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Strong ideal convergence in probabilistic metric spaces

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Abstract. In the present paper we introduce the concepts of strongly ideal convergent sequence and strong ideal Cauchy sequence in a probabilistic metric (PM) space endowed with the strong topology, and establish some basic facts. Next, we define the strong ideal limit points and the strong ideal cluster points of a sequence in this space and investigate some properties of these concepts.

Keywords. Probabilistic metric space; strong topology; strong ideal convergence; strong ideal Cauchy sequence; strong ideal limit point; strong ideal cluster point.

1. Introduction

Probabilistic metric (PM) spaces were first introduced by Menger [7] as a generalization of ordinary metric spaces. In this theory, the notion of distance has a probabilistic nature. Namely, the distance between two points p and q is represented by a distribution function F_{pq}; and for any positive number x, the value $F_{pq}(x)$ is interpreted as the probability that the distance from p to q is less than x. Such a probabilistic generalization is well adapted for the investigation of physiological thresholds and physical quantities. It has also important applications in nonlinear analysis [2].

The theory was brought to its present state by Schweizer and Sklar [10–13], Šerstnev [16], Tardiff [20] and Thorp [21] in a series of papers. There are also many others studying on PM spaces (see, for instance [3, 8, 15]). A clear and detailed history of the subject up to 1983 can be found in the famous book by Schweizer and Sklar [14].

PM spaces have nice topological properties. Many different topological structures may be defined on a PM space. The one that has received the most attention to date is the strong topology and it is the principal tool of this study. The convergence with respect to this topology is called strong convergence. Since the strong topology is first countable and Hausdorff, it can be completely specified in terms of the strong convergence of sequences.

The aim of this work is to introduce an important and applicable generalization of strong convergence, namely, the strong ideal convergence in a PM space endowed with the strong topology, and to obtain basic results. Recently we have introduced the notion of strong statistical convergence in such a PM space and investigated certain properties [19]. The current work also generalizes the one in [19]. Since the convergence of a sequence in a PM space is very important to probabilistic analysis, we feel that the concept of strong ideal convergence in a PM space would provide a more general framework for the analysis of PM spaces.

The concept of ideal convergence (hereafter we shall say $\mathcal{I}$-convergence, briefly) is a generalization of statistical convergence (see [5, 18]) and also a generalization of ordinary convergence, and it is based on the notion of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers. It was first introduced for sequences in an ordinary metric space by Kostyrko *et al* [6] and since then it has been discussed by many authors (see, for instance [1, 4, 9, 17]).

There are many pioneering works in the theory of $\mathcal{I}$-convergence. In this study we will consider the ones by Dems [4] and Kostyrko *et al* [6] in which the notions of $\mathcal{I}$-Cauchy sequence [4] and $\mathcal{I}$-convergence [6] were introduced in an ordinary metric space. We extend these notions to the setting of sequences in a PM space endowed with the strong topology and try to establish basic facts related to these concepts. Next, we introduce the concepts of strong $\mathcal{I}$-limit point and strong $\mathcal{I}$-cluster point of a sequence in this space and investigate their basic properties.

2. Preliminaries

First we recall some of the basic concepts related to the theory of PM spaces. All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [14].

DEFINITION 2.1

A *distribution function* is a nondecreasing function F defined on $R = [-\infty, +\infty]$, with $F(-\infty) = 0$ and $F(\infty) = 1$.

The set of all distribution functions that are left continuous on $(-\infty, \infty)$ is denoted by Δ.

The elements of Δ are partially ordered via

$$F \leq G \text{ iff } F(x) \leq G(x) \quad \text{for all } x \in R.$$

DEFINITION 2.2

For any a in R, ε_a, the *unit step at* a, is the function in Δ given by

$$\varepsilon_a(x) = \begin{cases} 0, & -\infty \leq x \leq a \\ 1, & a < x \leq \infty \end{cases} \quad \text{for } -\infty \leq a < \infty,$$

$$\varepsilon_\infty(x) = \begin{cases} 0, & -\infty \leq x < \infty \\ 1, & x = \infty \end{cases}.$$

DEFINITION 2.3

A sequence (F_n) of distribution functions *converges weakly* to a distribution function F (and we write $F_n \xrightarrow{w} F$) if and only if the sequence $(F_n(x))$ converges to $F(x)$ at each continuity point x of F.

