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# Divisors, measures and critical functions

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**Abstract.** In [4] we have introduced a new distance between Galois orbits over $\mathbb{Q}$. Using generalized divisors, we have extended the notion of trace of an algebraic number to other transcendental quantities. In this article we continue the work started in [4]. We define the critical function for a class of transcendental numbers, that generalizes the notion of minimal polynomial of an algebraic number. Our results extend the results obtained by Popescu *et al* [5].

**Keywords.** Trace function; divisors; minimal polynomial; critical function.

## 1. Introduction

We denote by $\bar{\mathbb{Q}}$ the algebraic closure of the field $\mathbb{Q}$ of rational numbers in $\mathbb{C}$, the field of complex numbers. Let $\alpha \in \bar{\mathbb{Q}}$ and let $K$ be a number field containing $\alpha$ with $m = [K : \mathbb{Q}]$. If $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the embeddings of $K$ into $\mathbb{C}$, we define the *trace* of $\alpha$ by

$$\mathrm{Tr}(\alpha) = \frac{1}{m}\sum_{j=1}^{m} \sigma_j(\alpha). \tag{1}$$

The number $\mathrm{Tr}(\alpha)$ depends only on $\alpha$, and not on a choice of the particular number field $K$ containing $\alpha$. The trace map Tr: $(\bar{\mathbb{Q}}, |\cdot|) \to (\mathbb{C}, |\cdot|)$ is not continuous as shown in the example below.

*Example* 1.1. Consider the sequence $(\alpha_n)_{n\in\mathbb{N}}$ of algebraic numbers given by $\alpha_n = 1 - \sqrt{1 - 1/n} \in \bar{\mathbb{Q}}$. The corresponding minimal polynomials over $\mathbb{Q}$ are $P_{\alpha_n}(x) = x^2 - 2x + 1/n$. It is easy to see that $\alpha_n \to 0$ in $(\bar{\mathbb{Q}}, |\cdot|)$, but $1 = \mathrm{Tr}(\alpha_n) \nrightarrow \mathrm{Tr}(0) = 0$ in $(\mathbb{C}, |\cdot|)$.

Thus the map Tr: $(\bar{\mathbb{Q}}, |\cdot|) \to (\mathbb{C}, |\cdot|)$ cannot be used to define a trace for a complex number that is not an algebraic number over $\mathbb{Q}$, but is equal to the limit in $(\mathbb{C}, |\cdot|)$ of a sequence of algebraic numbers.

However, if we consider the spectral norm $\|\cdot\|_s$ defined by

$$\|x\|_s = \max\{|\sigma(x)|,\ \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\},$$

for any $x \in \bar{\mathbb{Q}}$, the trace function Tr: $(\bar{\mathbb{Q}}, \|\cdot\|_s) \to (\mathbb{C}, |\cdot|)$ becomes continuous since for any $\alpha, \beta \in \bar{\mathbb{Q}}$ we have

$$|\mathrm{Tr}(\alpha) - \mathrm{Tr}(\beta)| \le \|\alpha - \beta\|_s.$$

In order to define a trace function for transcendental elements, one could consider the completion $\tilde{\bar{\mathbb{Q}}}$ of $\mathbb{Q}$ with respect to the spectral norm. The ring $\tilde{\bar{\mathbb{Q}}}$ has been recently studied in [3], [5–8].

*Example* 1.2. The sequence of algebraic numbers $\alpha_n = 1 - \sqrt{1 - 1/n}$ from Example 1.1 does not converge in $(\tilde{\bar{\mathbb{Q}}}, \|\cdot\|_s)$, yet the sequence of traces $(\mathrm{Tr}(\alpha_n))_{n\in\mathbb{N}}$ converges in $(\mathbb{C}, |\cdot|)$. Indeed, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is not Cauchy with respect to spectral norm, since for any positive integer $n$, we have

$$\|\alpha_{n+1} - \alpha_n\|_s = \sqrt{1 - \frac{1}{n}} + \sqrt{1 - \frac{1}{n+1}}\ ,$$

which converges to 2 as $n \to \infty$.

In this paper we continue the work started in [4] and we generalize results from [5]. Our goal is to generalize the critical function $F(\alpha; z) = [P_\alpha(z)]^{\frac{1}{n}}$ of an algebraic number $\alpha$ to a class of transcendental numbers. Here $P_\alpha(z) = \prod_{i=1}^{n}(z - \alpha_i)$ is the minimal polynomial of $\alpha \in \bar{\mathbb{Q}}$, and $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ are the conjugates of $\alpha$ over $\mathbb{Q}$. The principal branch of the complex function $F(\alpha; z)$ is analytic on any simply connected domain $\Omega \subset (\mathbb{C} \cup \{\infty\}) \backslash \{\alpha_1, \ldots, \alpha_n\}$, except at the point at infinity, where it has a pole of order 1.

