Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 4, September 2009, pp. 423–429.

Good points for diophantine approximation

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MS received 19 July 2007; revised 3 June 2009

Abstract. Given a sequence $(x_n)_{n=1}^{\infty}$ of real numbers in the interval [0, 1) and a sequence $(\delta_n)_{n=1}^{\infty}$ of positive numbers tending to zero, we consider the size of the set of numbers in [0, 1] which can be 'well approximated' by terms of the first sequence, namely, those $y \in [0, 1]$ for which the inequality $|y - x_n| < \delta_n$ holds for infinitely many positive integers n. We show that the set of 'well approximable' points by a sequence $(x_n)_{n=1}^{\infty}$, which is dense in [0, 1], is 'quite large' no matter how fast the sequence $(\delta_n)_{n=1}^{\infty}$ converges to zero. On the other hand, for any sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$ tending to zero, there is a well distributed sequence $(x_n)_{n=1}^{\infty}$ in the interval [0, 1] such that the set of 'well approximable' points y is 'quite small'.

Keywords. Uniform distribution; diophantine approximation; Hausdorff dimension.

1. Introduction

Usually, in diophantine approximation one has a set of numbers which are to be approximated and another set whose elements are used to approximate those numbers. The first set may consist of all numbers of some type, say, all real irrational numbers, all algebraic numbers, almost all numbers of some type, or perhaps some special numbers, like $\sqrt{2}, \pi, e, \zeta(3), \log 2$. The set used to approximate may consist of, say, all rational numbers of bounded denominator, all algebraic numbers of bounded degree and height, etc. The question is how well can numbers in the first set be approximated by numbers in the second.

In this paper, motivated by some work related to [2], we reverse the question. We start with a set of numbers used to approximate, and with a required degree of approximation. We inquire about the size of the set of numbers which can be approximated by numbers of the first set with the prescribed degree of accuracy. More precisely, let $\mathbf{x} = (x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ a sequence of positive numbers converging to 0. Denote by $G(\mathbf{x}, \boldsymbol{\delta})$ the set of those real numbers y, for which the inequality $|y - x_n| < \delta_n$ has infinitely many solutions in positive integers n. Below, we shall study the set $G(\mathbf{x}, \boldsymbol{\delta})$.

In particular, suppose that the sequence $\mathbf{x}$ is uniformly distributed in the interval [0, 1]. Then $\liminf_{n\to\infty} |y - x_n| = 0$ for any $y \in [0, 1]$. The question is whether at least for some y it is possible to claim something stronger than this, for instance, give an upper bound $|y - x_n| < \delta_n$ (with some 'explicit' $\delta_n \to 0$ as $n \to \infty$, e.g., $\delta_n = 1/n!$) which holds for infinitely many $n \in \mathbf{N}$.

- Is it true that, for any sequence $\mathbf{x}$ which is uniformly distributed in $[0, 1]$ and any sequence $\boldsymbol{\delta}$ of positive numbers tending to 0, the set $G(\mathbf{x}, \boldsymbol{\delta})$ is non-empty?
- Is there an explicit slowly decreasing sequence $\boldsymbol{\delta}$ such that $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]$ for any $\mathbf{x}$ which is uniformly distributed in $[0, 1]$?

In §2 we present the main results. In particular, the answers to these questions will turn out to be 'yes' and 'no', respectively. Section 3 is devoted to the proofs, and §4 to examples.

2. The main results

As stated above, we shall answer the first question in the affirmative. Moreover, we relax the condition on $\mathbf{x}$ from uniform distribution to just density and, at the same time, show that the set $G(\mathbf{x}, \boldsymbol{\delta})$ is not only non-empty, but also quite large, even if $\boldsymbol{\delta}$ decreases arbitrarily fast. This means that for 'many' numbers y the inequality $|y - x_n| < \delta_n$ has infinitely many solutions in $n \in \mathbf{N}$.

Theorem 2.1. *Let* $\mathbf{x} = (x_n)_{n=1}^{\infty}$ *be an arbitrary dense sequence in* $[0, 1]$, *and let* $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ *be an arbitrary sequence of positive numbers. Then the set* $G(\mathbf{x}, \boldsymbol{\delta})$ *is an uncountable dense subset of the interval* $[0, 1]$.