DEFINITION 2.4

The distance $d_L(F, G)$ between two functions $F, G \in \Delta$ is defined as the infimum of all numbers $h \in (0, 1]$ such that the inequalities

$$F(x-h) - h \leq G(x) \leq F(x+h) + h$$

and

$$G(x-h) - h \leq F(x) \leq G(x+h) + h$$

hold for every $x \in \left(-\frac{1}{h}, \frac{1}{h}\right)$.

It is known that d_L is a metric on Δ and, for any sequence (F_n) in Δ and $F \in \Delta$, we have

$$F_n \xrightarrow{w} F \text{ if and only if } d_L(F_n, F) \longrightarrow 0.$$

DEFINITION 2.5

A *distance distribution function* is a nondecreasing function F defined on $R^+ = [0, \infty]$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left continuous on $(0, \infty)$.

The set of all distance distribution functions is denoted by Δ^+.

Theorem 2.1. *Let $F \in \Delta^+$ be given. Then for any $t > 0$,*

$$F(t) > 1 - t \quad \textit{iff} \quad d_L(F, \varepsilon_0) < t.$$

Note 2.1. Geometrically, $d_L(F, \varepsilon_0)$ is the abscissa of the point of intersection of the line $y = 1 - x$ and the graph of F (completed, if necessary, by the addition of vertical segments at discontinuities).

DEFINITION 2.6

A *triangle function* is a binary operation τ on Δ^+, $\tau\colon \Delta^+ \times \Delta^+ \to \Delta^+$, that is commutative, associative, nondecreasing in each place, and has ε_0 as identity.

DEFINITION 2.7

A *probabilistic metric space* (briefly, a PM space) is a triple $(S, \mathfrak{F}, \tau)$ where S is a nonempty set (whose elements are the *points* of the space), $\mathfrak{F}$ is a function from $S \times S$ into Δ^+, τ is a triangle function, and the following conditions are satisfied for all p, q, r in S:

(i) $\mathfrak{F}(p, p) = \varepsilon_0$, (PM 1)
(ii) $\mathfrak{F}(p, q) \neq \varepsilon_0$ if $p \neq q$, (PM 2)
(iii) $\mathfrak{F}(p, q) = \mathfrak{F}(q, p)$, (PM 3)
(iv) $\mathfrak{F}(p, r) \geq \tau(\mathfrak{F}(p, q), \mathfrak{F}(q, r))$. (PM 4)

In the sequel we shall denote the distribution function $\mathfrak{F}(p, q)$ by F_{pq}.

DEFINITION 2.8

Let $(S, \mathfrak{F}, \tau)$ be a PM space. For $p \in S$ and $t > 0$, the *strong t-neighborhood of* p is defined by the set

$$\mathcal{N}_p(t) = \{q \in S\colon F_{pq}(t) > 1 - t\}.$$

The collection $\mathfrak{N}_p = \{\mathcal{N}_p(t) \colon t > 0\}$ is called the *strong neighborhood system at* p, and the union $\mathfrak{N} = \bigcup_{p \in S} \mathfrak{N}_p$ is said to be the *strong neighborhood system for* S.
Note that we can write $\mathcal{N}_p(t) = \{q \in S \colon d_L(F_{pq}, \varepsilon_0) < t\}$ by Theorem 2.1.

If τ is *continuous*, then the strong neighborhood system $\mathfrak{N}$ determines a *Hausdorff topology* for S. This topology is called the *strong topology for* S.

DEFINITION 2.9

Let $(S, \mathfrak{F}, \tau)$ be a PM space. Then for any $t > 0$, the subset $\mathcal{U}(t)$ of $S \times S$ given by

$$\mathcal{U}(t) = \{(p, q) \colon F_{pq}(t) > 1 - t\}$$

is called the *strong t-vicinity*.