We use the following procedure. To each algebraic number $\alpha \in \bar{\mathbb{Q}}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$, we associate a positive unitary divisor of the form

$$\alpha \overset{\psi}{\longmapsto} \left(\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n; \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

On the set $\mathcal{D}$ of positive unitary divisors we introduce a distance $d$. Some properties of $(\mathcal{D}, d)$ studied in [4] are recalled in §2. It is worth noticing that for any $\alpha, \beta \in \bar{\mathbb{Q}}$, one has

$$|\mathrm{Tr}(\alpha) - \mathrm{Tr}(\beta)| \leq d(\psi(\alpha), \psi(\beta)) \leq \|\alpha - \beta\|_s.$$

In §3 we construct a sequence of algebraic numbers $(\alpha_n)_{n\in\mathbb{N}}$ that does not converge in the spectral norm, while the corresponding sequence of unitary divisors $(\psi(\alpha_n))_{n\in\mathbb{N}}$ converges in $\tilde{\mathcal{D}}$, the completion of $\mathcal{D}$ with respect to $d$.

In §4 we extend some of the properties of the trace of unitary divisors from [4] to Lipschitzian functions on bounded domains in $\mathbb{C}$ using a Dirac measure.

In §5 we define the critical function associated to divisors with compact support. We show in Theorem 5.5 that given a generalized divisor $W \in \tilde{\mathcal{D}}$ with compact support and a sequence $(W_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$, $W_n \overset{d}{\to} W$, the sequence of analytic functions $z \mapsto F(W_n, z) - z$ converges uniformly on compacts in $\Omega$ to an analytic function $z \mapsto F(W, z) - z$, where $\infty \in \Omega \subset (\mathbb{C} \cup \{\infty\}) \backslash \mathcal{K}$, $\mathcal{K}$ compact.

## 2. Preliminaries

In this section we recall some definitions and results from [4]. The complex plane with the Euclidean distance is denoted by $\mathbb{C}$. A *complex divisor* is a mapping $U: \mathbb{C} \to \mathbb{C}$ taking values $\{a_1, a_2, \ldots, a_m\}$ at $\{u_1, u_2, \ldots, u_m\}$, respectively, and vanishing everywhere else on $\mathbb{C}$. The *support* of $U$ is the set of distinct points $\mathrm{supp}(U) = \{u_1, u_2, \ldots, u_m\}$, and the *coefficients (weights)* of $U$ are the complex numbers $a_1, a_2, \ldots, a_m$. We denote a complex divisor by $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$ and we define

$$\|\text{supp}(U)\| := \max\{|u_1|, |u_2|, \ldots, |u_m|\}. \tag{2}$$

We say that two divisors

$$U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m) \quad \text{and} \quad V = (v_1, v_2, \ldots, v_n; b_1, b_2, \ldots, b_n)$$

are equal if and only if the sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ coincide, and the corresponding coefficients are equal. We say that the divisor $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$ is *real* (*positive*) if $a_1, a_2, \ldots, a_m \in \mathbb{R}$ ($a_1 > 0, a_2 > 0, \ldots, a_m > 0$, respectively). The *degree* of $U$ is defined by $\deg(U) = a_1 + a_2 + \cdots + a_m$. We say that $U$ is *unitary* if $\deg(U) = 1$. Let $\mathcal{D}$ denote the set of all positive and unitary divisors, i.e. the set of all divisors $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$ with $u_1, u_2, \ldots, u_m$ distinct points in $\mathbb{C}$, and with $a_1 > 0, a_2 > 0, \ldots, a_m > 0$, such that $\sum_{j=1}^{m} a_j = 1$.

In [4] we have defined a distance on the set $\mathcal{D}$ of positive and unitary divisors as follows. Let $U, V \in \mathcal{D}$, where $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$, $u_1, u_2, \ldots, u_m$ distinct points in $\mathbb{C}$, $a_i > 0,\ i = 1, \ldots, m$, with $\sum_{i=1}^{m} a_i = 1$, and $V = (v_1, v_2, \ldots, v_n; b_1, b_2, \ldots, b_n)$, $v_1, v_2, \ldots, v_n$ distinct points in $\mathbb{C}$, $b_j > 0,\ j = 1, \ldots, n$ with $\sum_{j=1}^{n} b_j = 1$. The set $\mathcal{M}(U, V)$ of *admissible matrices* is defined as the set of all $m \times n$ matrices $X = (x_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ that satisfy the following properties:

(I) $x_{ij} \ge 0$ for all $1 \le i \le m$, $1 \le j \le n$;
(II) $\sum_{i=1}^{m} x_{ij} = b_j$, for any $j \in \{1, 2, \ldots, n\}$;
(III) $\sum_{j=1}^{n} x_{ij} = a_i$, for any $i \in \{1, 2, \ldots, m\}$.