Obviously, the faster the sequence $\boldsymbol{\delta}$ converges to 0, the smaller the set $G(\mathbf{x}, \boldsymbol{\delta})$ tends to be. The following theorem brings two results which make this claim more formal. We use here the notion of Hausdorff dimension, which lets us distinguish between 'large' and 'small' sets of Lebesgue measure zero. The Hausdorff dimension of a set E is denoted by $\dim_H(E)$. For the definition and properties of Hausdorff dimension we refer to [3].

Theorem 2.2. *Let* $\mathbf{x} = (x_n)_{n=1}^{\infty}$ *be an arbitrary sequence in* $[0, 1]$, *and let* $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ *be an arbitrary sequence of positive numbers.*

1. *If* $\sum_{n=1}^{\infty} \delta_n < \infty$, *then the set* $G(\mathbf{x}, \boldsymbol{\delta})$ *is of Lebesgue measure* 0.
2. *If* $\sum_{n=1}^{\infty} \delta_n^s < \infty$ *for some* $0 < s < 1$, *then* $\dim_H(G(\mathbf{x}, \boldsymbol{\delta})) \le s$.

The following example shows that the second part of the theorem is sharp. Consider the process of constructing Cantor's middle thirds set, where one starts with $[0, 1]$ and at each step removes the 'middle third' open interval from every remaining interval. Let $I_{k1} = [0, 1/3^k]$, $I_{k2} = [2/3^k, 1/3^{k-1}], \ldots, I_{k2^k} = [1-1/3^k, 1]$ be the intervals remaining after k steps. For $k \ge 1$ and $1 \le j \le 2^k$, let m_{kj} denote the middle point of I_{kj}. Finally, let $\mathbf{x}$ be the sequence consisting of all m_{kj}'s and let $\boldsymbol{\delta}$ be the sequence consisting of all $|I_{kj}|/2$, where $|I_{kj}|$ denotes the length of the corresponding interval I_{kj}. We have $x_1 = 1/6, x_2 = 5/6, x_3 = 1/18, \ldots$ and $\delta_1 = \delta_2 = 1/6, \delta_3 = 1/18, \ldots$. Thus $\sum_{n=1}^{\infty} \delta_n^s = \sum_{k=1}^{\infty} \mathrm{e}^{k(\log 2 - s\log 3) - s\log 2}$, which converges for $s > \log 2/\log 3$. Now the set $G(\mathbf{x}, \boldsymbol{\delta})$ is actually Cantor's set, with the exclusion of countably many points (the endpoints of the I_{kj}'s), so its Hausdorff dimension is $\log 2/\log 3$. Hence $\dim_H(G(\mathbf{x}, \boldsymbol{\delta})) = \log 2/\log 3$, and thus the inequality $\dim_H(G(\mathbf{x}, \boldsymbol{\delta})) \le s$ cannot be replaced by the strict inequality $\dim_H(G(\mathbf{x}, \boldsymbol{\delta})) < s$.

The next statement addresses the divergence case in the first part of the theorem.

Theorem 2.3. *For every* $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ *with* $\sum_{n=1}^{\infty} \delta_n = \infty$ *there exists a sequence* $\mathbf{x} = (x_n)_{n=1}^{\infty}$ *in* $[0, 1)$ *for which* $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]$.

As we already mentioned above, the answer to the second question stated in §1 is negative. Our next result claims that, even if $\mathbf{x}$ is very well behaved and $\boldsymbol{\delta}$ decreases arbitrarily slowly, the set $G(\mathbf{x}, \boldsymbol{\delta})$ may still be 'very small'. In particular, one can strengthen the condition on $\mathbf{x}$ from being uniformly distributed to being well distributed, as defined by Hlawka [4] and Petersen [8] (see, e.g., p. 40 of [6] or p. 5 of [11]). Recall that $\mathbf{x}$ is *well distributed* in [0, 1] (henceforward – w.d.) if

$$\frac{\#\{M \le n < M+N\colon x_n \in I\}}{N} \underset{N\to\infty}{\longrightarrow} |I|$$

uniformly in M for every interval $I \subseteq [0, 1)$, where $|I|$ denotes the length of I.

Theorem 2.4. *For any sequence of positive numbers* $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ *satisfying* $\lim_{n\to\infty} \delta_n = 0$, *there exists a w.d. sequence* $\mathbf{x} = (x_n)_{n=1}^{\infty}$ *such that* $\dim_H(G(\mathbf{x}, \boldsymbol{\delta})) = 0$.