Theorem 2.2. *Let* $(S, \mathfrak{F}, \tau)$ *be a PM space and* τ *is continuous. Then for any* $t > 0$, *there is an* $\eta > 0$ *such that* $\mathcal{U}(\eta) \circ \mathcal{U}(\eta) \subseteq \mathcal{U}(t)$, *where*

$$\mathcal{U}(\eta) \circ \mathcal{U}(\eta) = \{(p, r) \colon \textit{for some } q, \quad (p, q) \textit{ and } (q, r) \textit{ are in } \mathcal{U}(\eta)\}.$$

Note 2.2. Under the hypotheses of Theorem 2.2 we can say that for any $t > 0$, there is an $\eta > 0$ such that $d_L(F_{pr}, \varepsilon_0) < t$ whenever $d_L(F_{pq}, \varepsilon_0) < \eta$ and $d_L(F_{qr}, \varepsilon_0) < \eta$.

In a PM space $(S, \mathfrak{F}, \tau)$ where τ is *continuous,* the strong neighborhood system $\mathfrak{N}$ determines a Kuratowski closure operation which is called the *strong closure;* and, for any subset A of S the strong closure of A is denoted by $k(A)$. For any nonempty subset A of S, $k(A)$ is defined by

$$k(A) = \{p \in S \colon \text{for any } t > 0, \text{ there is a } q \in A \text{ such that } F_{pq}(t) > 1 - t\}.$$

Remark 2.1. Throughout the rest of the paper, when we speak about a PM space $(S, \mathfrak{F}, \tau)$, we always assume that τ is *continuous* and S is endowed with the *strong topology.*

DEFINITION 2.10

Let $(S, \mathfrak{F}, \tau)$ be a PM space. A sequence (p_n) in S is said to be *strongly convergent to a point* p *in* S, and we write $p_n \longrightarrow p$, if for any $t > 0$, there is an integer N such that p_n is in $\mathcal{N}_p(t)$ whenever $n \geq N$. It can easily be shown that

$$p_n \longrightarrow p \text{ iff } d_L(F_{p_n p}, \varepsilon_0) \longrightarrow 0.$$

Similarly, a sequence (p_n) in S is called a *strong Cauchy sequence* if for any $t > 0$, there is an integer N such that (p_m, p_n) is in $\mathcal{U}(t)$ whenever $m, n \geq N$.

In the following, we list some of the basic concepts related to the theory of $\mathcal{I}$-convergence and we refer to [4, 6] for more details.

If X is a non-empty set, then a family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is an ideal if and only if for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and each $B \subset A$ we have $B \in \mathcal{I}$. A non-empty family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is a filter on X if and only if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, and for each $A \in \mathcal{F}$ and each $B \supset A$ we have $B \in \mathcal{F}$. An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. $\mathcal{I} \subset \mathcal{P}(X)$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash A \colon A \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called admissible if and only if $\mathcal{I} \supset \{\{x\} \colon x \in X\}$. For instance, the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathcal{P}(\mathbb{N})$.

Throughout the rest of the paper, $\mathcal{I}$ will denote an admissible ideal of subsets of $\mathbb{N}$.

DEFINITION 2.11

Let (X, ρ) be a metric space. A sequence (x_n) of elements of X is said to be $\mathcal{I}$-*convergent* to an element $x \in X$ (written as $\mathcal{I}$-lim $x_n = x$), provided that for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n \in \mathbb{N}: \rho(x_n, x) \geq \varepsilon\}$$

belongs to $\mathcal{I}$. The element x is called the $\mathcal{I}$-limit of the sequence (x_n).

Note that (x_n) is $\mathcal{I}$-convergent to $x \in X$ iff $\mathcal{I}$-lim $\rho(x_n, x) = 0$.

Observe that the usual convergence in (X, ρ) coincides with $\mathcal{I}_f$-convergence, and that the usual convergence implies $\mathcal{I}$-convergence for any admissible ideal $\mathcal{I}$.

DEFINITION 2.12

Let (X, ρ) be a metric space. A sequence (x_n) in X is said to be an $\mathcal{I}$-*Cauchy sequence* provided that for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\{n \in \mathbb{N}: \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I}.$$

3. Strong $\mathcal{I}$-convergence

In this section we introduce the concepts of strongly $\mathcal{I}$-convergent sequence and strong $\mathcal{I}$-Cauchy sequence in a PM space $(S, \mathfrak{F}, \tau)$, and present some main results.

DEFINITION 3.1

Let $(S, \mathfrak{F}, \tau)$ be a PM space. A sequence (p_n) in S is *strongly* $\mathcal{I}$-*convergent to a point* p *in* S, and we write $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$, provided that

$$\{n \in \mathbb{N}: p_n \notin \mathcal{N}_p(t)\} \in \mathcal{I}$$

for each $t > 0$.

