Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 4, September 2009, pp. 411–421.

On the $2m$-th power mean of Dirichlet L-functions with the weight of trigonometric sums

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MS received 26 August 2008; revised 26 November 2008

Abstract. Let p be a prime, χ denote the Dirichlet character modulo p, $f(x) = a_0 + a_1 x + \cdots + a_k x^k$ is a k-degree polynomial with integral coefficients such that $(p, a_0, a_1, \ldots, a_k) = 1$, for any integer m, we study the asymptotic property of

$$\sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right) \right|^2 |L(1, \chi)|^{2m},$$

where $e(y) = \mathrm{e}^{2\pi i y}$. The main purpose is to use the analytic method to study the $2m$-th power mean of Dirichlet L-functions with the weight of the general trigonometric sums and give an interesting asymptotic formula. This result is an extension of the previous results.

Keywords. Dirichlet L-functions; trigonometric sums; congruence equation; asymptotic formula.

1. Introduction

Estimating the various forms of trigonometric sums is the most important research topic in analytic number theory. Let p be a prime, $f(x) = a_0 + a_1 x + \cdots + a_k x^k$ is a k-degree polynomial with integral coefficients such that $(p, a_0, a_1, \ldots, a_k) = 1$. Mordell [8] studied the trigonometric sums over primes and obtained the famous formula

$$\sum_{x=1}^{p-1} e\left(\frac{f(x)}{p}\right) \ll p^{1-\frac{1}{k}},$$

where $e(y) = \mathrm{e}^{2\pi i y}$.

Then Hua [4] and Min [7] extended this result to the case of two variables, that is

$$\sum_{x=1}^{p-1} \sum_{y=1}^{p-1} e\left(\frac{f(x, y)}{p}\right) \ll p^{2-\frac{2}{k}},$$

where $f(x, y)$ is a k-degree polynomial with the two variables x and y, but can not be transformed into one variable.

After that, Carlitz and Uchiyama [1] improved the result of [8] as

$$\left|\sum_{x=1}^{p-1} e\left(\frac{f(x)}{p}\right)\right| \le k\sqrt{p},$$

where $k \ge 2$. Some other results about trigonometric sums can be found in refs [13, 2, 5, 6].

In this paper, we define a general trigonometric sum as

$$\sum_{a=1}^{p} \chi(a) e\left(\frac{f(a)}{p}\right),$$

where χ denotes a Dirichlet character modulo p and $p \nmid (a_0, a_1, \dots, a_k)$. When $\chi = \chi_0$, we can see the trigonometric sums enjoy many good properties as mentioned above. When $f(a) = na$, the sums become Gauss sums

$$G(n, \chi) = \sum_{a=1}^{p} \chi(a) e\left(\frac{an}{p}\right).$$

In particular, for $n = 1$, we write $\tau(\chi) = \sum_{a=1}^{p} \chi(a) e\left(\frac{a}{p}\right)$. Maybe the most important property of $\tau(\chi)$ is that if χ is a primitive character modulo p, then $|\tau(\chi)| = \sqrt{p}$ (see [12] and [9]).

However, when χ varies, and $f(a)$ is a k-degree polynomial, the value of

$$\sum_{a=1}^{p} \chi(a) e\left(\frac{f(a)}{p}\right)$$

is extremely irregular. But it is surprising that if the trigonometric sums are weighted by some arithmetic functions, such as Dirichlet L-functions, they have good mean value properties. There are many results about the mean value of Dirichlet L-functions with weight (see [3], [10], [11], [15]). For example, Yi and Zhang [14] gave the asymptotic formula of Dirichlet L-functions with the weight of $\tau(\chi)$,

$$\begin{aligned}
&\sum_{\chi \ne \chi_0} |\tau(\chi)|^m |L(1, \chi)|^{2k} \\
&\quad = N^{\frac{m}{2}-1} \phi^2(N) \zeta^{2k-1}(2) \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) \\
&\qquad \times \prod_{p \mid M} \left(p^{\frac{m}{2}+1} - 2p^{\frac{m}{2}} + 1\right) + O\left(q^{\frac{m}{2}+\epsilon}\right),
\end{aligned}$$

where q is an integer ≥ 3 and $q = MN$, $(M, N) = 1$, $M = \prod_{\substack{p \mid q \\ p^2 \nmid q}} p$.