One can visualize an admissible matrix as

$$X = \begin{array}{cccc|c} x_{11} & x_{12} & \cdots & x_{1n} & a_1 \\ x_{21} & x_{22} & \cdots & x_{2n} & a_2 \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots} \\ x_{m1} & x_{m2} & \cdots & x_{mn} & a_m \\ \hline b_1 & b_2 & \cdots & b_n & 1 \end{array}$$

For each admissible matrix $X$, a weighted average of the distances between the points in the supp$(U)$ and the points in the supp$(V)$ is defined as

$$H(U, V, X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} |u_i - v_j|. \tag{3}$$

The distance $d$ on the set of positive and unitary divisors $\mathcal{D}$ is defined by

$$d(U, V) = \inf_{X \in \mathcal{M}(U,V)} H(U, V, X). \tag{4}$$

Note that $X \mapsto H(U, V, X)$ is a continuous mapping from the set $M_{m,n}(\mathbb{R})$ into the set of nonnegative real numbers. Since $\mathcal{M}(U, V)$ is compact, the infimum above is attained, so there exists $X \in \mathcal{M}(U, V)$ such that $d(U, V) = H(U, V, X)$.

Let $\tilde{\mathcal{D}}$ denote the completion of $\mathcal{D}$ with respect to our distance $d$. The elements of $\tilde{\mathcal{D}}$ are called *generalized divisors*. The distance on $\tilde{\mathcal{D}}$ will also be denoted by $d$.

In [5] a metric $d_1$ on the set $\mathcal{S}_1$ of *simple finite sets* was defined (a finite simple set is a set $U = \{u_1, u_2, \ldots, u_m\}$ where $u_1, u_2, \ldots, u_m$ are distinct points in the complex plane $\mathbb{C}$). Let us note that there exists an injection $\nu\colon \mathcal{S}_1 \to \mathcal{D}$ defined by

$$U = \{u_1, u_2, \ldots, u_m\} \overset{\nu}{\longmapsto} U_m = \left(u_1, u_2, \ldots, u_m; \frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right).$$

The following inequality between the above distances has been proved in [4].

*Lemma* 2.1. *With the notations above, we have*

$$d(U_m, V_n) \le d_1(U, V), \quad \textit{for any} \quad U, V \in \mathcal{S}_1. \tag{5}$$

For any divisor $U \in \mathcal{D}$, $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$, the *trace* of $U$ is defined by (see [4])

$$\mathrm{Tr}(U) = \sum_{i=1}^{m} a_i u_i. \tag{6}$$

Note that the trace satisfies the following inequality.

*Lemma* 2.2 [4]. *For any $U, V \in \mathcal{D}$ we have*

$$|\mathrm{Tr}(U) - \mathrm{Tr}(V)| \le d(U, V). \tag{7}$$

An immediate consequence of the above lemma is the following.

**Theorem 2.3 [4].** *For any $W \in \tilde{\mathcal{D}}$, and any sequence $(W_n)_{n\in\mathbb{N}}$ of elements from $\mathcal{D}$ converging to $W$ in $\tilde{\mathcal{D}}$, the sequence*

$$(\mathrm{Tr}(W_n))_{n\in\mathbb{N}} \tag{8}$$

*converges in $\mathbb{C}$. Moreover, the limit depends on $W$ only, and not on the choice of the sequence $(W_n)_{n\in\mathbb{N}}$.*

For any generalized divisor $W \in \tilde{\mathcal{D}}$ we define

$$\mathrm{Tr}(W) = \lim_{n\to\infty} \mathrm{Tr}(W_n), \tag{9}$$

where $(W_n)_{n\in\mathbb{N}}$ is any sequence of elements from $\mathcal{D}$ converging to $W$. We call this number the *trace* of $W$.

Let $\alpha$ be an algebraic number, and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be its conjugates over $\mathbb{Q}$. Let $\phi\colon (\bar{\mathbb{Q}}, \|\cdot\|_s) \to (\mathcal{S}_1, d_1)$ be the mapping that associates each algebraic number with its set of conjugates, that is, $\phi(\alpha) = \{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n\}$. Then the composition mapping $\psi\colon (\bar{\mathbb{Q}}, \|\cdot\|_s) \to (\mathcal{D}, d)$ defined by $\psi = \nu \circ \phi$, i.e.

$$\psi(\alpha) = \left(\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n; \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right), \tag{10}$$

is 1-Lipschitzian, as shown in the following.