Finally, we discuss the 'typical' situation. Namely, suppose the sequence $\boldsymbol{\delta}$ is given, but the sequence $\mathbf{x}$ is chosen randomly, each x_n being uniformly distributed and independent of the other x_i's. How large is $G(\mathbf{x}, \boldsymbol{\delta})$? Here we shall refer to the interval [0, 1) as the circle, so that the inequality $|y - x_n| < \delta_n$ will be understood to hold modulo 1.

Theorem 2.5. *Let* $\mathbf{x}$ *be a sequence of uniformly distributed independent points in* [0, 1] *and* $\boldsymbol{\delta}$ *a fixed sequence of positive numbers.*

1. $G(\mathbf{x}, \boldsymbol{\delta})$ *is almost surely* (*i.e., with probability* 1) *of Lebesgue measure* 1 *if and only if* $\sum_{n=1}^{\infty} \delta_n = \infty$.
2. $G(\mathbf{x}, \boldsymbol{\delta})$ *is almost surely the whole interval* [0, 1] *if and only if*

$$\sum_{n=1}^{\infty} \frac{\mathrm{e}^{2(\delta_1+\delta_2+\cdots+\delta_n)}}{n^2} = \infty.$$

3. Proofs

Proof of Theorem 2.1. We shall show a little more than required, namely, that $G(\mathbf{x}, \boldsymbol{\delta}) \cap I$ is uncountable for any closed subinterval I of [0, 1]. In fact, without loss of generality we may assume that the sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ lies in [0, 1). Since $\mathbf{x}$ is everywhere dense in the interval [0, 1], any closed subinterval $I \subseteq [0, 1]$ contains infinitely many points $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$. If $\limsup_{i\to\infty} \delta_{n_i} > 0$, then $G(\mathbf{x}, \boldsymbol{\delta})$ contains a subinterval of I of positive length, and the proof is finished immediately. Alternatively, using $\lim_{i\to\infty} \delta_{n_i} = 0$ we can find integers n_j and n_k such that the intervals $[x_{n_j} - \delta_{n_j}/2, x_{n_j} + \delta_{n_j}/2]$ and $[x_{n_k} - \delta_{n_k}/2, x_{n_k} + \delta_{n_k}/2]$ are non-overlapping, and both contained in I. Denote these intervals by I_1 and I_2, respectively. All points y of the union $I_1 \cup I_2$ satisfy one of the required inequalities, i.e., $|y - x_n| < \delta_n$ for $n = n_j$ or $n = n_k$. Similarly, we find non-overlapping subintervals $I_{11}, I_{12} \subset I_1$ and $I_{21}, I_{22} \subset I_2$, such that, at each point of these four intervals $y \in I_{11} \cup I_{12} \cup I_{21} \cup I_{22}$, the inequality $|y - x_n| < \delta_n$ holds for at least two different positive integers n. Continuing in this way, we obtain a Cantor-type uncountable set Y contained in I such that for each $y \in Y$ the inequality $|y - x_n| < \delta_n$ holds for infinitely many $n \in \mathbf{N}$. Hence $Y \subset G(\mathbf{x}, \boldsymbol{\delta})$. This completes the proof of the theorem.

Proof of Theorem 2.2. The first part follows readily from the Borel-Cantelli lemma. Under the assumptions of the second part, $\lim_{n\to\infty}\delta_n = 0$. For every n_0, the set $G(\mathbf{x},\boldsymbol{\delta})$ is covered by the collection of intervals $(x_n - \delta_n, x_n + \delta_n)$, where $n = n_0, n_0 + 1, \ldots$. For sufficiently large n_0, all these intervals are arbitrarily small. Hence the s-dimensional Hausdorff measure of $G(\mathbf{x},\boldsymbol{\delta})$ is 0, and in particular $\dim_H(G(\mathbf{x},\boldsymbol{\delta})) \le s$.