In this article, we combine the general trigonometric sums as the weight function with the Dirichlet L-functions and study the asymptotic relation for the twisted general sums

$$\sum_{\chi \ne \chi_0} \left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right)\right|^2 |L(1, \chi)|^{2m}.$$

It leads to some relationship between the general trigonometric sums and the Dirichlet L-functions. More precisely, we prove the following:

Theorem. *Let* $p \geq 3$ *be a prime and* χ *denote the Dirichlet character modulo* p. *Let* $f(x) = \sum_{i=0}^{k} a_i x^i$ *be a polynomial such that* $\deg(f(x)) = k$ *and* $p \nmid (a_0, a_1, \ldots, a_k)$. *Then for any positive integer* m *and* k, *we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right) \right|^2 |L(1, \chi)|^{2m}$$

$$= p^2 \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{2-\frac{1}{k}+\epsilon}),$$

where $\prod_{p_0}$ *denotes the product over all primes different from* p, $C_m^n = \frac{m!}{n!(m-n)!}$, *and the* O *constant only depends on* k *and* ϵ. *If we let* $f(a) = a$, *we have the mean value of Dirichlet* L*-functions with the weight of* $\tau(\chi)$. *In fact, it is also a corollary of* [14].

Corollary. *Let* $p \geq 3$ *be a prime and* χ *denotes the Dirichlet character modulo* p. *Then for any positive integer* m, *we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |\tau(\chi)|^2 \, |L(1, \chi)|^{2m}$$

$$= p^2 \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{1+\epsilon}).$$

For the general case of an odd number $q \geq 3$, we believe that there exists an asymptotic formula for

$$\sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{q-1} \chi(a) e\left(\frac{f(a)}{q}\right) \right|^2 |L(1, \chi)|^{2m},$$

though it remains an open problem.

2. Some lemmas

To complete the proof of the above theorem, we need the following several lemmas. First, we establish an identity of the trigonometric sums with the polynomial.

Lemma 1. *Let* $f(x)$ *be a polynomial with integer coefficients as* $f(x) = a_0 + a_1 x + \cdots + a_k x^k$, *and* χ *be the Dirichlet character modulo* p. *Then we have*

$$\left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right) \right|^2 = p - 1 + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{g(b, a)}{b}\right),$$

where $g(b, a) = f(ab) - f(b) = \sum_{i=0}^{k} a_i (a^i - 1) b^i$.

Proof. Note that for $1 \le b \le p-1$, we have $(b, p) = 1$. According to the properties of characters, we have

$$\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right)\right|^2 = \sum_{a,\, b=1}^{p-1} \chi(a)\bar{\chi}(b) e\left(\frac{f(a)-f(b)}{p}\right)$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(ab)\bar{\chi}(b) e\left(\frac{f(ab)-f(b)}{p}\right).$$

Let

$$g(b, a) = f(ab) - f(b) = \sum_{i=0}^{k} a_i(a^i - 1)b^i.$$

We get

$$\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right)\right|^2 = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) e\left(\frac{g(b, a)}{p}\right)$$

$$= \sum_{b=1}^{p-1} \chi(1) e\left(\frac{g(b, 1)}{p}\right) + \sum_{a=2}^{p-1}\sum_{b=1}^{p-1} \chi(a) e\left(\frac{g(b, a)}{p}\right)$$

$$= p - 1 + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{g(b, a)}{b}\right).$$

This proves Lemma 1.

