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Moment convergence rates in the law of the logarithm for dependent sequences

KE-ANG FU and XIAO-RONG YANG

School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310 018, China
E-mail: statzju@tom.com; yangxr110@gmail.com

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Abstract. Let $\{X_n; n \geq 1\}$ be a strictly stationary sequence of negatively associated random variables with mean zero and finite variance. Set $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{k\leq n} |S_k|$, $n \geq 1$. Suppose $\sigma^2 = \mathsf{E}X_1^2 + 2\sum_{k=2}^{\infty} \mathsf{E}X_1 X_k$ $(0 < \sigma < \infty)$. In this paper, the exact convergence rates of a kind of weighted infinite series of $\mathsf{E}\{M_n - \sigma\varepsilon\sqrt{n\log n}\}_+$ and $\mathsf{E}\{|S_n| - \sigma\varepsilon\sqrt{n\log n}\}_+$ as $\varepsilon \searrow 0$ and $\mathsf{E}\{\sigma\varepsilon\sqrt{\frac{\pi^2 n}{8\log n}} - M_n\}_+$ as $\varepsilon \nearrow \infty$ are obtained.

Keywords. The law of the logarithm; Chung-type law of the logarithm; negative association; moment convergence; tail probability.

1. Introduction and main results

Let $\{X_n; n \geq 1\}$ be a sequence of random variables with common distribution, $\mathsf{E}X_1 = 0$ and $0 < \mathsf{E}X_1^2 < \infty$. Set $S_n = \sum_{k=1}^{n} X_k$, $M_n = \max_{k\leq n} |S_k|$, $n \geq 1$. Denote $\log n = \ln(n \vee e)$, and $x_+ = \max\{x, 0\}$. When $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, Chow [3] first discussed the complete moment convergence, and got the following result.

Theorem A. *Suppose that* $\mathsf{E}X = 0$. *Assume* $p \geq 1, \alpha > 1/2, p\alpha > 1$ *and* $\mathsf{E}(|X|^p + |X|\log(1 + |X|)) < \infty$. *Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha}\mathsf{E}\{\max_{j\leq n} |S_j| - \varepsilon n^{\alpha}\}_+ < \infty.$$

Recently, Jiang *et al* [6], established the following precise rates in the law of the logarithm for the moment convergence of i.i.d. random variables via strong approximation methods.

Theorem B. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with* $\mathsf{E}X = 0, \mathsf{E}X^2 = \sigma^2 < \infty$ *and* $\mathsf{E}(|X|^{2r}/(\log |X|)^r) < \infty$. *Set* $S_n = \sum_{k=1}^{n} X_k, n \geq 1$. *Then for* $r > 1$, *we have*

$$\lim_{\varepsilon\searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} n^{r-2-1/2}\mathsf{E}\left\{|S_n| - \sigma\sqrt{2(\varepsilon + \sqrt{r-1})^2 n\log n}\right\}_+ = \frac{\sigma}{(r-1)\sqrt{2\pi}}.$$

Inspired by Chow [3] and Jiang *et al* [6], here we consider the exact convergence rates in the law of the logarithm and Chung-type law of the logarithm for negatively associated (NA) random variables including partial sums and the maximum of the partial sums. First, we shall give the definition of negatively associated random variables:

DEFINITION 1

A finite sequence of random variables $\{X_k; 1 \le k \le n\}$ is said to be negatively associated (NA), if for every disjoint subsets A and B of $\{1, 2, \dots, n\}$, we have

$$\mathrm{Cov}\{f(X_i; i \in A), g(X_j; j \in B)\} \le 0,$$

whenever f and g are coordinate-wise increasing and the covariance exists. An infinite sequence of random variables is NA if every finite subsequence is NA.

The notion of NA was first introduced by Alam and Saxena [1]. Joag-Dev and Proschan [7] showed that many well known multivariate distributions possess the NA property. Because of its wide application in multivariate statistical analysis and system reliability, the notion of NA has received considerable attention recently. We refer to Joag-Dev and Proschan [7] for fundamental properties, Shao and Su [10] for the law of the iterated logarithm, and Shao [9] for the moment equalities and the maximum of the partial sums inequalities.

In the sequel, let $\{X_n; n \ge 1\}$ be a sequence of strictly stationary NA random variables, $\mathsf{E}X_1 = 0, 0 < \mathsf{E}X_1^2 < \infty$, and set $0 < \sigma^2 = \mathsf{E}X_1^2 + 2\sum_{k=2}^{\infty} \mathsf{E}X_1X_k < \infty$ $(0 < \sigma < \infty)$ unless it is specially mentioned. Now we state our results as follows.