Proof of Theorem 2.3. Replacing the δ_n's by smaller numbers, we may assume that $\delta_n \underset{n\to\infty}{\longrightarrow} 0$. Permuting the δ_n's, we may further assume that $\boldsymbol{\delta}$ is non-increasing. For every $n \ge 1$, put $x_n = \left\{c + \sum_{i=1}^{n-1}\delta_i\right\}$, where $c \in \mathbf{R}$ is a fixed number and $\{\cdot\}$ stands for the fractional part of a number. Since the series $\sum_{n=1}^{\infty}\delta_n$ is divergent, by assumption, for any $y \in [0, 1)$ and any large enough positive integer m there is an $n \in \mathbf{N}$ such that the number $y + m$ lies in

$$\left[c + \sum_{i=1}^{n-1}\delta_i, c + \sum_{i=1}^{n}\delta_i\right) \subseteq [m, m+1).$$

Indeed, if n is so large that $\delta_n < 1 - y$ then we shall not go through the next integer $m+1$ by adding δ_n to $y + m$. It follows that $x_n \le y < x_n + \delta_n$. Similarly, for $y = 1$ and m large enough, there is an $n \in \mathbf{N}$ such that $c + \sum_{i=1}^{n-1}\delta_i \le 1 + m = y + m < c + \sum_{i=1}^{n}\delta_i$. It follows that for any $y \in [0, 1]$ the inequality $|y - x_n| < \delta_n$ has infinitely many solutions. This proves the theorem.

Consider the sequences $\mathbf{x}$ and $\boldsymbol{\delta}$, defined by $x_n = \{\log n\}$ and $\delta_n = 1/n$, where $n \ge 1$. Then (see the proof of Theorem 2.3 with c being Euler's constant γ) we can show that $G(\mathbf{x},\boldsymbol{\delta}) = [0, 1]$. The same holds for $x_n = \{\log\log(n+1)\}$ and $\delta_n = 1/(n\log n)$, where $n \ge 1$.

Lemma 3.1. *Let $(x_n)_{n=1}^{\infty}$ be a w.d. sequence. Then, for any sequence $(n_k)_{k=1}^{\infty}$ of non-negative integers in which infinitely many terms are strictly positive, the sequence consisting of n_1 occurrences of x_1, then n_2 occurrences of the block (x_1, x_2), then n_3 occurrences of the block (x_1, x_2, x_3), and so on, is also w.d.*

More formally, we can define the resulting sequence $(\bar{x}_n)_{n=1}^{\infty}$ as follows. Given n, first take k such that $\sum_{i=1}^{k-1} in_i < n \le \sum_{i=1}^{k} in_i$, and then put $\bar{x}_n := x_r$, where $r \in \{1, 2, \ldots, k\}$ satisfies the congruence $r \equiv \left(n - \sum_{i=1}^{k-1} in_i\right) \bmod k$.

The lemma is obvious, by Weyl's criterion for a sequence $\mathbf{x}$ to be w.d. modulo 1 (see, e.g. [6]).

Proof of Theorem 2.4. Replacing each δ_n by $\sup_{m\ge n}\delta_m$, we may assume that the sequence $\boldsymbol{\delta}$ is non-increasing. (Indeed, if the result is valid for the enlarged values, then it is certainly so for the original values.) Take any w.d. sequence $(x_n)_{n=1}^{\infty}$, and let $\bar{\mathbf{x}} = (\bar{x}_n)_{n=1}^{\infty}$ be the w.d. sequence generated from it by applying the mechanism described in Lemma 3.1, where the sequence $(n_k)_{k=1}^{\infty}$ of positive integers will be determined later. We will show that if n_k grows sufficiently fast, then $\dim_H(G(\bar{\mathbf{x}},\boldsymbol{\delta})) = 0$.

Put $N_k := \left\{\sum_{i=1}^{k-1} in_i + 1, \sum_{i=1}^{k-1} in_i + 2, \ldots, \sum_{i=1}^{k} in_i\right\}$ for each k. By enlarging the entries of $\boldsymbol{\delta}$ we may assume that, for each k, the δ_n's are the same for all n's in N_k. Denote this common value by $\bar{\delta}_k$, i.e., $\delta_n = \bar{\delta}_k$ for every $n \in N_k$. A number y belongs to $G(\bar{\mathbf{x}},\boldsymbol{\delta})$ if and only if there exist infinitely many k's, for each of which there exists some $n \in N_k$ such that $|y - \bar{x}_n| < \bar{\delta}_k$. However, $\bar{x}_n \in \{x_1, x_2, \ldots, x_k\}$ for $n \in N_k$. So the set of y's for which this inequality holds for a fixed k can be covered by a collection of k intervals of length $2\bar{\delta}_k$ each. More specifically, we have

$$G(\bar{\mathbf{x}}, \boldsymbol{\delta}) = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \bigcup_{i=1}^{k} (x_i - \bar{\delta}_k, x_i + \bar{\delta}_k).$$