Lemma 2. *Let $f(x)$ satisfy the conditions of Lemma* 1. *Let $g(x) = g(x, a) = f(ax) - f(x) = \sum_{i=0}^{k} a_i(a^i - 1)x^i$. Then we have the following estimate*:

$$\left|\sum_{b=1}^{p-1} e\left(\frac{g(b, a)}{p}\right)\right| \begin{cases} \ll p^{1-\frac{1}{k}}, & p \nmid (b_0, b_1, \dots, b_k) \\ = p - 1, & p | (b_0, b_1, \dots, b_k) \end{cases}$$

where $b_i = a_i(a^i - 1)$, $i = 0, 1, \dots, k$ and k is the degree of the polynomial $f(x)$.

Proof. The result is apparent if $p | (b_0, b_1, \dots, b_k)$. If $p \nmid (b_0, b_1, \dots, b_k)$, according to the definition of $g(x, a)$, we have (see ref. [1]),

$$\left|\sum_{b=1}^{p-1} e\left(\frac{g(b, a)}{p}\right)\right| \ll p^{1-\frac{1}{k}}.$$

This proves Lemma 2.

Lemma 3. *Let p be a prime with $p \ge 3$ and $d_m(n)$ denote the m-th divisor function. Then for any complex variable s with* $\mathrm{Re}(s) > 1$, *we have*

$$\sum_{\substack{n=1 \\ (n,\, p)=1}}^{\infty} \frac{d_m^2(n)}{n^s} = \zeta^{2m-1}(s)\left(1 - \frac{1}{p^s}\right)^{2m-1} \prod_{p_0 \ne p}\left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_{p_0 \neq p}$ denotes the product over all primes different from p, $C_m^n = \frac{m!}{n!(m-n)!}$.

Proof. See Lemma 3 in ref. [15], and let $q = p$ be a prime.

Lemma 4. *Let p be a prime with $p \geq 3$ and χ a Dirichlet character modulo p. Denote $A(y,\chi) = A(y,\chi,m) = \sum_{N \leq n \leq y} \chi(n) d_m(n)$. Then we have the estimate*

$$\sum_{\chi \neq \chi_0} |A(y,\chi)|^2 \ll y^{2-\frac{4}{2m}+\epsilon} p^2,$$

where ϵ is any fixed positive number.

Proof. See Lemma 4, and let q be a prime [15].

Lemma 5. *Let $p \geq 3$ be a prime and χ be a character modulo p. Then for $1 < a < p$ and any positive integer m, we have*

$$\sum_{\chi \neq \chi_0} \chi(a) |L(1,\chi)|^{2m} = \frac{p\, d_m(a)}{a} \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{\epsilon}),$$

where ϵ is any fixed positive number, $\prod_{p_0}$ denotes the product over all primes different from p.

Proof. For convenience, we put

$$A(\chi, y) = \sum_{\frac{p}{a} \leq n \leq y} \chi(n) d_m(n), \quad B(\chi, y) = \sum_{p \leq n \leq y} \chi(n) d_m(n).$$

Then for $s > 1$, the series $L(s,\chi)$ is absolutely convergent. So applying Abel's identity we have

$$\begin{aligned} L^m(s,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n) d_m(n)}{n^s} \\ &= \sum_{n=1}^{\frac{p}{a}} \frac{\chi(n) d_m(n)}{n^s} + s \int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^{s+1}} \mathrm{d}y \\ &= \sum_{n=1}^{p} \frac{\chi(n) d_m(n)}{n^s} + s \int_{p}^{+\infty} \frac{B(\chi,y)}{y^{s+1}} \mathrm{d}y. \end{aligned}$$