Theorem 1.1. *For any $b > -1/2$, if $\mathsf{E}|X|^{2+\delta} < \infty (0 < \delta \le 1)$, then we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log n)^{b-1/2}}{n^{3/2}} \mathsf{E}\{M_n - \sigma\varepsilon\sqrt{n \log n}\}_+ = \frac{2\sigma \mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2(b+1)}} \tag{1.1}$$

and

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log n)^{b-1/2}}{n^{3/2}} \mathsf{E}\{|S_n| - \sigma\varepsilon\sqrt{n \log n}\}_+ = \frac{\sigma \mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}, \tag{1.2}$$

where N is a standard normal random variable.

Theorem 1.2. *For any $b > -1/2$, if $0 < \mathsf{E}X_1^2 < \infty$, then we have*

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\left\{\sigma\varepsilon\sqrt{\frac{\pi^2 n}{8 \log n}} - M_n\right\}_+ = \frac{\sigma\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}}, \tag{1.3}$$

where $\Gamma(\cdot)$ is the gamma function.

Remark 1.1. It is well-known that Theorems A and B investigated the complete moment convergence and exact moment convergence rates for i.i.d. random variables, respectively. While our main results extend them to dependent (NA) random variables, and further we obtain the precise moment convergence rates of the maximum of the partial sums by universal logarithm law and Chung's logarithm law, which extend the result of Fu and Zhang [4].

Remark 1.2. As we know, to obtain this kind of moment convergence results, one way is using strong approximation methods (c.f. [6]), but this method is not applicable here. Another way is using the Berry–Esseen's inequality (c.f. [8]), and we do not take this approach either.

2. The proof of Theorem 1.1

From this section on, we begin to prove the theorems, and in the sequel, let M, C etc. denote positive constants whose values possibly vary from place to place. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$, and $[x]$ denotes the largest integer $\leq x$. Now we give some lemmas which will be used in the following proofs.

Lemma 2.1 (*p.* 79–80 *of* [2]). *Let* $\{W(t); t \geq 0\}$ *be a standard Wiener process, and let* N *be a standard normal variable. Then for any* $x > 0$,

$$\mathsf{P}\left\{\sup_{0\leq s\leq 1} |W(s)| \geq x\right\} = 1 - \sum_{k=-\infty}^{\infty} (-1)^k \mathsf{P}\{(2k-1)x \leq N \leq (2k+1)x\}$$

$$= 4\sum_{k=0}^{\infty}(-1)^k \mathsf{P}\{N \geq (2k+1)x\}$$

$$= 2\sum_{k=0}^{\infty}(-1)^k \mathsf{P}\{|N| \geq (2k+1)x\}.$$

In particular,

$$\mathsf{P}\left\{\sup_{0\leq s\leq 1} W(s) \geq x\right\} \sim 2\mathsf{P}(N \geq x) \sim \frac{2}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right), \quad as \ x \to \infty.$$

Also, *for any* $x > 0$,

$$\mathsf{P}\left(\sup_{0\leq s\leq 1} |W(s)| \leq x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\exp\left\{-\frac{\pi^2(2k+1)^2}{8x^2}\right\}$$

and

$$\mathsf{P}\left(\sup_{0\leq s\leq 1} |W(s)| \leq x\right) \sim \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right), \quad as \ x \to 0.$$

Lemma 2.2 [12]. *Suppose that* $\{X_n; n \geq 1\}$ *be a sequence of strictly stationary NA random variables with* $\mathsf{E}X_1 = 0$, $0 < \mathsf{E}X_1^2 < \infty$ *and* $0 < \sigma^2 = \mathsf{E}X_1^2 + 2\sum_{k=2}^{\infty}\mathsf{E}X_1X_k < \infty (0 < \sigma < \infty)$. *Then we have*

$$\frac{M_n}{\sigma\sqrt{n}} \to \sup_{0\leq s\leq 1}|W(s)| \ \ and \ \ \frac{S_n}{\sigma\sqrt{n}} \to N \ in\ distribution.$$

Lemma 2.3 [9]. *Let $\{Y_i; 1 \le i \le n\}$ be a sequence of NA random variables with mean zero and finite variance. Denote $S_k = \sum_{i=1}^{k} Y_i$, $i \le k \le n$, $B_n = \sum_{i=1}^{n} \mathsf{E}Y_i^2$. Then for any $z > 0$, $y > 0$,*

$$\mathsf{P}\left(\max_{k\le n} |S_k| \ge z\right) \le 2\mathsf{P}\left(\max_{k\le n} |Y_k| \ge y\right) + 4\exp\left\{-\frac{z^2}{8B_n}\right\} + 4\left(\frac{B_n}{4(zy+B_n)}\right)^{z/(12y)}.$$

Set $b(\varepsilon) = \exp(M/\varepsilon^2)$, $M > 4$ and $0 < \varepsilon < 1/4$, say. Without loss of generality, assume $\sigma = 1$.