In particular, for each m, we have $G(\bar{\mathbf{x}}, \boldsymbol{\delta}) \subseteq \bigcup_{k=m}^{\infty} \bigcup_{i=1}^{k} (x_i - \bar{\delta}_k, x_i + \bar{\delta}_k)$, where the lengths of all intervals on the right-hand side become arbitrarily small as m grows. Given any $s > 0$, we can now estimate the sum of s-th powers of the lengths of these intervals:

$$\sum_{k=m}^{\infty} \sum_{i=1}^{k} (2\bar{\delta}_k)^s = \sum_{k=m}^{\infty} k(2\bar{\delta}_k)^s.$$

Now, as $\bar{\delta}_k = \delta_n$ with $n = \sum_{i=1}^{k} i n_i$, where $\lim_{n\to\infty} \delta_n = 0$, by letting n_k increase sufficiently fast, we can make the numbers $\bar{\delta}_k$ decrease fast enough (say, $\bar{\delta}_k \le 2^{-k}$ for $k \ge 2$), so that the series $\sum_{k=2}^{\infty} k(2\bar{\delta}_k)^s$ will converge for every $s > 0$. In particular, for any $\varepsilon > 0$, we have $\sum_{k=m}^{\infty} k(2\bar{\delta}_k)^s < \varepsilon$ provided that m is large enough. Thus, for every $\varepsilon, \delta, s > 0$, the set $G(\bar{\mathbf{x}}, \boldsymbol{\delta})$ can be covered a family of intervals $\{I_k : k \in \mathbf{N}\}$, each of which is of length at most $\boldsymbol{\delta}$, such that $\sum_{k=1}^{\infty} |I_k|^s < \varepsilon$. This implies that $\dim_H(G(\bar{\mathbf{x}}, \boldsymbol{\delta})) = 0$, and thereby completes the proof.

In the proof of Theorem 2.5, we shall use the following result of Shepp, which we state here explicitly for convenience (for a detailed treatment of this result in [5]).

Theorem [10]. Suppose open arcs of lengths $l_1 \ge l_2 \ge l_3 \ge \dots$, where $l_1 < 1$, are thrown independently and uniformly on a circumference of length 1. Then their union covers the circumference with probability 1 if and only if $\sum_{n=1}^{\infty} n^{-2} e^{l_1 + l_2 + \dots + l_n} = \infty$.

Proof of Theorem 2.5. Suppose first that the intervals $(x_n - \delta_n, x_n + \delta_n)$ are taken modulo 1, namely, that we only require that a point y be at a distance of at most δ_n from x_n modulo 1. Then the first part of the theorem follows readily from Borel-Cantelli's lemma, and the second from the above result of Shepp, taking arc lengths $l_n = 2\delta_n$. This easily implies that the conditions in both parts are necessary.

Suppose $\sum_{n=1}^{\infty} \delta_n = \infty$. We may assume that $\delta_n \underset{n\to\infty}{\longrightarrow} 0$. By the above, the intersection of $G(\mathbf{x}, \boldsymbol{\delta})$ with any interval of the form $[\varepsilon, 1-\varepsilon]$ with $\varepsilon > 0$ is almost surely of full measure in that interval, which implies that $G(\mathbf{x}, \boldsymbol{\delta})$ is almost surely of full measure in $[0, 1]$.

Next suppose the condition in the second part is satisfied. We may again assume that $\delta_n \underset{n\to\infty}{\longrightarrow} 0$. Shepp's result implies that $G(\mathbf{x}, \boldsymbol{\delta}) \supseteq [\varepsilon, 1-\varepsilon]$ almost surely for any $\varepsilon > 0$, and therefore $G(\mathbf{x}, \boldsymbol{\delta}) \supseteq (0, 1)$ almost surely. Borel-Cantelli's lemma shows that the points 0 and 1 also belong to $G(\mathbf{x}, \boldsymbol{\delta})$ almost surely, which concludes the proof of the second part.