*Lemma* 2.4. *For any algebraic numbers $\alpha$ and $\beta$, we have*

$$d(\psi(\alpha), \psi(\beta)) \leq \|\alpha - \beta\|_s. \tag{11}$$

*Proof.* Let $\alpha, \beta \in \bar{\mathbb{Q}}$ of degrees $m$ and $n$ respectively. Then $\phi(\alpha) = \{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_m\}$ and $\phi(\beta) = \{\beta_1 = \beta, \beta_2, \ldots, \beta_n\}$. The application $\phi$ is 1-*Lipschitzian* (as shown in Proposition 1.5, Corollary 1.6 of [5]), that is,

$$d_1(\phi(\alpha), \phi(\beta)) \leq \|\alpha - \beta\|_s.$$

Lemma 2.1 and the above inequality yield

$$d(\psi(\alpha), \psi(\beta)) \leq d_1(\phi(\alpha), \phi(\beta)) \leq \|\alpha - \beta\|_s,$$

which proves the Lemma. ■

*Remark* 2.5. If $\alpha$ is any algebraic number, then its trace defined by (1) coincides with the trace of the associated divisor $\psi(\alpha)$ defined by (6),

$$\mathrm{Tr}(\alpha) = \mathrm{Tr}(\psi(\alpha)). \tag{12}$$

Using (7) and (11), we have the following string of inequalities for $\alpha, \beta \in \bar{\mathbb{Q}}$:

$$|\mathrm{Tr}(\alpha) - \mathrm{Tr}(\beta)| \leq d(\psi(\alpha), \psi(\beta)) \leq \|\alpha - \beta\|_s. \tag{13}$$

*Remark* 2.6. If $x$ is any element in $\tilde{\bar{\mathbb{Q}}}$, then its trace coincides with the trace of the generalized divisor $\psi(x)$, i.e.,

$$\mathrm{Tr}(x) = \mathrm{Tr}(\psi(x)). \tag{14}$$

To see this, let $(x_n)_{n\in N}$ be a sequence of elements from $\bar{\mathbb{Q}}$ converging to $x$ in $\tilde{\bar{\mathbb{Q}}}$ with respect to the spectral norm $\|\cdot\|_s$. Then we can write $\psi(x_n) = \left(x_n^{(1)}, \ldots, x_n^{(r_n)}; \frac{1}{r_n}, \ldots, \frac{1}{r_n}\right)$, where $x_n^{(j)}$, $1 \leq j \leq r_n$, are all the conjugates of $x_n$ over $\mathbb{Q}$. Recall that the trace of an element in $\bar{\mathbb{Q}}$ is the arithmetic mean of all its conjugates over $\mathbb{Q}$, while the trace of a divisor was defined by (6). Thus we have

$$\begin{aligned}\mathrm{Tr}(x) &= \lim_{n\to\infty} \mathrm{Tr}(x_n) = \lim_{n\to\infty} \frac{1}{r_n} \sum_{j=1}^{r_n} x_n^{(j)} \\ &= \lim_{n\to\infty} \mathrm{Tr}(\psi(x_n)) \\ &= \mathrm{Tr}(\psi(x)),\end{aligned}$$

since by the continuity of $\psi$ (Lemma 2.4), $\psi(x_n) \to \psi(x)$ as $n \to \infty$.

## 3. An example

First let us return to Example 1.2. We have seen that the sequence of algebraic numbers $\alpha_n = 1 - \sqrt{1 - 1/n} \in \bar{\mathbb{Q}}$, $n \in \mathbb{N}$, does not converge in $(\tilde{\bar{\mathbb{Q}}}, \|\cdot\|_s)$, yet $1 = \mathrm{Tr}(\alpha_n) =$

$\mathrm{Tr}(\psi(\alpha_n)) \to \mathrm{Tr}(A) = 1$. However, the sequence of the associated positive and unitary divisors $(\psi(\alpha_n))_{n\in\mathbb{N}} \in \mathcal{D}$ converges to a divisor $A \in \mathcal{D}$ with respect to the distance $d$ defined by (4). More precisely,

$$\psi(\alpha_n) = \left(1 - \sqrt{1 - \frac{1}{n}}, 1 + \sqrt{1 - \frac{1}{n}}; \frac{1}{2}, \frac{1}{2}\right) \xrightarrow{d} \left(0, 2; \frac{1}{2}, \frac{1}{2}\right) = A.$$

To see this, for each $n \in \mathbb{N}$, consider the admissible matrix $X_n \in \mathcal{M}(\psi(\alpha_n), A)$, $X_n = \left[\begin{smallmatrix} 1/2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$. Then

$$d(\psi(\alpha_n), A) \le H(\psi(\alpha_n), A, X_n) = 1 - \sqrt{1 - \frac{1}{n}} \to 0, \quad \text{as } n \to \infty.$$

For the remainder of this section we focus on the construction of a class of sequences $(\alpha_n)_{n\in\mathbb{N}} \in \bar{\mathbb{Q}}$ with the property that $(\alpha_n)_{n\in\mathbb{N}}$ does not converge in the spectral norm, i.e., it is not a Cauchy sequence in $\tilde{\bar{\mathbb{Q}}}$, while the corresponding sequence $(\psi(\alpha_n))_{n\in\mathbb{N}}$ converges to an element $W \in \tilde{\mathcal{D}}$.

In this case, the sequence of traces $\mathrm{Tr}(\alpha_n)$ converges, namely we have

$$\lim_{n\to\infty} \mathrm{Tr}(\alpha_n) = \lim_{n\to\infty} \mathrm{Tr}(\psi(\alpha_n)) = \mathrm{Tr}(W). \tag{15}$$

So there are instances when one can talk about the limiting trace of a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of algebraic numbers, although the sequence $(\alpha_n)_{n\in\mathbb{N}}$ does not converge in the spectral norm.