Lemma 2.4. *For any $M > 4$, we have*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \times \left| n^{-1/2}\mathsf{E}\{M_n - \varepsilon\sqrt{n\log n}\}_+ - \mathsf{E}\left\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n}\right\}_+ \right| = 0$$

and

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \times |n^{-1/2}\mathsf{E}\{|S_n| - \varepsilon\sqrt{n\log n}\}_+ - \mathsf{E}\{|N| - \varepsilon\sqrt{\log n}\}_+| = 0. \tag{2.1}$$

Proof. We only give the proof of the former, since the proof of the latter is similar.

Note that

$$\begin{aligned}
&\varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \times \left| n^{-1/2}\mathsf{E}\{M_n - \varepsilon\sqrt{n\log n}\}_+ - \mathsf{E}\left\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n}\right\}_+ \right| \\
&= \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \left| n^{-1/2}\int_0^\infty \mathsf{P}(M_n \ge x + \varepsilon\sqrt{n\log n})\mathrm{d}x - \int_0^\infty \mathsf{P}\left(\sup_{0\le s\le 1} |W(s)| \ge x + \varepsilon\sqrt{\log n}\right)\mathrm{d}x \right| \\
&= \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log n)^{b}}{n} \left| \int_0^\infty \mathsf{P}(M_n \ge (x+\varepsilon)\sqrt{n\log n})\mathrm{d}x - \int_0^\infty \mathsf{P}\left(\sup_{0\le s\le 1} |W(s)| \ge (x+\varepsilon)\sqrt{\log n}\right)\mathrm{d}x \right| \\
&\le \varepsilon^{2b+1} \sum_{n\le b(\varepsilon)} \frac{(\log n)^{b}}{n} \int_0^\infty \left| \mathsf{P}(M_n \ge (x+\varepsilon)\sqrt{n\log n}) - \mathsf{P}\left(\sup_{0\le s\le 1} |W(s)| \ge (x+\varepsilon)\sqrt{\log n}\right) \right| \mathrm{d}x \\
&=: \text{(I)} + \text{(II)},
\end{aligned}$$

where

$$(\mathrm{I}) := \varepsilon^{2b+1} \sum_{n \le b(\varepsilon)} \frac{(\log n)^b}{n} \int_0^{\Gamma_n} \Big| \mathsf{P}(M_n \ge (x+\varepsilon)\sqrt{n \log n}) - \mathsf{P}\Big(\sup_{0 \le s \le 1} |W(s)| \ge (x+\varepsilon)\sqrt{\log n}\Big) \Big| \, \mathrm{d}x,$$

$$(\mathrm{II}) := \varepsilon^{2b+1} \sum_{n \le b(\varepsilon)} \frac{(\log n)^b}{n} \int_{\Gamma_n}^{\infty} \Big| \mathsf{P}(M_n \ge (x+\varepsilon)\sqrt{n \log n}) - \mathsf{P}\Big(\sup_{0 \le s \le 1} |W(s)| \ge (x+\varepsilon)\sqrt{\log n}\Big) \Big| \, \mathrm{d}x,$$

$\Gamma_n = (\log n)^{-1/2} \Delta_n^{-1/2}$ and $\Delta_n = \sup_x |\mathsf{P}(M_n \ge x\sqrt{n}) - \mathsf{P}(\sup_{0 \le s \le 1} |W(s)| \ge x)|$. For (I), we can easily get that $\Delta_n \to 0$ as $n \to \infty$ by Lemma 2.2. So, via Toeplitz lemma, p. 120 of [11],

$$\frac{1}{(\log m)^{b+1/2}} \sum_{n=1}^{m} \frac{\Delta_n^{1/2} (\log n)^{b-1/2}}{n} \to 0, \quad \text{as} \quad m \to \infty.$$

Hence,

$$\begin{aligned}
(\mathrm{I}) &\le \varepsilon^{2b+1} \sum_{n \le b(\varepsilon)} \frac{\Delta_n^{1/2} (\log n)^{b-1/2}}{n} \\
&= \varepsilon^{2b+1} (\log[b(\varepsilon)])^{b+1/2} \frac{1}{(\log[b(\varepsilon)])^{b+1/2}} \sum_{n \le b(\varepsilon)} \frac{\Delta_n^{1/2} (\log n)^{b-1/2}}{n} \\
&\le M^{b+1/2} \frac{1}{(\log b(\varepsilon)])^{b+1/2}} \sum_{n \le b(\varepsilon)} \frac{\Delta_n^{1/2} (\log n)^{b-1/2}}{n} \to 0, \quad \text{as} \quad \varepsilon \searrow 0.
\end{aligned}$$