4. Additional examples

We shall consider the sequence (see, e.g. p. 59 of [11])

$$\mathbf{x} = (1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, \dots),$$

(or, more formally, $x_n = (2i-1)/2^k$ for $n = 2^{k-1} + i - 1$, where $k = 1, 2, 3, \dots$ and $i = 1, 2, \dots, 2^{k-1}$) which is everywhere dense in $[0, 1]$.

As already pointed out, the faster a sequence $\boldsymbol{\delta}$ converges to 0, the smaller the set $G(\mathbf{x}, \boldsymbol{\delta})$ is. We first show that, for the sequence $\boldsymbol{\delta}$ defined by $\delta_n = 1/n$ for each n, we have $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]$. Indeed, one needs to show that, for each $y \in [0, 1]$, the inequality $|y - x_n| < 1/n$ holds for infinitely many $n \in \mathbf{N}$. For $y = 0$, we have $x_{2^{k-1}} = 1/2^k$, so $|0 - x_n| < 1/n$ for every n of the form 2^k. Similarly, using $x_{2^k-1} = (2^k - 1)/2^k$, we see that $|1 - x_n| < 1/n$ for every n of the form $2^k - 1$. Hence both 0 and 1 lie in $G(\mathbf{x}, \boldsymbol{\delta})$. Any $y \in (0, 1)$ belongs to some interval $[x_n, x_{n+1}) = [(2i - 1)/2^k, (2i + 1)/2^k)$, where $n = 2^{k-1} + i - 1$. Moreover, we can assume that $i \le 2^{k-1} - 1$ if n is large enough. Since $x_n \le y < x_{n+1}$, either $|y - x_n|$ or $|y - x_{n+1}|$ is at most $(x_{n+1} - x_n)/2 = 1/2^k$. But $n = 2^{k-1} + i - 1 \le 2^k - 2$, so $1/2^k \le 1/(n + 2) < 1/(n + 1)$. It follows that $\min\{|y - x_n|, |y - x_{n+1}|\} < 1/(n+1)$, so the inequality $|y - x_n| < 1/n$ holds for infinitely many $n \in \mathbf{N}$.

As an alternative example, we take the following slightly faster decreasing sequence $\delta_n = 1/n^{1+\varepsilon}$, $n = 1, 2, 3, \dots$, where $\varepsilon > 0$ is fixed. By Theorem 2.1, the set $G(\mathbf{x}, \boldsymbol{\delta})$ is an uncountable dense subset of the interval [0, 1]. On the other hand, by Theorem 2.2, the Lebesgue measure of $G(\mathbf{x}, \boldsymbol{\delta})$ is zero. In particular, we will show that $G(\mathbf{x}, \boldsymbol{\delta})$ contains no algebraic numbers. Suppose that $y \in G(\mathbf{x}, \boldsymbol{\delta})$. By the definition of $G(\mathbf{x}, \boldsymbol{\delta})$, the inequality

$$|y - x_n| = |y - (2i - 1)/2^k| < 1/n^{1+\varepsilon}$$

holds for infinitely many $n = 2^{k-1} + i - 1$. Since $n \ge 2^{k-1}$, setting $c = 2^{1+\varepsilon}$, we obtain that the inequality $|y - (2i - 1)/2^k| < c/(2^k)^{1+\varepsilon}$ has infinitely many solutions in $i, k \in \mathbf{N}$. This implies that the inequality $|y - p/q| < c/q^{1+\varepsilon}$ has infinitely many solutions in coprime positive integers $p = 2i - 1$ and $q = 2^k$. However, since q is power of a fixed prime number, by a theorem of Ridout [9], no algebraic number y satisfies this inequality for infinitely many such pairs $p \in \mathbf{N}$, $q = 2^k$. So $G(\mathbf{x}, \boldsymbol{\delta}) \subset (0, 1)$ is not only of Lebesgue measure zero, but it also contains no algebraic numbers.

Finally, we shall consider the following uniformly distributed sequence of Farey fractions

$$\mathbf{x} = (0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6, \dots)$$

consisting of consecutive blocks $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \dots$, where $\mathbf{F}_1 = 0$ and $\mathbf{F}_q$, $q \ge 2$, stands for the block of $\varphi(q)$ fractions p/q, where p runs over the elements of $\{1, 2, \dots, q - 1\}$ relatively prime to q.