We proceed as follows. For each $n \in \mathbb{N}$ we choose a real number $t_n > 1$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n/n = 0$. Then, for each $n$ take $\varepsilon_n > 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. For each fixed $n$ we choose a real algebraic number $\alpha_n$ of degree $r_n = \deg(\alpha_n) \ge n$, with $|\alpha_n - t_n| \le \varepsilon_n$, such that all its other conjugates over $\mathbb{Q}$ are situated in the disk of center 1 and radius $\varepsilon_n$. This is possible by the Artin–Whaples approximation theorem (p. 467 of [2]), and by Proposition 3.2 (p. 477 of [2]). The sequence $(\alpha_n)_{n\in\mathbb{N}}$ does not converge in the spectral norm, since for any $n, m \in \mathbb{N}$ we have

$$\begin{aligned} \|\alpha_n - \alpha_m\| &= \sup_{\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} |\sigma(\alpha_n) - \sigma(\alpha_m)| \\ &\ge |\alpha_n - \alpha_m| \\ &\ge |t_n - t_m| - \varepsilon_n - \varepsilon_m. \end{aligned}$$

If the sequence $(\alpha_n)_{n\in\mathbb{N}}$ was a Cauchy sequence, then for any $\varepsilon > 0$ we could find an $m_\varepsilon$ such that

$$\|\alpha_n - \alpha_m\| \le \varepsilon, \quad \text{for } n, m \ge m_\varepsilon,$$

and thus

$$|t_n - t_m| \le \varepsilon + \varepsilon_n + \varepsilon_m, \quad \text{for } n, m \ge m_\varepsilon.$$

In particular,

$$|t_n - t_{m_\varepsilon}| \le \varepsilon + \varepsilon_n + \varepsilon_{m_\varepsilon}, \quad \text{for any } n \ge m_\varepsilon,$$

which contradicts the fact that $t_n \to \infty$ as $n \to \infty$.

Next we prove that the sequence of divisors $(\psi(\alpha_n))_{n\in\mathbb{N}}$ converges to an element $W$ in $\tilde{\mathcal{D}}$ in the topology induced by $d$.

Let us write $U_n = \psi(\alpha_n) = \left(\alpha_n = \alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(r_n)}; \frac{1}{r_n}, \ldots, \frac{1}{r_n}\right)$ for all $n$. By the choice of the $\alpha_n$'s, we know that all the points in $\mathrm{supp}(U_n)$ are inside the disk of radius $\varepsilon_n$ centered at 1, except for $\alpha_n$ which belongs to the disk of radius $\varepsilon_n$ centered at $t_n$.

We want to show that the distance $d(U_n, U_m) \to 0$ as $n, m \to \infty$. First we can estimate the distance between points in $\mathrm{supp}(U_n)$ and points in $\mathrm{supp}(U_m)$ as follows:

$$\begin{aligned}
|\alpha_n^{(1)} - \alpha_m^{(1)}| &= |\alpha_n - \alpha_m| \\
&\le |\alpha_n - t_n| + |t_n - t_m| + |\alpha_m - t_m| \\
&\le \varepsilon_n + t_n + t_m + \varepsilon_m, \\
|\alpha_n^{(1)} - \alpha_m^{(j)}| &= |\alpha_n - \alpha_m^{(j)}| \\
&\le |\alpha_n - t_n| + |1 - \alpha_m^{(j)}| + |t_n - 1| \\
&\le \varepsilon_n + \varepsilon_m + t_n, \quad \text{for any } j \in \{2, \ldots, r_m\},
\end{aligned}$$

and similarly,

$$\begin{aligned}
|\alpha_m^{(1)} - \alpha_n^{(i)}| &= |\alpha_m - \alpha_n^{(i)}| \\
&\le \varepsilon_m + \varepsilon_n + t_m, \quad \text{for any } i \in \{2, \ldots, r_n\}.
\end{aligned}$$

For $2 \le i \le r_n$ and $2 \le j \le r_m$ we have

$$|\alpha_n^{(i)} - \alpha_m^{(j)}| \le |\alpha_n^{(i)} - 1| + |1 - \alpha_m^{(j)}| \le \varepsilon_n + \varepsilon_m.$$