It is clear that the above formula also holds for $s = 1$ and $\chi \neq \chi_0$. Hence according to the definition of the Dirichlet L-function, for any positive integer $a \neq 1$, we have

$$\begin{aligned} &\sum_{\chi \neq \chi_0} \chi(a) |L(1,\chi)|^{2m} \\ &\quad = \sum_{\chi \neq \chi_0} \chi(a) \left| \sum_{n=1}^{\infty} \frac{\chi(n) d_m(n)}{n} \right|^2 \end{aligned}$$

$$= \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{n=1}^{\infty} \frac{\chi(n)d_m(n)}{n}\right) \left(\sum_{l=1}^{\infty} \frac{\bar{\chi}(l)d_m(l)}{l}\right)$$

$$= \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{1\leq n\leq \frac{p}{a}} \frac{\chi(n)d_m(n)}{n} + \int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^2}\mathrm{d}y\right)$$

$$\times \left(\sum_{n=1}^{p} \frac{\bar{\chi}(n)d_m(n)}{n} + \int_{p}^{+\infty} \frac{B(\bar{\chi},y)}{y^2}\mathrm{d}y\right)$$

$$= \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{n=1}^{\frac{p}{a}} \frac{\chi(n)d_m(n)}{n}\right) \left(\sum_{l=1}^{p} \frac{\bar{\chi}(l)d_m(l)}{l}\right)$$

$$+ \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{n=1}^{\frac{p}{a}} \frac{\chi(n)d_m(n)}{n}\right) \left(\int_{p}^{+\infty} \frac{B(\bar{\chi},y)}{y^2}\mathrm{d}y\right)$$

$$+ \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{l=1}^{p} \frac{\bar{\chi}(l)d_m(l)}{l}\right) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^2}\mathrm{d}y\right)$$

$$+ \sum_{\chi\neq\chi_0} \chi(a) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^2}\mathrm{d}y\right) \left(\int_{p}^{+\infty} \frac{B(\bar{\chi},y)}{y^2}\mathrm{d}y\right)$$

$$\equiv M_1 + M_2 + M_3 + M_4 \quad \text{(say)}.$$

Now we will estimate each term of the above.
(i) From the orthogonality relation for character sums modulo p, we know that for $(p,n)=1$, we have the identity

$$\sum_{\chi \bmod p} \chi(n)\bar{\chi}(l) = \begin{cases} \phi(p), & \text{if } n\equiv l \bmod p\,; \\ 0, & \text{otherwise.} \end{cases}$$

Then according to Lemma 3, we can easily get

$$M_1 = \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{n=1}^{\frac{p}{a}} \frac{\chi(n)d_m(n)}{n}\right) \left(\sum_{l=1}^{p} \frac{\bar{\chi}(l)d_m(l)}{l}\right)$$

$$= \sum_{\chi \bmod p} \sum_{n=1}^{\frac{p}{a}} \sum_{l=1}^{p} \frac{\chi(an)\bar{\chi}(l)d_m(n)d_m(l)}{nl} - \sum_{n=1}^{\frac{p}{a}} \sum_{l=1}^{p} \frac{d_m(n)d_m(l)}{nl}$$

$$= \phi(p) \mathop{{\sum}'}_{\substack{n=1 \\ an\equiv l(\bmod p)}}^{\frac{p}{a}} \mathop{{\sum}'}_{l=1}^{p} \frac{d_m(n)d_m(l)}{nl} + O(p^{\epsilon})$$

$$= \phi(p) \sum_{n=1}^{\frac{p}{a}}{}' \frac{d_m^2(n) d_m(a)}{an^2} + O(p^{\epsilon})$$

$$= \frac{d_m(a)\phi(p)}{a} \sum_{n=1}^{\frac{p}{a}}{}' \frac{d_m^2(n)}{n^2} + O(p^{\epsilon})$$

$$= \frac{d_m(a)\phi(p)}{a} \left(\sum_{n=1}^{\infty}{}' \frac{d_m^2(n)}{n^2} - \sum_{n=\frac{p}{a}}^{\infty}{}' \frac{d_m^2(n)}{n^2} \right) + O(p^{\epsilon})$$