We claim that $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]$ for the sequence $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ defined by $\delta_n = (\sqrt{3}/\pi + \varepsilon)/\sqrt{n}$, where ε is an arbitrary positive number. Similarly, for $\delta_n = (3/(\pi^2\sqrt{5}) + \varepsilon)/n$, we shall prove that $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]\backslash\mathbf{Q}$. Moreover, in both cases, one cannot replace ε by $-\varepsilon$.

Indeed, let $q \ge 2$ be the integer defined by the inequalities $\sum_{j=1}^{q-1} \varphi(j) < n \le \sum_{j=1}^{q} \varphi(j)$. Then $x_n = p/q$ with some $p \in \{1, \dots, q - 1\}$ satisfying $\gcd(p, q) = 1$. Using the asymptotic formula

$$\sum_{j=1}^{m} \varphi(j) = 3m^2/\pi^2 + O(m \log m)$$

for the sum of values of Euler's function (cf. pp. 24–26 of [7]), we find that $q \sim \pi\sqrt{n/3}$. By Hurwitz's theorem (cf. [1]) for every irrational number $y \in [0, 1]$ and $\varepsilon > 0$ there are infinitely many Farey fractions $x_n = p/q$ satisfying the inequality

$$|y - x_n| = |y - p/q| < 1/(\sqrt{5}q^2) < (3/(\pi^2\sqrt{5}) + \varepsilon)/n.$$

Since Hurwitz's theorem is sharp in the sense that $\sqrt{5}$ in the inequality $|y - p/q| < 1/(\sqrt{5}q^2)$ cannot be replaced by any number strictly greater than $\sqrt{5}$ (e.g., for $y = (\sqrt{5}-1)/2$), we obtain that not only for any irrational number $y \in [0, 1]$ does the inequality $|y - x_n| < (3/(\pi^2\sqrt{5}) + \varepsilon)/n$ have infinitely many solutions in $n \in \mathbf{N}$, but also that ε cannot be replaced by $-\varepsilon$ for some irrational y.

Suppose now that $y = u/v$ is a rational number in [0, 1]. Then, for any $x_n = p/q$ with n large enough, we have $|y - x_n| \geq 1/(vq)$. Moreover, there are infinitely many $n \in \mathbf{N}$ for which $|y - x_n| = 1/(vq)$. Using $q \sim \pi\sqrt{n/3}$, we find that $1/(vq) \sim \sqrt{3}/(v\pi\sqrt{n})$. It follows that, for such y, the inequality $|y - x_n| < (\sqrt{3}/(v\pi) + \varepsilon)/\sqrt{n}$ has infinitely many solutions in $n \in \mathbf{N}$. Hence $y = u/v$ is in $G(\mathbf{x}, \boldsymbol{\delta})$ for $\delta_n = (\sqrt{3}/\pi + \varepsilon)/\sqrt{n}$, giving $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]$ in this case.

On the other hand, the inequality $|y - x_n| > (\sqrt{3}/(v\pi) - \varepsilon)/\sqrt{n}$, where $y = u/v$, holds for every sufficiently large n. Therefore, it is impossible that, for any given $y \in \mathbf{Q}$ and $c > 0$, the inequality $|y - x_n| < c/n$ holds for infinitely many $n \in \mathbf{N}$. This completes the proof of $G(\mathbf{x}, \boldsymbol{\delta}) = [0, 1]\backslash\mathbf{Q}$ for $\delta_n = (3/(\pi^2\sqrt{5}) + \varepsilon)/n$.

Finally, if, say, $y = 0$, i.e., $u = 0$, $v = 1$, then $|y - x_n| = |x_n| < (\sqrt{3}/\pi + \varepsilon)/\sqrt{n}$ holds for infinitely many $n \in \mathbf{N}$, whereas the inequality $|x_n| < (\sqrt{3}/\pi - \varepsilon)/\sqrt{n}$ holds for at most finitely many n's. This implies that $0 \notin G(\mathbf{x}, \boldsymbol{\delta})$ for $\delta_n = (\sqrt{3}/\pi - \varepsilon)/\sqrt{n}$.

Acknowledgments

We express our gratitude to Zhiqiang Xu, who brought [2] to our attention, and whose questions led us to study the problems considered in this paper, and to Michael Boshernitzan, who drew our attention to [5], and to the referee for his helpful comments on the paper.

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