Note that the matrix $X$ with entries $x_{ij} = \frac{1}{r_n r_m}$, $1 \le i \le r_n$, $1 \le j \le r_m$, belongs to $\mathcal{M}(U_n, U_m)$. Therefore

$$\begin{aligned}
d(U_n, U_m) &\le H(U_n, U_m, X) \\
&= \frac{1}{r_n r_m} \sum_{i=1}^{r_n} \sum_{j=1}^{r_m} |\alpha_n^{(i)} - \alpha_m^{(j)}| \\
&= \frac{1}{r_n r_m} \left( |\alpha_n^{(1)} - \alpha_m^{(1)}| + \sum_{j=2}^{r_m} |\alpha_n^{(1)} - \alpha_m^{(j)}| \right. \\
&\qquad \left. + \sum_{i=2}^{r_n} |\alpha_n^{(i)} - \alpha_m^{(1)}| + \sum_{i=2}^{r_n} \sum_{j=2}^{r_m} |\alpha_n^{(i)} - \alpha_m^{(j)}| \right) \\
&\le \frac{1}{r_n r_m} [(\varepsilon_n + \varepsilon_m + t_n + t_m) + (r_m - 1)(\varepsilon_n + \varepsilon_m + t_n) \\
&\qquad + (r_n - 1)(\varepsilon_n + \varepsilon_m + t_m) + (r_n - 1)(r_m - 1)(\varepsilon_n + \varepsilon_m)] \\
&= \varepsilon_n + \varepsilon_m + \frac{t_n}{r_n} + \frac{t_m}{r_m}.
\end{aligned}$$

Since the sequences were chosen in such a way that $\lim_{n\to\infty} \varepsilon_n = 0$, $r_n \ge n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} t_n/n = 0$, we conclude that

$$d(U_n, U_m) \to 0, \quad \text{as} \quad m, n \to \infty. \tag{16}$$

This means that the sequence $(\psi(\alpha_n))_{n\in\mathbb{N}}$ is Cauchy, and thus it converges to an element $W$ in $\tilde{\mathcal{D}}$.

Then, by the previous discussion, we know that

$$\mathrm{Tr}(W) = \lim_{n\to\infty} \mathrm{Tr}(\psi(\alpha_n)) = \lim_{n\to\infty} \mathrm{Tr}(\alpha_n).$$

For the algebraic numbers $\alpha_n$ constructed above we have $\lim_{n\to\infty} \mathrm{Tr}(\alpha_n) = 1$. Indeed, the trace of $\alpha_n$, as the arithmetic mean of its conjugates $\alpha_n^{(1)} = \alpha_n, \alpha_n^{(2)}, \ldots, \alpha_n^{(r_n)}$, can be estimated as follows:

$$\begin{aligned}
|\mathrm{Tr}(\alpha_n) - 1| &= \left| \frac{\alpha_n^{(1)} + \alpha_n^{(2)} + \cdots + \alpha_n^{(r_n)}}{r_n} - 1 \right| \\
&= \left| \frac{(\alpha_n^{(1)} - 1) + (\alpha_n^{(2)} - 1) + \cdots + (\alpha_n^{(r_n)} - 1)}{r_n} \right| \\
&\leq \frac{|\alpha_n^{(1)} - 1| + |\alpha_n^{(2)} - 1| + \cdots + |\alpha_n^{(r_n)} - 1|}{r_n} \\
&\leq \frac{(t_n + \varepsilon_n) + \varepsilon_n + \cdots + \varepsilon_n}{r_n} \\
&= \frac{t_n}{r_n} + \varepsilon_n \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

We conclude that the trace of $W$ is

$$\mathrm{Tr}(W) = \lim_{n\to\infty} \mathrm{Tr}(\psi(\alpha_n)) = \lim_{n\to\infty} \mathrm{Tr}(\alpha_n) = 1.$$

## 4. Measures and Lipschitzian functions on a bounded domain in $\mathbb{C}$

In this section we associate a measure $\mu_U$ to each divisor $U \in \mathcal{D}$. Next, we use this measure to extend Lemma 2.2 and Theorem 2.3 (involving the trace of divisors in $\mathcal{D}$) to the case of a Lipschitzian function on a bounded domain in $\mathbb{C}$.

### 4.1 *Measures*

To each divisor $U \in \mathcal{D}$, $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$, we associate a probability measure $\mu_U$, defined by

$$\mu_U = \sum_{i=1}^{m} a_i \delta_{u_i}, \tag{17}$$

where $\delta_{u_i}$ is the (normalized) Dirac measure centered at $u_i$.

Consider a bounded domain $\Omega$ that contains the support of $U$. For any $f\colon \Omega \to \mathbb{C}$ the integral of $f$ with respect to $\mu_U$ is equal to

$$\int_{\Omega} f \, \mathrm{d}\mu_U = \sum_{i=1}^{m} a_i f(u_i). \tag{18}$$

For simplicity, we will denote $\int_{\Omega} f \, \mathrm{d}\mu_U$ by $\int_{\Omega} f \, \mathrm{d}U$.