We call p as the *strong* $\mathcal{I}$-*limit of* (p_n).

Using Theorem 2.1 and Definition 2.11, we can say that the following statements are equivalent:

(i) $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$,
(ii) $\{n \in \mathbb{N}: d_L(F_{p_n p}, \varepsilon_0) \geq t\} \in \mathcal{I}$ for each $t > 0$,
(iii) $\mathcal{I} - \lim\ d_L(F_{p_n p}, \varepsilon_0) = 0$.

Note that every strongly convergent sequence is strongly $\mathcal{I}$-convergent but the converse is not true in general as can be seen in the following example.

Example 3.1. Let (S, d) be the Euclidean line and $G(x) = 1 - \mathrm{e}^{-x}$ where $G \in \Delta^+$. Consider the simple space (S, d, G) which is generated by (S, d) and G. Then this space becomes a PM space $(S, \mathfrak{F})$ under the continuous triangle function τ_M, which is in fact a Menger space, where $\mathfrak{F}$ is defined on $S \times S$ by

$$\mathfrak{F}(p, q)(x) = F_{pq}(x) = G(x/d(p, q)) = 1 - \mathrm{e}^{-\frac{x}{|p-q|}}$$

for all $p, q \in S$ and $x \in R^+$. Here we make the convention that $G(x/0) = G(\infty) = 1$ for $x > 0$, and $G(0/0) = G(0) = 0$.

Now let (p_n) be a sequence in $(S, \mathfrak{F}, \tau_M)$ defined by

$$p_n = \begin{cases} n, & \text{if } n \text{ is prime} \\ \frac{1}{n}, & \text{otherwise} \end{cases}.$$

Consider the function $F_{p_n 0}$ defined by

$$F_{p_n 0}(x) = \begin{cases} 1 - \mathrm{e}^{-\frac{x}{n}}, & \text{if } n \text{ is prime} \\ 1 - \mathrm{e}^{-xn}, & \text{otherwise} \end{cases}.$$

Let us take for $\mathcal{I}$ the family $\mathcal{I}_d = \{A \subset \mathbb{N}\colon d(A) = 0\}$ where

$$d(A) = \lim_{n\to\infty} \left(\frac{1}{n}\right) \cdot \mathrm{card}(\{k \in A\colon k \leq n\})$$

is the natural density of a set $A \subset \mathbb{N}$ provided that this limit exists. Then $\mathcal{I}_d$ is an admissible ideal. Thus, we get $\mathcal{I}_d - \lim\ d_L(F_{p_n 0}, \varepsilon_0) = 0$, which shows that $p_n \overset{\text{str-}\mathcal{I}_d}{\longrightarrow} 0$.

Note that the strong $\mathcal{I}$-limit is uniquely determined since $(S, \mathfrak{F}, \tau)$ is Hausdorff. To see this, assume that the sequence (p_n) has two strong $\mathcal{I}$-limits $p, q \in S$ with $p \neq q$. Thus, we can find a number $t > 0$ such that $\mathcal{N}_p(t) \cap \mathcal{N}_q(t) = \emptyset$. Now put $A_1 = \{n \in \mathbb{N}\colon p_n \notin \mathcal{N}_p(t)\}$ and $A_2 = \{n \in \mathbb{N}\colon p_n \notin \mathcal{N}_q(t)\}$. Then we have $(\mathbb{N}\backslash A_2) \subset A_1$, but since $A_1 \in \mathcal{I}$ we should have $(\mathbb{N}\backslash A_2) \in \mathcal{I}$, which is a contradiction. This proves our assertion.