For (II),

$$\begin{aligned}
(\mathrm{II}) &\le \varepsilon^{2b+1} \sum_{n \le b(\varepsilon)} \frac{(\log n)^b}{n} \int_{\Gamma_n}^{\infty} \Big(\mathsf{P}(M_n \ge (x+\varepsilon)\sqrt{n \log n}) + \mathsf{P}\Big(\sup_{0 \le s \le 1} |W(s)| \ge (x+\varepsilon)\sqrt{\log n}\Big) \Big) \, \mathrm{d}x \\
&=: \varepsilon^{2b+1} \sum_{n \le b(\varepsilon)} \frac{(\log n)^b}{n} \int_{\Gamma_n}^{\infty} ((\mathrm{II}_1) + (\mathrm{II}_2)) \, \mathrm{d}x.
\end{aligned}$$

Now we deal with (II_1) and (II_2), respectively. For (II_1), denote $\theta = \sqrt{1/\mathsf{E}X_1^2}$, by using Lemma 2.3 (where $z = (x+\varepsilon)\sqrt{n \log n}$ and $y = \beta z$, $\beta > 0$ to be specialized later), and we have that

$$\int_{\Gamma_n}^{\infty}(\mathrm{II}_1)\mathrm{d}x \le \int_{\Gamma_n}^{\infty}\Bigg(2n\mathsf{P}\{|X_1| \ge \beta(x+\varepsilon)\sqrt{n\log n}\} + 4\exp\left\{-\frac{\theta^2(x+\varepsilon)^2\log n}{8}\right\} + 4\left(\frac{n/\theta^2}{4\beta(x+\varepsilon)^2 n\log n}\right)^{1/(12\beta)}\Bigg)\mathrm{d}x$$

$$=: \int_{\Gamma_n}^{\infty}((\mathrm{II}_{11}) + (\mathrm{II}_{12}) + (\mathrm{II}_{13}))\,dx.$$

Thus it follows that

$$(\mathrm{II}_{11}) \le C(\log n)^{-1}(x+\varepsilon)^{-2} \quad \text{(by Markov's inequality)},$$

$$(\mathrm{II}_{12}) \le C(\log n)^{-1}(x+\varepsilon)^{-2} \quad (\text{since } e^y \ge y,\, y \ge 0)$$

and

$$(\mathrm{II}_{13}) \le C(\log n)^{-1}(x+\varepsilon)^{-2} \quad (\text{taking } \beta = 1/12).$$

Hence,

$$\int_{\Gamma_n}^{\infty}(\mathrm{II}_1)\mathrm{d}x \le C(\log n)^{-1}\int_{\Gamma_n}^{\infty}(x+\varepsilon)^{-2}\mathrm{d}x \le C(\log n)^{-1/2}{\Delta_n}^{1/2}.$$

For (II_2), it follows from Lemma 2.1 that, for any $m \ge 1$ and $x > 0$,

$$2\sum_{k=0}^{2m+1}(-1)^k\mathsf{P}\{|N| \ge (2k+1)x\} \le \mathsf{P}\left\{\sup_{0\le s\le 1}|W(s)| \ge x\right\} \le 2\sum_{k=0}^{2m}(-1)^k\mathsf{P}\{|N| \ge (2k+1)x\}. \tag{2.2}$$

Thus it follows that

$$\begin{aligned}\int_{\Gamma_n}^{\infty}(\mathrm{II}_2)\mathrm{d}x &\le \int_{\Gamma_n}^{\infty}\mathsf{P}\left(\sup_{0\le s\le 1}|W(s)| \ge (x+\varepsilon)\sqrt{\log n}\right)\mathrm{d}x\\ &\le 2\sum_{k=0}^{2m}(-1)^k\int_{\Gamma_n}^{\infty}\mathsf{P}(|N| \ge (2k+1)(x+\varepsilon)\sqrt{\log n})\mathrm{d}x\\ &\le 2C\sum_{k=0}^{2m}\frac{1}{(2k+1)^2}(\log n)^{-1}\int_{\Gamma_n}^{\infty}(x+\varepsilon)^{-2}\mathrm{d}x\\ &\le C(\log n)^{-1/2}{\Delta_n}^{1/2}.\end{aligned}$$

And by Toeplitz lemma (p. 120 of [11]), we get that

$$(\mathrm{II}) \le C\varepsilon^{2b+1}\sum_{n\le b(\varepsilon)}\frac{(\log n)^{b-1/2}}{n}\Delta_n^{1/2} \to 0 \quad \text{as } \varepsilon \to 0.$$

Consequently, the proof is completed. □

Remark 2.1. The proof of (2.1) can be shortened by using Markov's inequality directly in the latter part.

