of extensions. The classical case involving a coassociative coalgebra V and a V-bicomodule M is worked out in [10]. Most of the definitions and proofs in this section are modeled after the corresponding ones in [10].

The following is the main result of this section.

Theorem 7.1. *There is a bijection*

$$\bar{H}^2_{\mathcal{P}}(M, V) \cong \mathrm{Ext}(M, V),$$

where $\mathrm{Ext}(M, V)$ *denotes the set of equivalence classes of singular* $\mathcal{P}$*-coalgebra extensions of* V *by* M.

A sketch of the proof, which is a slight modification of standard arguments (see, for e.g., §9.3 of [21]), will be given after the following list of definitions.

DEFINITION 7.2

(1) Let (V, π) and (V', π') be two $\mathcal{P}$-coalgebras. A *morphism* of $\mathcal{P}$-coalgebras ϕ: $(V, \pi) \to (V', \pi')$ is a k-linear map ϕ: $V \to V'$ such that for any $\mu \in \mathcal{P}(n)$,

$$\pi'(\mu) \circ \phi = \phi^{\otimes n} \circ \pi(\mu).$$

(2) A morphism α: $V \to D$ is called a *coretraction* if there exists a morphism γ: $D \to V$ such that $\gamma\alpha = \mathrm{Id}_V$.
(3) The morphisms $V' \xrightarrow{i} V \xrightarrow{j} V''$ are called a *sequence* if $ji = 0$. The morphism j is called a *cokernel* of i if $ji = 0$ and if for every morphism f: $V \to W$ satisfying $fi = 0$, f can be factorized uniquely as $f = gj$ for some g: $V'' \to W$.
(4) A sequence $V' \xrightarrow{i} V \xrightarrow{j} V''$ is called *coexact* if the morphism j can be factored as $V \xrightarrow{k} \bar{V} \xrightarrow{l} V''$, where k is a cokernel of i and l is a monomorphism.
(5) A *singular* $\mathcal{P}$*-coalgebra extension of* V *by* M is a coexact sequence of coalgebras $0 \to V \xrightarrow{\alpha} D \xrightarrow{\beta} M \to 0$, where the comultiplication on M is the zero morphism and α is a coretraction.

(6) Two singular $\mathcal{P}$-coalgebra extensions $0 \to V \to D \to M \to 0$ and $0 \to V \to \bar{D} \to M \to 0$ of V by M are said to be *equivalent* if there exists a morphism $\phi\colon D \to \bar{D}$ of $\mathcal{P}$-coalgebras such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}_V} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}_M} & & \\ 0 & \longrightarrow & V & \longrightarrow & \bar{D} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is commutative.

(7) The set of equivalence classes of singular $\mathcal{P}$-coalgebra extensions of V by M is denoted by $\mathrm{Ext}(M, V)$.

The following lemma is standard.

Lemma 7.3. *The following statements are equivalent*:

(1) *The morphism $\alpha\colon V \to D$ of $\mathcal{P}$-coalgebras is a coretraction.*

(2) *There is a commutative diagram of $\mathcal{P}$-coalgebras*:

$$\begin{array}{ccccc} V & \xrightarrow{\alpha} & D & \longrightarrow & M \\ & {\scriptstyle i_1}\searrow & \cong\downarrow{\scriptstyle a} & \nearrow{\scriptstyle p_2} & \\ & & V \oplus M & & \end{array}$$

in which i_1 and p_2 are the inclusion into the first factor and the projection onto the second factor, *respectively.*

If either one of the two equivalent statements in Lemma 7.3 is true, then for every element $\mu \in \mathcal{P}(2)$, the comultiplication $\pi_D\colon \mathcal{P}(2) \to \mathrm{Hom}(D, D^{\otimes 2})$ on D determines a morphism

$$\pi_D(\mu) = \begin{pmatrix} \pi(\mu) & h(\mu) \\ 0 & m_l(\mu) \\ 0 & m_r(\mu) \\ 0 & m(\mu) \end{pmatrix} : V \oplus M \to V^{\otimes 2} \oplus (V \otimes M) \oplus (M \otimes V) \oplus M^{\otimes 2}. \quad (7.3.1)$$

Here $\pi(\mu)\colon V \to V^{\otimes 2}$ and $m(\mu)\colon M \to M^{\otimes 2}$ are from the comultiplications of V and M, respectively, and $h(\mu)\colon M \to V \otimes V$, $m_l(\mu)\colon M \to V \otimes M$ and $m_r(\mu)\colon M \to M \otimes V$.