## 4.2 *Lipschitzian functions on a bounded domain in* $\mathbb{C}$

Let $\Omega$ be an arbitrary bounded domain in $\mathbb{C}$. Let $\mathrm{Lip}(\Omega)$ denote the set of all Lipschtzian functions defined on $\Omega$ with values in $\mathbb{C}$. Then $\mathrm{Lip}(\Omega)$ is a vector space over $\mathbb{C}$, and for any divisor $U \in \mathcal{D}$, with $\mathrm{supp}(U) \subset \Omega$, there exists a linear functional from $\mathrm{Lip}(\Omega)$ to $\mathbb{C}$, given by

$$f \mapsto \int_\Omega f \,\mathrm{d}U. \tag{19}$$

We prove the following.

**Theorem 4.1.** *Let $W \in \tilde{\mathcal{D}}$ and let $(W_n)_{n\in\mathbb{N}}$ be a sequence of elements from $\mathcal{D}$ converging to $W$ in $\tilde{\mathcal{D}}$. For any bounded domain $\Omega$ in $\mathbb{C}$ containing the support of $W_n$, for all $n \in \mathbb{N}$, and any Lipschitzian function $f\colon \Omega \to \mathbb{C}$ the sequence*

$$\left(\int_\Omega f \,\mathrm{d}W_n\right)_{n\in\mathbb{N}} \tag{20}$$

*converges in $\mathbb{C}$. Moreover, the limit depends only on $W$ and $f$, and not on the choice of the sequence $(W_n)_{n\in\mathbb{N}}$.*

The proof of the theorem follows immediately from the lemma below.

*Lemma* 4.2. *Let $U, V \in \mathcal{D}$, and let $\Omega$ be an arbitrary bounded domain in $\mathbb{C}$ containing the supports of $U$ and $V$. If $f\colon \Omega \to \mathbb{C}$ is a $\lambda$-Lipschitzian map, then*

$$\left|\int_\Omega f \,\mathrm{d}U - \int_\Omega f \,\mathrm{d}V\right| \le \lambda \mathrm{d}(U, V). \tag{21}$$

*Proof.* To prove the lemma, we choose a matrix $X \in \mathcal{M}(U, V)$ such that $d(U, V) = H(U, V, X)$. Let $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$ and $V = (v_1, v_2, \ldots, v_n; b_1, b_2, \ldots, b_n)$. Then we have

$$\int_\Omega f \,\mathrm{d}U = \sum_{i=1}^m a_i f(u_i) \quad \text{and} \quad \int_\Omega f \,\mathrm{d}V = \sum_{j=1}^n b_j f(v_j).$$

Since

$$x_{ij} \ge 0,\ \sum_{i=1}^m x_{ij} = b_j,\ \sum_{j=1}^n x_{ij} = a_i, \tag{22}$$

we deduce that

$$\begin{aligned}
\int_\Omega f \,\mathrm{d}U - \int_\Omega f \,\mathrm{d}V &= \sum_{i=1}^m a_i f(u_i) - \sum_{j=1}^n b_j f(v_j) \\
&= \sum_{i=1}^m f(u_i) \sum_{j=1}^n x_{ij} - \sum_{j=1}^n f(v_j) \sum_{i=1}^m x_{ij} \\
&= \sum_{i=1}^m \sum_{j=1}^n x_{ij}(f(u_i) - f(v_j)).
\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_\Omega f\mathrm{d}U - \int_\Omega f\mathrm{d}V\right| &\le \sum_{i=1}^m\sum_{j=1}^n x_{ij}|f(u_i) - f(v_j)| \\ &\le \lambda \sum_{i=1}^m\sum_{j=1}^n x_{ij}|u_i - v_j| \\ &= \lambda H(U, V, X) \\ &= \lambda d(U, V),\end{aligned}$$

which proves the lemma, and then the theorem. ■

Let $W \in \tilde{\mathcal{D}}$ be a generalized divisor and let $(W_n)_{n\in\mathbb{N}}$ be any sequence of elements from $\mathcal{D}$ converging to $W$. If $\Omega$ is a bounded domain in $\mathbb{C}$ with $\bigcup_{n\in\mathbb{N}} \mathrm{supp}(W_n) \subset \Omega$, then for any function $f\colon \Omega \to \mathbb{C}$ that is Lipschitzian we can define

$$\int_\Omega f\mathrm{d}W = \lim_{n\to\infty}\int_\Omega f\mathrm{d}W_n, \tag{23}$$

We denote the set of all divisors in $\mathcal{D}$ whose support is contained in $\Omega$ by

$$\mathcal{D}_\Omega = \{U \in \mathcal{D}\colon \mathrm{supp}(U) \subset \Omega\}. \tag{24}$$

Let $\tilde{\mathcal{D}}_\Omega$ be the topological closure of $\mathcal{D}_\Omega$ in $\tilde{\mathcal{D}}$. Thus $\tilde{\mathcal{D}}_\Omega$ is a closed metric subspace of $\tilde{\mathcal{D}}$. Note that all the polynomial functions $z \mapsto z^m$, $m \ge 0$ are Lipschitzian in any bounded domain $\Omega$. Then for any $W \in \tilde{\mathcal{D}}_\Omega$ and any $m \ge 0$ we can define

$$\int_\Omega z^m\mathrm{d}W = \lim_{n\to\infty}\int_\Omega z^m\mathrm{d}W_n, \tag{25}$$

where $(W_n)_{n\in\mathbb{N}}$ is any sequence from $\mathcal{D}_\Omega$ converging to $W$. We make the observation that this integral is well-defined, since the above limit does not depend on the choice of the convergent sequence $(W_n)_{n\in\mathbb{N}}$.