Lemma 2.5. *For* $0 < \varepsilon < 1/4$, *we have uniformly*

$$\lim_{M\to\infty} \varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \mathsf{E}\left\{ \sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n} \right\}_+ = 0.$$

Proof. Noting that by Lemma 2.1, we have that, for k large enough,

$$\begin{aligned} &\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \int_0^\infty \mathsf{P}\left\{ \sup_{0\le s\le 1} |W(s)| \ge \varepsilon\sqrt{\log n} + x \right\} \mathrm{d}x \\ &\le C\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n} \int_0^\infty \mathsf{P}\{N \ge (x+\varepsilon)\sqrt{\log n}\}\mathrm{d}x \\ &\le C\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n} \int_0^\infty \frac{\mathsf{E}|N|^k}{(x+\varepsilon)^k (\log n)^{k/2}} \mathrm{d}x \\ &\le C\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^{b-k/2}}{n} \varepsilon^{-k+1} \\ &= CM^{b+1-k/2} \to 0, \end{aligned}$$

when $M \to \infty$, uniformly for $0 < \varepsilon < 1/4$. □

Lemma 2.6. *For* $b > -1$, *if* $\mathsf{E}|X_1|^{2+\delta} < \infty (0 < \delta \le 1)$, *then*

$$\lim_{M\to\infty} \lim_{\varepsilon\searrow 0} \varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n^{3/2}} \mathsf{E}\{M_n - \varepsilon\sqrt{n\log n}\}_+ = 0.$$

Proof. Denote $\theta = \sqrt{1/\mathsf{E}X_1^2}$. By Lemma 2.3 (where $z = (x+\varepsilon)\sqrt{n\log n}$ and $y = \beta' z$), we have

$$\begin{aligned} &\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^{b-1/2}}{n^{3/2}} \mathsf{E}\{M_n - \varepsilon\sqrt{n\log n}\}_+ \\ &= \varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n} \int_0^\infty \mathsf{P}\{M_n \ge (x+\varepsilon)\sqrt{n\log n}\}\mathrm{d}x \\ &\le 2\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} (\log n)^b \int_0^\infty \mathsf{P}\{|X_1| \ge \beta'(x+\varepsilon)\sqrt{n\log n}\}\mathrm{d}x \\ &\quad + 4\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n} \int_0^\infty \exp\left\{ -\frac{\theta^2 (x+\varepsilon)^2 \log n}{8} \right\} \mathrm{d}x \\ &\quad + 4\varepsilon^{2b+1} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n} \int_0^\infty \left(\frac{n/\theta^2}{4\beta'(x+\varepsilon)^2 n\log n} \right)^{1/(12\beta')} \mathrm{d}x \\ &=: (\mathrm{III}_1) + (\mathrm{III}_2) + (\mathrm{III}_3). \end{aligned}$$

Now, we begin to deal with (III_1), (III_2) and (III_3). Note that

$$
\begin{aligned}
(III_1) &\le C\varepsilon^{2b-\delta}(\log b(\varepsilon))^{-1-\delta/2} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n^{1+\delta/2}} \\
&= C\varepsilon^{2b+2} M^{-1-\delta/2} \sum_{n>b(\varepsilon)} \frac{(\log n)^b}{n^{1+\delta/2}},
\end{aligned}
$$

$$
\begin{aligned}
(III_2) &\le C\varepsilon^{2b+1} \int_{b(\varepsilon)}^{\infty} \frac{(\log y)^b}{y} \int_0^{\infty} \exp\left\{-\frac{\theta^2(x+\varepsilon)^2 \log y}{8}\right\} \mathrm{d}x\mathrm{d}y \\
&\le C\varepsilon^{2b+1} \int_{b(\varepsilon)}^{\infty} \frac{(\log y)^{b-1/2}}{y} \int_{\varepsilon^2\theta^2 \log y/8}^{\infty} s^{-1/2}\mathrm{e}^{-s}\mathrm{d}s\mathrm{d}y \\
&\le C\varepsilon^2 \int_{\theta^2 M/8}^{\infty} t^{b-1/2} \int_t^{\infty} s^{-1/2}\mathrm{e}^{-s}\mathrm{d}s\mathrm{d}t \\
&= C\varepsilon^2 \int_{\theta^2 M/8}^{\infty} s^{-1/2}\mathrm{e}^{-s} \int_{\theta^2 M/8}^{s} t^{b-1/2}\mathrm{d}t\mathrm{d}s \\
&\le C\varepsilon^2 \int_{\theta^2 M/8}^{\infty} s^b \mathrm{e}^{-s}\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
(III_3) &\le C\varepsilon^{-2} \sum_{n>b(\varepsilon)} \frac{1}{n(\log n)^2} \\
&\le C\varepsilon^{-2} \frac{1}{\log b(\varepsilon)} = CM^{-1} \quad \left(\text{taking } \beta' = \frac{1}{12(b+2)}\right).
\end{aligned}
$$

Thus we have the desired result by letting $\varepsilon \to 0$ and $M \to \infty$. □

PROPOSITION 2.1

For any $b > -1/2$, we have

$$
\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log n)^{b-1/2}}{n} \mathsf{E}\left\{ \sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n} \right\}_+ \\
&\quad = \frac{2\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2(b+1)}} \qquad (2.3)
\end{aligned}
$$

and

$$
\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log n)^{b-1/2}}{n} \mathsf{E}\{|N| - \varepsilon\sqrt{\log n}\}_+ = \frac{\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}, \qquad (2.4)
$$

where N is a standard normal random variable.