$$= \frac{p\, d_m(a)}{a} \zeta^{2m-1}(2) \left(1 - \frac{1}{p^2}\right)^{2m-1} \prod_{p_0 \neq p} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{\epsilon}).$$

$$= \frac{p\, d_m(a)}{a} \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{\epsilon}),$$

where the $'$ from $\sum_n'$ indicates that the sum is extended over those n relatively prime to p.
(ii) According to Lemma 3 and some properties of characters, we have

$$M_2 = \sum_{\chi \neq \chi_0} \chi(a) \left(\sum_{n=1}^{\frac{p}{a}} \frac{\chi(n) d_m(n)}{n} \right) \left(\int_p^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} \mathrm{d}y \right)$$

$$= \sum_{\chi \neq \chi_0} \chi(a) \left(\sum_{n=1}^{\frac{p}{a}} \frac{\chi(n) d_m(n)}{n} \right) \left(\int_p^{p^{3(2^m-2)}} \frac{\sum_{p \leq n \leq y} \bar{\chi}(n) d_m(n)}{y^2} \mathrm{d}y \right)$$

$$+ \sum_{\chi \neq \chi_0} \chi(a) \left(\sum_{n=1}^{\frac{p}{a}} \frac{\chi(n) d_m(n)}{n} \right) \left(\int_{p^{3(2^m-2)}}^{+\infty} \frac{\sum_{p \leq n \leq y} \bar{\chi}(n) d_m(n)}{y^2} \mathrm{d}y \right)$$

$$\ll \int_p^{p^{3(2^m-2)}} \frac{1}{y^2} \left| \sum_{n=1}^{\frac{p}{a}} \sum_{l=p}^{y} \frac{d_m(n) d_m(l)}{n} \sum_{\chi \neq \chi_0} \chi(an) \bar{\chi}(l) \right| \mathrm{d}y$$

$$+ p^{\epsilon} \int_{p^{3(2^m-2)}}^{+\infty} \frac{1}{y^2} \sum_{\chi \neq \chi_0} |B(\bar{\chi}, y)| \mathrm{d}y.$$

Applying the Cauchy inequality and Lemma 4 we can easily get

$$\sum_{\chi \neq \chi_0} |B(\bar{\chi}, y)| \leq \phi^{\frac{1}{2}}(p) \left(\sum_{\chi \neq \chi_0} |B(\bar{\chi}, y)|^2 \right)^{\frac{1}{2}} \leq p^{\frac{3}{2}} y^{1 - \frac{2}{2^m} + \epsilon}.$$

Therefore, we have

$$
\begin{aligned}
M_2 &\ll \int_p^{p^{3(2^m-2)}} \frac{\phi(p)}{y^2} \left| \sideset{}{'}\sum_{\substack{n=1\\ an\equiv l(p)}}^{\frac{p}{a}} \sideset{}{'}\sum_{l=p}^{y} \frac{d_m(n)d_m(l)}{n} \right| \mathrm{d}y \\
&\quad + O\left(p^{\frac{3}{2}} \int_{p^{3(2^m-2)}}^{+\infty} y^{-1-\frac{2}{2^m}+\epsilon} \mathrm{d}y \right) \\
&= O\left(\int_p^{p^{3(2^m-2)}} \frac{\phi(p)}{y^2} \sideset{}{'}\sum_{n=1}^{\frac{p}{a}} \frac{1}{n} \frac{y}{p} \left(\frac{p}{a}\right)^{\epsilon} \mathrm{d}y \right) + O(p^{\epsilon}) \\
&= O(p^{\epsilon}).
\end{aligned}
$$

(iii) Similarly, we also have

$$
M_3 = \sum_{\chi\neq\chi_0} \chi(a) \left(\sum_{l=1}^{p} \frac{\bar{\chi}(l)d_m(l)}{l} \right) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} \mathrm{d}y \right) = O(p^{\epsilon}),
$$

$$
M_4 = \sum_{\chi\neq\chi_0} \chi(a) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} \mathrm{d}y \right) \left(\int_{p}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} \mathrm{d}y \right) = O(p^{\epsilon}).
$$