Note that there are isomorphisms

$$\begin{aligned} \mathcal{P}^!(2) \otimes_{\Sigma_2} (V^{\otimes 2} \otimes \mathrm{sgn}_2) &\cong \mathrm{Hom}_{\mathbf{k}[\Sigma_2]}((\mathcal{P}^!)^{\#}(2), (V^{\otimes 2} \otimes \mathrm{sgn}_2)) \\ &\cong \mathrm{Hom}_{\mathbf{k}[\Sigma_2]}(\mathcal{P}(2), (V^{\otimes 2} \otimes \mathrm{sgn}_2)), \end{aligned}$$

which implies that

$$\begin{aligned} \bar{C}^2_{\mathcal{P}}(M, V) &\cong \mathrm{Hom}(M, \mathrm{Hom}_{\mathbf{k}[\Sigma_2]}((\mathcal{P}(2), (V^{\otimes 2} \otimes \mathrm{sgn}_2)))) \\ &\cong \mathrm{Hom}_{\mathbf{k}[\Sigma_2]}(M \otimes \mathcal{P}(2), V^{\otimes 2} \otimes \mathrm{sgn}_2). \end{aligned} \quad (7.3.2)$$

Now let $f \in \bar{C}^2_{\mathcal{P}}(M, V)$ be a 2-cocycle. Then f gives rise to a singular $\mathcal{P}$-coalgebra extension of V by M as follows. As a vector space, $D = V \oplus M$. For any $\mu \in \mathcal{P}(2)$, we set $\pi_D(\mu)\colon D \to D^{\otimes 2}$ as the matrix in (7.3.1) with

$$m(\mu) = 0, \quad h(\mu) = f(-, \mu),$$
$$m_l(\mu) = \eta(\mu), \quad m_r(\mu) = \eta^\tau(\mu).$$

Here we are using the isomorphisms in (7.3.2) when considering $f(-, \mu)$, and $\eta(\mu)$ and $\eta^\tau(\mu)$ are as in (3.1.2). The assembled map

$$\pi_D\colon \mathcal{P}(2) \to \mathrm{Hom}(D, D^{\otimes 2})$$

gives a $\mathcal{P}$-coalgebra structure on $D = V \oplus M$ because f is a cocycle, and

$$0 \to V \xrightarrow{i_1} D \xrightarrow{p_2} M \to 0$$

is a singular $\mathcal{P}$-coalgebra extension of V by M. Moreover, it is not hard to check that cohomologous 2-cocycles give rise to equivalent singular $\mathcal{P}$-coalgebra extensions.

Similarly, let us start with a singular $\mathcal{P}$-coalgebra extension

$$0 \to V \xrightarrow{\alpha} D \to M \to 0.$$

This defines a 2-cocycle $f \in \bar{C}^2_{\mathcal{P}}(M, V)$ as follows. Using the isomorphism (7.3.2), we set

$$f(-, \mu) = h(\mu)$$

for $\mu \in \mathcal{P}(2)$, where $h(\mu)$ is as in the matrix in (7.3.1). It is standard that equivalent extensions give rise to cohomologous 2-cocycles. This establishes Theorem 7.1.

8. Relations with $\mathcal{P}$-algebra cohomology

The purpose of this section is to show that our $\mathcal{P}$-coalgebra cohomology can be identified with Balavoine's $\mathcal{P}$-algebra cohomology [1]. The main result in this section (Corollary 8.4) asserts that the cochain complex $\bar{C}^*_{\mathcal{P}}(M, V)$ of a $\mathcal{P}$-coalgebra V with coefficients in a V-corepresentation M is canonically isomorphic, via dualization, to the cochain complex $C^*_{\mathcal{P}}(V^\#, M^\#)$ of the dual $\mathcal{P}$-algebra $V^\#$ with coefficients in the dual $V^\#$-representation $M^\#$.