Proof. Note that for any $m \geq 1$ and $x > 0$, (2.2) holds. Thus, it follows that

$$\begin{aligned} \mathsf{E}\Big\{\sup_{0\le s\le 1}|W(s)|-t\Big\}_+ &= \int_0^\infty \mathsf{P}\Big(\sup_{0\le s\le 1}|W(s)|\ge t+x\Big)\mathrm{d}x \\ &\le 2\sum_{k=0}^{2m}(-1)^k\int_0^\infty \mathsf{P}(|N|\ge (2k+1)(t+x))\mathrm{d}x \\ &= 2\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}\int_0^\infty \mathsf{P}(|N|\ge (2k+1)t+x)\mathrm{d}x \\ &= 2\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}\mathsf{E}\{|N|-(2k+1)t\}_+ \end{aligned}$$

and

$$\mathsf{E}\Big\{\sup_{0\le s\le 1}|W(s)|-t\Big\}_+ \ge 2\sum_{k=0}^{2m+1}\frac{(-1)^k}{2k+1}\mathsf{E}\{|N|-(2k+1)t\}_+.$$

So, it suffices to show that for any $q \geq 1$ and $b > -1/2$,

$$\begin{aligned} &\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^{\infty}\frac{(\log n)^{b-1/2}}{n}\mathsf{E}\{|N|-q\varepsilon\sqrt{\log n}\}_+ \\ &= q^{-2b-1}\frac{\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}. \end{aligned} \tag{2.5}$$

Obviously,

$$\begin{aligned} &\lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\sum_{n=1}^{\infty}\frac{(\log n)^{b-1/2}}{n}\mathsf{E}\{|N|-q\varepsilon\sqrt{\log n}\}_+ \\ &= \lim_{\varepsilon\searrow 0}\varepsilon^{2b+1}\int_e^\infty\frac{(\log y)^{b-1/2}}{y}\int_{q\varepsilon\sqrt{\log y}}^\infty \mathsf{P}\{|N|\ge x\}\mathrm{d}x\mathrm{d}y \\ &= 2q^{-2b-1}\lim_{\varepsilon\searrow 0}\int_{q\varepsilon}^\infty z^{2b}\int_z^\infty \mathsf{P}\{|N|\ge x\}\mathrm{d}x\mathrm{d}z \\ &= 2q^{-2b-1}\lim_{\varepsilon\searrow 0}\int_{q\varepsilon}^\infty \mathsf{P}\{|N|\ge x\}\int_{q\varepsilon}^x z^{2b}\mathrm{d}z\mathrm{d}x \\ &= q^{-2b-1}\frac{2}{2b+1}\lim_{\varepsilon\searrow 0}\int_{q\varepsilon}^\infty \mathsf{P}\{|N|\ge x\}x^{2b+1}\mathrm{d}x \\ &= q^{-2b-1}\frac{\mathsf{E}|N|^{2(b+1)}}{(b+1)(2b+1)}. \end{aligned} \tag{2.6}$$

Thus the proposition is now proved by taking $q = 2k + 1$ and $q = 1$, respectively. □

PROPOSITION 2.2

For any $b > -1/2$, *if* $\mathsf{E}|X|^{2+\delta}$ $(0 < \delta \le 1)$, *then we have*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log n)^{b-1/2}}{n} \times \left| n^{-1/2}\mathsf{E}\{M_n - \varepsilon\sqrt{n \log n}\}_+ - \mathsf{E}\left\{\sup_{0\le s\le 1} |W(s)| - \varepsilon\sqrt{\log n}\right\}_+ \right| = 0$$

and

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+1} \sum_{n=1}^{\infty} \frac{(\log n)^{b-1/2}}{n} \times |n^{-1/2}\mathsf{E}\{|S_n| - \varepsilon\sqrt{n \log n}\}_+ - \mathsf{E}\{|N| - \varepsilon\sqrt{\log n}\}_+| = 0.$$

Proof. It is trivial via Lemmas 2.4–2.6. □

Proof of Theorem 1.1. By using Propositions 2.1 and 2.2, we can easily get the conclusions. □

3. The proof of Theorem 1.2

Similarly, we state some lemmas before showing the proof of Theorem 1.2.