Combining the estimates of (i), (ii) and (iii) we immediately obtain

$$
\sum_{\chi\neq\chi_0} \chi(a)|L(1,\chi)|^{2m} = \frac{p\, d_m(a)}{a}\, \zeta^{2m-1}(2) \prod_{p_0\neq p} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2} \right) + O(p^{\epsilon}).
$$

This completes the proof of Lemma 5.

Lemma 6. *Let $p \geq 3$ be a prime and χ be a character modulo p. Then for any positive integer m we have*

$$
\sum_{\chi\neq\chi_0} |L(1,\chi)|^{2m} = p\, \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2} \right) + O(p^{\epsilon}),
$$

where ϵ is any fixed positive number, $\prod_{p_0}$ denotes the product over all primes different from p.

Proof. See Lemma 6 and let q be a prime [15].

3. Proof of the theorem

In this section, we shall complete the proof of the theorem. First, from Lemma 1 we have

$$\sum_{\chi\neq\chi_0}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{f(a)}{p}\right)\right|^2|L(1,\chi)|^{2m}$$

$$=\sum_{\chi\neq\chi_0}\left(p-1+\sum_{a=2}^{p-1}\chi(a)\sum_{b=1}^{p-1}e\left(\frac{g(b,a)}{p}\right)\right)|L(1,\chi)|^{2m}$$

$$=(p-1)\sum_{\chi\neq\chi_0}|L(1,\chi)|^{2m}+\sum_{a=2}^{p-1}\sum_{b=1}^{p-1}e\left(\frac{g(b,a)}{p}\right)\sum_{\chi\neq\chi_0}\chi(a)|L(1,\chi)|^{2m}$$

$$=(p-1)\sum_{\chi\neq\chi_0}|L(1,\chi)|^{2m}$$

$$+\sum_{a=2}^{p-1^*}\sum_{b=1}^{p-1}e\left(\frac{g(b,a)}{p}\right)\sum_{\chi\neq\chi_0}\chi(a)|L(1,\chi)|^{2m}$$

$$+\sum_{a=2}^{p-1^{**}}\sum_{b=1}^{p-1}e\left(\frac{g(b,a)}{p}\right)\sum_{\chi\neq\chi_0}\chi(a)|L(1,\chi)|^{2m},$$

where $g(b,a)=\sum_{i=0}^{k}a_i(a^i-1)b^i$, $b_i=a_i(a^i-1)$ and $\sum_{a=2}^{p-1^*}\sum_{b=1}^{p-1}$ and $\sum_{a=2}^{p-1^{**}}\sum_{b=1}^{p-1}$ means $p\nmid(b_0,b_1,\ldots,b_k)$ and $p|(b_0,b_1,\ldots,b_k)$ respectively. Then we will estimate the two sums respectively.

(1) When $p\nmid(b_0,b_1,\ldots,b_k)$, according to Lemmas 2 and 5, we have

$$\left|\sum_{a=2}^{p-1^*}\sum_{b=1}^{p-1}e\left(\frac{g(b,a)}{p}\right)\sum_{\chi\neq\chi_0}\chi(a)|L(1,\chi)|^{2m}\right|$$

$$\ll\sum_{a=2}^{p-1}p^{1-\frac{1}{k}+\epsilon}\left|\sum_{\chi\neq\chi_0}\chi(a)|L(1,\chi)|^{2m}\right|$$

$$\ll\sum_{a=2}^{p-1}\frac{p^{2-\frac{1}{k}+\epsilon}d_m(a)}{a}$$

$$\ll p^{2-\frac{1}{k}+\epsilon},$$

where $d_m(a)\ll a^\epsilon$.