Passing to cohomology, the duality isomorphism (Corollary 8.5) identifies $\bar{H}^*_{\mathcal{P}}(M, V)$ with $H^*_{\mathcal{P}}(V^\#, M^\#)$. If one restricts to the case when $\mathcal{P} = \mathrm{As}$ (Example 8.6), the associative algebras operad, then one recovers the duality isomorphism of Hochschild (coalgebra) cohomology modules that was first observed by Parshall and Wang [20].

We begin by recalling Balavoine's notion of cohomology of a $\mathcal{P}$-algebra with coefficients.

8.1 *$\mathcal{P}$-algebra cohomology*

Recall from §3.3 of [1] that the cohomology $H^*_{\mathcal{P}}(A, N)$ of the $\mathcal{P}$-algebra (A, π) with coefficients in the representation (N, ψ) is defined by the cochain complex $C^*_{\mathcal{P}}(A, N)$ with

$$C^n_{\mathcal{P}}(A, N) = \mathrm{Hom}((\mathcal{P}^!)^\#(n) \otimes_{\Sigma_n} A^{\otimes n}, N).$$

It is considered a submodule of $C^n_{\mathcal{P}}(A \oplus N) = C^n_{\mathcal{P}}(A \oplus N, A \oplus N)$ via the inclusion

$$\begin{aligned} C^n_{\mathcal{P}}(A, N) &\xrightarrow{\iota} C^n_{\mathcal{P}}(A \oplus N) \\ f &\mapsto \iota f \end{aligned} \tag{8.1.1}$$

defined by

$$(\iota f)(\mu \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} 0, & \text{if } a_i \in N \text{ for some } i, \\ f(\mu \otimes a_1 \otimes \cdots \otimes a_n), & \text{if every } a_i \in A. \end{cases}$$

The submodule $C^n_{\mathcal{P}}(A, N)$ is closed under the differential δ^*_ψ in $C^*_{\mathcal{P}}(A \oplus N)$ (Proposition 3.3.1 of [1]), so $C^*_{\mathcal{P}}(A, N)$ is a subcomplex of $C^*_{\mathcal{P}}(A \oplus N)$. The differential in $C^*_{\mathcal{P}}(A, N)$, which is the restriction of δ^*_ψ, is denoted by d^*_ψ.

8.2 *Duality*

We are still assuming that (M, η) is a finite dimensional corepresentation of the finite dimensional $\mathcal{P}$-coalgebra (V, π) and that $C = V \oplus M$ is the $\mathcal{P}$-coalgebra with structural morphism $\theta = \eta_C(2) \in \bar{C}^2_{\mathcal{P}}(C)$ (eq. (3.1.2)).

According to Corollary 4.8, $(M^\#, \eta^\#)$ is a representation of the $\mathcal{P}$-algebra $(V^\#, \pi^\#)$. In other words, the linear dual $C^\# = V^\# \oplus M^\#$ is a $\mathcal{P}$-algebra with structural morphism $\theta^\# = \xi^2\theta$ (eq. (4.6.1)).

In this framework, we have the submodule inclusions

$$\begin{aligned} \bar{C}^n_{\mathcal{P}}(M, V) &\subseteq \bar{C}^n_{\mathcal{P}}(C) \quad \text{(by (6.7.1))}, \\ C^n_{\mathcal{P}}(V^\#, M^\#) &\subseteq C^n_{\mathcal{P}}(C^\#) \quad \text{(by (8.1.1))}. \end{aligned}$$

Recall that the dualization isomorphism (eq. (5.1) of [15])

$$\xi^n \colon \bar{C}^n_{\mathcal{P}}(C) \xrightarrow{\cong} C^n_{\mathcal{P}}(C^\#)$$

is given by

$$(\xi^n\varphi)(\mu \otimes \alpha)(x) = \sum \langle \mu, \varphi(x)_{(1)} \rangle \langle \alpha, \varphi(x)_{(2)} \rangle, \tag{8.2.1}$$

for $\varphi \in \bar{C}^n_{\mathcal{P}}(C)$, $\mu \in (\mathcal{P}^!)^\#(n)$, $\alpha \in (C^\#)^{\otimes n} \cong (C^{\otimes n})^\#$, and $x \in C$. Here

$$\varphi(x) = \sum \varphi(x)_{(1)} \otimes \varphi(x)_{(2)} \in \mathcal{P}^!(n) \otimes_{\Sigma_n} C^{\otimes n}.$$