Theorem 3.1. *Let $(S, \mathfrak{F}, \tau)$ be a PM space. If (p_n) and (q_n) are sequences in S such that $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$ and $q_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} q$, then we have*

$$\mathcal{I} - \lim\ d_L(F_{p_n q_n}, F_{pq}) = 0.$$

Proof. It is known that $\mathfrak{F}$ is a uniformly continuous mapping from $S \times S$ into Δ^+ if τ is continuous and S is endowed with the strong topology (see [14]). Namely, for any $t > 0$ there is an $\eta > 0$ such that $d_L(F_{pq}, F_{p'q'}) < t$ whenever $p' \in \mathcal{N}_p(\eta)$ and $q' \in \mathcal{N}_q(\eta)$. Now assume that $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$ and $q_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} q$. Then we have

$$\{n \in \mathbb{N}\colon d_L(F_{p_n q_n}, F_{pq}) \geq t\} \subseteq (\{n \in \mathbb{N}\colon p_n \notin \mathcal{N}_p(\eta)\} \cup \{n \in \mathbb{N}\colon q_n \notin \mathcal{N}_q(\eta)\}) \tag{3.1}$$

for any $t > 0$, and hence

$$\{n \in \mathbb{N}\colon d_L(F_{p_n q_n}, F_{pq}) \geq t\} \in \mathcal{I}$$

for each $t > 0$ since each set on the right-hand side of the relation (3.1) belongs to $\mathcal{I}$. Thus, by Definition 2.11 and the discussion following it, we get $\mathcal{I} - \lim\ d_L(F_{p_n q_n}, F_{pq}) = 0$. ■

We now introduce the concept of strong $\mathcal{I}^*$-convergence in a PM space, which is analogous to the $\mathcal{I}^*$-convergence in an ordinary metric space defined in [6].

DEFINITION 3.2

Let $(S, \mathfrak{F}, \tau)$ be a PM space. A sequence (p_n) in S is *strongly $\mathcal{I}^*$-convergent to a point p in S*, and we write $p_n \overset{\text{str-}\mathcal{I}^*}{\longrightarrow} p$, provided that there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $p_{m_k} \to p$ as $k \to \infty$.

Theorem 3.2. *Let $(S, \mathfrak{F}, \tau)$ be a PM space, (p_n) be a sequence in S. If $p_n \overset{\text{str-}\mathcal{I}^*}{\longrightarrow} p$, then $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$.*

Proof. By assumption, there exists a set $H \in \mathcal{I}$ such that for $M = \mathbb{N} \backslash H = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ we have $p_{m_k} \to p$ as $k \to \infty$. Let $t > 0$. Then there exists a $k_0 \in \mathbb{N}$ such that $p_{m_k} \in \mathcal{N}_p(t)$ for each $k > k_0$. Thus, we have

$$A(t) = \{n \in \mathbb{N}\colon p_n \notin \mathcal{N}_p(t)\} \subset (H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}). \tag{3.2}$$

The set on the right-hand side of (3.2) belongs to $\mathcal{I}$ since $\mathcal{I}$ is admissible. Hence $A(t) \in \mathcal{I}$, which completes the proof. ■

The converse implication does not hold in general, and it depends on the structure of the PM space.

Theorem 3.3. *Let $(S, \mathfrak{F}, \tau)$ be a PM space such that all of its points are isolated. Then strong $\mathcal{I}$-convergence and strong $\mathcal{I}^*$-convergence coincide for each admissible ideal $\mathcal{I}$.*

Proof. Let $p \in S$ and $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$. By Theorem 3.2 it suffices to show that $p_n \overset{\text{str-}\mathcal{I}^*}{\longrightarrow} p$. Since each point of S is isolated, there exists an $\eta > 0$ such that $\mathcal{N}_p(\eta) = \{p\}$. By assumption, we have $\{n \in \mathbb{N}\colon p_n \notin \mathcal{N}_p(\eta)\} \in \mathcal{I}$. Hence

$$\{n \in \mathbb{N}\colon p_n \in \mathcal{N}_p(\eta)\} = \{n \in \mathbb{N}\colon p_n = p\} \in \mathcal{F}(\mathcal{I}),$$

and we get $p_n \overset{\text{str-}\mathcal{I}^*}{\longrightarrow} p$. ■

Theorem 3.4. *Suppose that $\mathcal{I}$ contains an infinite set $M \subset \mathbb{N}$. Then the strong $\mathcal{I}$-convergence is not metrizable.*

Proof. We know that the strong topology is metrizable (see [14]). Now assume that there exists a metric ρ on S such that $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$ iff $\lim_{n\to\infty} \rho(p_n, p) = 0$. Take $q_1, q_2 \in S$, $q_1 \neq q_2$ and write $p_n = q_1$ if $n \in M$, $p_n = q_2$ if $n \notin M$. Then we get $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} q_2$ and thus $\lim_{n\to\infty} \rho(p_n, q_2) = 0$, which implies that $\rho(q_1, q_2) = 0$. This contradicts to $q_1 \neq q_2$. ■