(2) When $p|(b_0,b_1,\ldots,b_k)$, i.e. $p|b_0$, $p|b_1,\ldots,p|b_k$, since $p\nmid(a_0,a_1,\ldots,a_k)$, there is at least one a_l such that $p\nmid a_l$. Then for this l we must have $p|(a^l-1)$, i.e. $a^l\equiv 1(\bmod p)$.

But in the set $\{2, 3, \ldots, p-1\}$, there are at most $l-1$ numbers a such that $p|(a^l-1)$. Also $l-1 < l \leq k$, $a^l > a^l - 1 \geq p$, so $a > p^{\frac{1}{l}} \geq p^{\frac{1}{k}}$. Then from Lemmas 2 and Lemma 5 we have

$$\begin{aligned}
&\left|\sum_{a=2}^{p-1^{**}} \sum_{b=1}^{p-1} e\left(\frac{g(b,a)}{p}\right) \sum_{\chi\neq\chi_0} \chi(a)|L(1,\chi)|^{2m}\right| \\
&\leq \sum_{a=2}^{p-1^{**}} \left|\sum_{b=1}^{p-1} e\left(\frac{g(b,a)}{p}\right)\right| \left|\sum_{\chi\neq\chi_0} \chi(a)|L(1,\chi)|^{2m}\right| \\
&\ll \sum_{a=2}^{p-1^{**}} (p-1)\frac{p\, d_m(a)}{a} \\
&\ll p^2 \left(\max_{p^{1/k}\leq a<p} \left(\frac{d_m(a)}{a}\right)\right) \\
&\quad \times \sharp\{a\colon a \in \{2, 3, \ldots, p-1\} \quad \text{with } a^l \equiv 1 (\bmod\ p)\} \\
&\ll kp^{2-\frac{1}{k}+\epsilon},
\end{aligned}$$

where $d_m(a)$ is the same as that of (1).

Therefore, combining (1), (2) and Lemma 6, we can get

$$\begin{aligned}
&\sum_{\chi\neq\chi_0} \left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right)\right|^2 |L(1,\chi)|^{2m} \\
&= \left|(p-1) \sum_{\chi\neq\chi_0} |L(1,\chi)|^{2m}\right| \\
&\quad + O\left(\left|\sum_{a=2}^{p-1^{*}} \sum_{b=1}^{p-1} e\left(\frac{g(b,a)}{p}\right) \sum_{\chi\neq\chi_0} \chi(a)|L(1,\chi)|^{2m}\right|\right) \\
&\quad + O\left(\left|\sum_{a=2}^{p-1^{**}} \sum_{b=1}^{p-1} e\left(\frac{g(b,a)}{p}\right) \sum_{\chi\neq\chi_0} \chi(a)|L(1,\chi)|^{2m}\right|\right) \\
&= p^2 \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{2-\frac{1}{k}+\epsilon}),
\end{aligned}$$

where $\sum_{a=2}^{p-1^{*}} \sum_{b=1}^{p-1}$ and $\sum_{a=2}^{p-1^{**}} \sum_{b=1}^{p-1}$ denotes the conditions $p \nmid (b_0, b_1, \ldots, b_k)$ and $p|(b_0, b_1, \ldots, b_k)$ respectively, and $\prod_{p_0}$ denotes the product over all primes different from p. The O constant depends upon k and ϵ. Therefore, we get the asymptotic formula

$$\sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right) \right|^2 |L(1,\chi)|^{2m}$$
$$= p^2 \zeta^{2m-1}(2) \prod_{p_0} \left(1 - \frac{1 - C_{2m-2}^{m-1}}{p_0^2}\right) + O(p^{2-\frac{1}{k}+\epsilon}).$$

This completes the proof of the theorem.

Acknowledgement

The authors greatly appreciate Prof. Wenpeng Zhang, Yuan Yi and the referee for their comments and suggestions. This work is supported by NSF (10601039) and PNSF of People's Republic of China.

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