Theorem 8.3. *Under the dualization isomorphism*

$$\xi^n \colon \bar{C}^n_{\mathcal{P}}(C) \xrightarrow{\cong} C^n_{\mathcal{P}}(C^\#),$$

the image of $\bar{C}^n_{\mathcal{P}}(M, V)$ is exactly $C^n_{\mathcal{P}}(V^\#, M^\#)$.

Proof. Pick $f \in \bar{C}^n_{\mathcal{P}}(M, V)$, $\mu \in (\mathcal{P}^!)^\#(n)$, $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \in (C^\#)^{\otimes n} \cong (C^{\otimes n})^\#$, $v \in V$, and $m \in M$. Write

$$f(m) = \sum f(m)_{(1)} \otimes f(m)_{(2)} \in \mathcal{P}^!(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

Then we have

$$(\xi^n f)(\mu \otimes \alpha)(v + m) = (\xi^n f)(\mu \otimes \alpha)(m) \\ = \sum \langle \mu, f(m)_{(1)} \rangle \langle \alpha, f(m)_{(2)} \rangle.$$

In particular, if $\alpha_i \in M^{\#}$ for some $i \in \{1, \ldots, n\}$ (i.e., $\alpha_i(V) = 0$), then

$$\langle \alpha, f(m)_{(2)} \rangle = 0$$

because $f(m)_{(2)} \in V^{\otimes n}$. Since this holds for any element $v + m \in C$, it follows that

$$(\xi^n f)(\mu \otimes \alpha_1 \otimes \cdots \otimes \alpha_n) = 0$$

whenever $\alpha_i \in M^{\#}$ for some $i \in \{1, \ldots, n\}$. This shows that

$$\xi^n(\bar{C}^n_{\mathcal{P}}(M, V)) \subseteq C^n_{\mathcal{P}}(V^{\#}, M^{\#}).$$

The inclusion in the other direction is proved by a similar argument. ■

COROLLARY 8.4

The dualization isomorphism restricts to an isomorphism

$$\xi\colon (\bar{C}^*_{\mathcal{P}}(M, V), \bar{d}^*_{\eta}) \xrightarrow{\cong} (C^*_{\mathcal{P}}(V^{\#}, M^{\#}), d^*_{\eta^{\#}})$$

of cochain complexes.

Proof. It is known that the dualization isomorphism commutes with the differentials in $\bar{C}^*_{\mathcal{P}}(C)$ and $C^*_{\mathcal{P}}(C^{\#})$ (Corollary 5.4 of [15]):

$$\xi^{n+1}\bar{\delta}^n_{\theta} = \delta^n_{\theta^{\#}}\xi^n.$$

Since the differentials $\bar{d}^*_{\eta}$ and $d^*_{\eta^{\#}}$ are the restrictions of $\bar{\delta}^*_{\theta}$ and $\delta^*_{\theta^{\#}}$, respectively, the proof now finishes by applying Theorem 8.3. ■

Passing to cohomology, we obtain the following result.

COROLLARY 8.5

There is a duality isomorphism

$$\xi^*\colon \bar{H}^*_{\mathcal{P}}(M, V) \xrightarrow{\cong} H^*_{\mathcal{P}}(V^{\#}, M^{\#})$$

of cohomology modules.

Example 8.6. When $\mathcal{P} = \mathrm{As}$, the duality isomorphism in Corollary 8.5 takes the form

$$H^*_c(M, V) \cong H^*_h(V^{\#}, M^{\#}),$$

where $M^{\#}$ is the dual bimodule over the dual associative algebra $V^{\#}$ and H^*_h denotes Hochschild cohomology [9]. In this case, we recover the result in Proposition 2.8 of [20].

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