Lemma 3.1. *For any* $M > 4$, *we have*

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n \le b(1/\varepsilon)} \frac{(\log n)^b}{n} \left| n^{-1/2}\mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log n}} - M_n\right\}_+ - \mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2}{8\log n}} - \sup_{0\le s\le 1} |W(s)|\right\}_+ \right| = 0.$$

Proof. Take $\Delta_n = \sup_x |\mathsf{P}(M_n \le x\sqrt{n}) - \mathsf{P}(\sup_{0\le s\le 1} |W(s)| \le x)|$, and by Lemma 2.2 we have $\Delta_n \to 0$. And via Toeplitz Lemma (p. 120 of [11]), again, we conclude that

$$\varepsilon^{-2(b+1)} \sum_{n \le b(1/\varepsilon)} \frac{(\log n)^b}{n} \left| n^{-1/2}\mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log n}} - M_n\right\}_+ - \mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2}{8\log n}} - \sup_{0\le s\le 1} |W(s)|\right\}_+ \right|$$

$$= \varepsilon^{-2(b+1)} \sum_{n \le b(1/\varepsilon)} \frac{(\log n)^b}{n} \left| \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log n}}} \mathsf{P}(M_n/\sqrt{n} \le x)\mathrm{d}x - \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log n}}} \mathsf{P}\left(\sup_{0\le s\le 1} |W(s)| \le x\right)\mathrm{d}x \right|$$

$$\leq \varepsilon^{-2(b+1)} \sum_{n \leq b(1/\varepsilon)} \frac{(\log n)^b}{n} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log n}}}$$

$$\times \left| \mathsf{P}(M_n/\sqrt{n} \leq x) - \mathsf{P}\Big(\sup_{0\leq s\leq 1} |W(s)| \leq x \Big) \right| \mathrm{d}x$$

$$\leq C\varepsilon^{-2b-1} \sum_{n \leq b(1/\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \Delta_n$$

$$= C\varepsilon^{-2b-1} (\log[b(1/\varepsilon)])^{b+1/2} \frac{1}{(\log[b(1/\varepsilon)])^{b+1/2}}$$

$$\times \sum_{n \leq b(1/\varepsilon)} \frac{\Delta_n (\log n)^{b-1/2}}{n}$$

$$\leq CM^{b+1/2} \frac{1}{(\log[b(\varepsilon)])^{b+1/2}}$$

$$\times \sum_{n \leq b(1/\varepsilon)} \frac{\Delta_n (\log n)^{b-1/2}}{n} \to 0, \quad \text{as} \quad \varepsilon \nearrow \infty.$$

Thus the proof is now completed. □

Lemma 3.2. *For* $\varepsilon > 0$ *sufficiently large*, *we have*

$$\lim_{M\to\infty} \varepsilon^{-2(b+1)} \sum_{n > b(1/\varepsilon)} \frac{(\log n)^b}{n} \mathsf{E} \left\{ \varepsilon\sqrt{\frac{\pi^2}{8\log n}} - \sup_{0\leq s\leq 1} |W(s)| \right\}_+ = 0,$$

uniformly in ε.

Proof. By Lemma 2.1, we have that

$$\varepsilon^{-2(b+1)} \sum_{n > b(1/\varepsilon)} \frac{(\log n)^b}{n} \mathsf{E} \left\{ \varepsilon\sqrt{\frac{\pi^2}{8\log n}} - \sup_{0\leq s\leq 1} |W(s)| \right\}_+$$

$$= \varepsilon^{-2(b+1)} \sum_{n > b(1/\varepsilon)} \frac{(\log n)^b}{n} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log n}}} \mathsf{P}\Big\{ \sup_{0\leq s\leq 1} |W(s)| \leq t \Big\} \mathrm{d}t$$

$$\leq C\varepsilon^{-2(b+1)} \int_{b(1/\varepsilon)}^{\infty} \frac{(\log x)^b}{x} \int_0^{\varepsilon\sqrt{\frac{\pi^2}{8\log x}}} \exp\left\{ -\frac{\pi^2}{8t^2} \right\} \mathrm{d}t\mathrm{d}x$$

$$\leq C\varepsilon^{-2(b+1)} \int_{b(1/\varepsilon)}^{\infty} \frac{(\log x)^b}{x} \int_{\log x/\varepsilon^2}^{\infty} y^{-3/2} \exp(-y)\mathrm{d}y\mathrm{d}x$$

$$\leq C \int_M^{\infty} s^b \int_s^{\infty} y^{-3/2} \exp(-y)\mathrm{d}y\mathrm{d}s$$

$$\leq C \int_M^\infty y^{-3/2} \exp(-y) \int_M^y s^b \mathrm{d}s\mathrm{d}y$$

$$\leq C \int_M^\infty y^{b-1/2} \mathrm{e}^{-y} \mathrm{d}y \to 0 \quad \text{as} \quad M \to \infty. \qquad \square$$

Lemma 3.3 [4]. *There exist constants* $\lambda > 0$ *and* $C > 0$ *such that for any* $x \geq 1$ *and* $n \geq 1$,

$$\mathsf{P}\{M_n \leq x\sqrt{n/\log n}\} \leq C \exp\left(-\frac{\lambda \log n}{2x^2}\right).$$

Lemma 3.4. *For* $b > -1/2$ *and* $\varepsilon > 0$ *sufficiently large*, *we have*

$$\lim_{M\to\infty} \varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log n}} - M_n\right\}_+ = 0,$$

uniformly in ε.