DEFINITION 3.3

Let $(S, \mathfrak{F}, \tau)$ be a PM space. A sequence (p_n) in S is *strong $\mathcal{I}$-Cauchy* provided that for each $t > 0$ there exists an $N \in \mathbb{N}$ such that

$$\{n \in \mathbb{N}\colon p_n \notin \mathcal{N}_{p_N}(t)\} \in \mathcal{I}.$$

Note that for $\mathcal{I} = \mathcal{I}_f$, this yields the strong Cauchy condition.

Theorem 3.5. *In a PM space $(S, \mathfrak{F}, \tau)$, every strongly $\mathcal{I}$-convergent sequence is strong $\mathcal{I}$-Cauchy.*

Proof. Let (p_n) be a sequence in S such that $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$. By Theorem 2.2 we can say that for any $t > 0$ there is an $\eta > 0$ such that

$$d_L(F_{p_n p_N}, \varepsilon_0) < t \;\text{ whenever }\; d_L(F_{p_n p}, \varepsilon_0) < \eta \text{ and } d_L(F_{p p_N}, \varepsilon_0) < \eta. \tag{3.3}$$

Now let $t > 0$ and choose $\eta > 0$ such that (3.3) holds. By assumption we have

$$B(\eta) = \{n \in \mathbb{N}\colon d_L(F_{p_n p}, \varepsilon_0) < \eta\} \in \mathcal{F}(\mathcal{I}).$$

Now pick an $N \in B(\eta)$. Thus, for any $t > 0$ there is an $\eta(t) > 0$ and hence there is an $N(\eta) = N(t) \in \mathbb{N}$ such that $d_L(F_{p_n p_N}, \varepsilon_0) < t$ whenever $n \in B(\eta)$. Therefore, for any $t > 0$ there exists an $N(t) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N}\colon d_L(F_{p_n p_N}, \varepsilon_0) < t\} \in \mathcal{F}(\mathcal{I}).$$

This shows that (p_n) is strong $\mathcal{I}$-Cauchy. ■

Theorem 3.6. *Let $(S, \mathfrak{F}, \tau)$ be a PM space. If every strong $\mathcal{I}$-Cauchy sequence in S is strongly $\mathcal{I}$-convergent in S, then S is complete in the strong topology, i.e., every strong Cauchy sequence in S is strongly convergent to a point in S.*

Proof. Let (p_n) be a strong Cauchy sequence in S. Since $\mathcal{I}$ is admissible, (p_n) is a strong $\mathcal{I}$-Cauchy sequence. Thus, by assumption, we have $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$ for some $p \in S$. Now put $k_0 = 0$ and for $t_n = \frac{1}{n}$, $n \in \mathbb{N}$, pick inductively $k_n \in \mathbb{N}\backslash(\{0, \ldots, k_{n-1}\} \cup A(t_n))$ where $A(t_n) = \{m \in \mathbb{N}\colon p_m \notin \mathcal{N}_p(t_n)\}$. Thus $d_L(F_{p_{k_n} p}, \varepsilon_0) < \frac{1}{n}$ for every n, which implies that $p_{k_n} \longrightarrow p$ as $n \to \infty$. Consequently, we have $p_n \longrightarrow p$. ■

Remark 3.1. A subsequence of a strong $\mathcal{I}$-Cauchy sequence can not be strong $\mathcal{I}$-Cauchy as can be seen in the following example (see [4]).

Example 3.2. Consider the equilateral PM space $(S, \mathfrak{F}, \mathbf{M})$ where $\mathfrak{F}$ is defined by

$$F_{pq} = \begin{cases} F, & p \neq q \\ \varepsilon_0, & p = q \end{cases}$$

and $\mathbf{M}$ is the maximal triangle function. Here $F \in \Delta^+$ is fixed and distinct from ε_0 and ε_∞.

Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be such that there exists a partition of $\mathbb{N}$ into pairwise disjoint infinite sets such that $A \in \mathcal{I}$ and $B \notin \mathcal{I}$, $C \notin \mathcal{I}$. Let $A = \{m_n\colon n \in \mathbb{N}\}$, $B \cup C = \{k_n\colon n \in \mathbb{N}\}$ with m_n and k_n strictly increasing. Define a sequence (p_n) in S as follows. Put $p_{k_n} = p$ for all $n \in \mathbb{N}$. Let

$$p_{m_n} = \begin{cases} p, & \text{if } n \in A \cup B \\ q, & \text{if } n \in C \end{cases}$$

where $p \neq q$. Observe that $p_n \overset{\text{str-}\mathcal{I}}{\longrightarrow} p$, thus (p_n) is strong $\mathcal{I}$-Cauchy by Theorem 3.5. However, the subsequence (p_{m_n}) is not strong $\mathcal{I}$-Cauchy.

4. Strong $\mathcal{I}$-limit points and strong $\mathcal{I}$-cluster points

In this section we extend the concepts of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points of a sequence in an ordinary metric space introduced in [6] to the setting of sequences in a PM space.

Throughout the following S denotes the PM space $(S, \mathfrak{F}, \tau)$ and $\mathcal{I}$ is an admissible ideal. If (p_n) is a sequence in S, then we will call a point $p \in S$ a *strong limit point of* (p_n) provided that there is a subsequence of (p_n) that strongly converges to p. We will denote the set of all strong limit points of (p_n) by $L_s(p_n)$.

DEFINITION 4.1

Let (p_n) be a sequence in S. Then an element $q \in S$ is a strong $\mathcal{I}$-limit point of (p_n) provided that there is a set $M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $p_{m_k} \to q$ as $k \to \infty$. We denote the set of all strong $\mathcal{I}$-limit points of (p_n) by $\mathcal{I}(\Lambda_s(p_n))$.

DEFINITION 4.2

Let (p_n) be a sequence in S. Then an element $r \in S$ is a strong $\mathcal{I}$-cluster point of (p_n) provided that

$$\{n \in \mathbb{N}\colon p_n \in \mathcal{N}_r(t)\} \notin \mathcal{I}$$

for each $t > 0$. We denote the set of all strong $\mathcal{I}$-cluster points of (p_n) by $\mathcal{I}(\Gamma_s(p_n))$.

Theorem 4.1. *For any sequence (p_n) in S, we have*

$$\mathcal{I}(\Lambda_s(p_n)) \subseteq \mathcal{I}(\Gamma_s(p_n)) \subseteq L_s(p_n).$$

Proof. Let $q \in \mathcal{I}(\Lambda_s(p_n))$. Then there exists a set $M = \{m_1 < m_2 < \cdots\} \notin \mathcal{I}$ such that $p_{m_k} \to q$ as $k \to \infty$. Now take $\eta > 0$. Then there exists a $k_0 \in \mathbb{N}$ such that $F_{p_{m_k}q}(\eta) > 1 - \eta$ for $k > k_0$. Hence

$$\{n \in \mathbb{N}\colon F_{p_n q}(\eta) > 1 - \eta\} \supset (M \backslash \{m_1, \ldots, m_{k_0}\})$$

and so $\{n \in \mathbb{N}\colon F_{p_n q}(\eta) > 1 - \eta\} \notin \mathcal{I}$, which means that $q \in \mathcal{I}(\Gamma_s(p_n))$. The second inclusion relationship is clear from the definitions above. ■

Theorem 4.2. *For each sequence (p_n) in S, the set $\mathcal{I}(\Gamma_s(p_n))$ is closed with respect to the strong topology.*

Proof. Let $q \in k(\mathcal{I}(\Gamma_s(p_n)))$. Let $t > 0$. Then there exists a $p_0 \in \mathcal{I}(\Gamma_s(p_n)) \cap \mathcal{N}_q(t)$. Choose $\eta > 0$ such that $\mathcal{N}_{p_0}(\eta) \subset \mathcal{N}_q(t)$. Then we have

$$\{n \in \mathbb{N}\colon p_n \in \mathcal{N}_q(t)\} \supset \{n \in \mathbb{N}\colon p_n \in \mathcal{N}_{p_0}(\eta)\}.$$

Hence $\{n \in \mathbb{N}\colon p_n \in \mathcal{N}_q(t)\} \notin \mathcal{I}$, which shows that $q \in \mathcal{I}(\Gamma_s(p_n))$. Hence the proof is complete. ■

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