Proof. For any ε large enough, it follows from Lemma 3.3 that

$$\varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log n)^b}{n^{3/2}} \mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2 n}{8\log n}} - M_n\right\}_+$$

$$= \varepsilon^{-2(b+1)} \sum_{n>b(1/\varepsilon)} \frac{(\log n)^b}{n^{3/2}} \int_0^{\varepsilon\sqrt{\frac{\pi^2 n}{8\log n}}} \mathsf{P}(M_n \leq t)\mathrm{d}t$$

$$\leq C\varepsilon^{-2b-1} \sum_{n>b(1/\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \mathsf{P}\left(M_n \leq \varepsilon\sqrt{\frac{\pi^2 n}{8\log n}}\right)$$

$$\leq C\varepsilon^{-2b-1} \sum_{n>b(1/\varepsilon)} \frac{(\log n)^{b-1/2}}{n} \exp\left(-\frac{4\lambda \log n}{\varepsilon^2\pi^2}\right)$$

$$\leq C\varepsilon^{-2b-1} \int_{b(1/\varepsilon)}^\infty \frac{(\log x)^{b-1/2}}{x} \exp\left(-\frac{4\lambda \log x}{\varepsilon^2\pi^2}\right) \mathrm{d}x$$

$$\leq C \int_M^\infty y^{b-1/2} \exp\left(-\frac{4\lambda}{\pi^2} y\right) \mathrm{d}y \to 0 \quad \text{as} \quad M \to \infty. \qquad \square$$

PROPOSITION 3.1

For any $b > -1/2$, *we have*

$$\lim_{\varepsilon\nearrow\infty} \varepsilon^{-2(b+1)} \sum_{n=1}^\infty \frac{(\log n)^b}{n} \mathsf{E}\left\{\varepsilon\sqrt{\frac{\pi^2}{8\log n}} - \sup_{0\leq s\leq 1} |W(s)|\right\}_+$$

$$= \frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{2b+2}}.$$

Proof. It follows from Lemma 2.4 of [5] and Lemma 2.1 that

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \mathsf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2}{8 \log n}} - \sup_{0 \le s \le 1} |W(s)| \right\}_+$$

$$= \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \int_0^{\varepsilon \sqrt{\frac{\pi^2}{8 \log n}}} \mathsf{P} \left\{ \sup_{0 \le s \le 1} |W(s)| \le t \right\} \mathrm{d}t$$

$$= \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \int_0^{\varepsilon \sqrt{\frac{\pi^2}{8 \log n}}}$$

$$\times \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{ -\frac{\pi^2 (2k+1)^2}{8t^2} \right\} \mathrm{d}t$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_e^{\infty} \frac{(\log x)^b}{x}$$

$$\times \int_0^{\varepsilon \sqrt{\frac{\pi^2}{8 \log x}}} \exp\left\{ -\frac{\pi^2 (2k+1)^2}{8t^2} \right\} \mathrm{d}t \mathrm{d}x$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_e^{\infty} \frac{(\log x)^b}{x}$$

$$\times \int_{(2k+1)^2 \log x / \varepsilon^2}^{\infty} y^{-3/2} \mathrm{e}^{-y} \mathrm{d}y \mathrm{d}x \qquad \left(t = \frac{\pi(2k+1)}{2\sqrt{2}} y^{-1/2} \right)$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \lim_{\varepsilon \nearrow \infty} \int_{(2k+1)^2/\varepsilon^2}^{\infty} s^b \int_s^{\infty} y^{-3/2} \mathrm{e}^{-y} \mathrm{d}y \mathrm{d}s$$

$$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \lim_{\varepsilon \nearrow \infty} \int_{(2k+1)^2/\varepsilon^2}^{\infty} y^{-3/2} \mathrm{e}^{-y} \int_{(2k+1)^2/\varepsilon^2}^{y} s^b \mathrm{d}s \mathrm{d}y$$

$$= \frac{1}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \lim_{\varepsilon \nearrow \infty} \int_{(2k+1)^2/\varepsilon^2}^{\infty} y^{b-1/2} \mathrm{e}^{-y} \mathrm{d}y$$

$$= \frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}}.$$

Thus we terminate the proof. □

PROPOSITION 3.2

For any $b > -1/2$, we have

$$\limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} \left| n^{-1/2} \mathsf{E} \left\{ \varepsilon\sigma \sqrt{\frac{n\pi^2}{8 \log n}} - M_n \right\}_+ \right.$$
$$\left. - \mathsf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2}{8 \log n}} - \sup_{0 \le s \le 1} |W(s)| \right\}_+ \right| = 0.$$

Proof. It can be readily seen via Lemmas 3.1, 3.2 and 3.4. □

The proof of Theorem 1.2. By Propositions 3.1 and 3.2, we complete the proof immediately. □

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