In particular, since the function $f(z) = z$ is Lipschitzian on $\Omega$, the number $\int_{\mathbb{C}} z\mathrm{d}W$ is well-defined. We call this number the *trace* of $W$, and we denote it by

$$\mathrm{Tr}(W) = \int_\Omega z\mathrm{d}W. \tag{26}$$

Note that for elements $W \in \mathcal{D}$, the above definition of trace coincides with that given by relation (6).

We now give the following

DEFINITION 4.3

We say that $W \in \tilde{\mathcal{D}}$ is a *generalized divisor with compact support* if there exists a sequence $(W_n)_{n\in\mathbb{N}}$ converging to $W$ with respect to the distance $d$, such that the set

$$\bigcup_{n\in\mathbb{N}} \mathrm{supp}(W_n)$$

is bounded in $\mathbb{C}$.

Thus the relation (25) holds for any generalized divisor $W$ with compact support, for any sequence $(W_n)_{n\in\mathbb{N}}$ converging to $W$ in $\tilde{\mathcal{D}}$ such that $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(W_n)$ is bounded, and for any domain $\Omega \subset \mathbb{C}$ containing the support of $W_n$, for all $n \in \mathbb{N}$.

*Remark* 4.4. For simple finite sets $W_n$ converging in the distance $d_1$ to an element $W \in \tilde{\mathcal{S}}_1$, the union of supports of all $W_n$ is always a bounded set, a fact that was used in the study of critical functions in [5].

## 5. Critical functions associated with divisors with compact support

We firstly define the critical function associated with a positive and unitary divisor $U \in \mathcal{D}$. Next, we construct a similar function for any generalized divisor with compact support, by taking the limit of a sequence of critical functions associated with a converging sequence of divisors from $(\mathcal{D}, d)$ .

Let $U = (u_1, u_2, \dots, u_m; a_1, a_2, \dots, a_m)$ be a positive and unitary divisor. For $k \geq 0$ we denote by $U^k$ the divisor $(u_1^k, u_2^k, \dots, u_m^k; a_1, a_2, \dots a_m)$, with the convention that if $u_1^k = u_2^k$, then we only write this element once, with the coefficient $a_1 + a_2$. Hence, $U^0 = (1; 1)$, $U^1 = U$, and $U^n \in \mathcal{D}$ for any $n \in \mathbb{N}$. Note that $\|\operatorname{supp}(U^n)\| = \|\operatorname{supp}(U)\|^n$ for any $n \in \mathbb{N}$.

Recall that for any $U \in \mathcal{D}$ we have defined its trace by

$$\operatorname{Tr}(U) = \sum_{i=1}^{m} a_i u_i.$$

By the same definition,

$$\operatorname{Tr}(U^k) = \sum_{i=1}^{m} a_i u_i^k, \quad \text{for any } k \geq 0.$$

If $\mu_U$ denotes the probability measure associated with $U$ (see §4.1), then we see that

$$\operatorname{Tr}(U^k) = \sum_{i=1}^{m} a_i u_i^k = \int_{\operatorname{supp}(U)} z^k \mathrm{d}\mu_U(z), \quad k \geq 0. \tag{27}$$

*Lemma* 5.1. *Let U and V be unitary divisors, and let $p \geq 1$ be an integer. Then*

$$|\operatorname{Tr}(U^p) - \operatorname{Tr}(V^p)| \leq p[\max\{\|\operatorname{supp}(U)\|, \|\operatorname{supp}(V)\|\}]^{p-1} d(U, V).$$

*Proof.* By applying Lemma 2.2 to $U^p$ and $V^p$ we obtain

$$|\operatorname{Tr}(U^p) - \operatorname{Tr}(V^p)| \leq d(U^p, V^p).$$

Therefore, in what follows, it is enough to show that

$$d(U^p, V^p) \leq p[\max\{\|\operatorname{supp}(U)\|, \|\operatorname{supp}(V)\|\}]^{p-1} d(U, V).$$

Consider two unitary divisors $U = (u_1, u_2, \dots, u_m; a_1, a_2, \dots, a_m)$, and $V = (v_1, v_2, \dots, v_n; b_1, b_2, \dots, b_n)$. Then $U^p = (u_1^p, u_2^p, \dots, u_m^p; a_1, a_2, \dots, a_m)$, $V^p = (v_1^p, v_2^p, \dots, v_n^p; b_1, b_2, \cdots, b_n)$, and

$$\begin{aligned} |u_i^p - v_j^p| &= |u_i - v_j|\, |u_i^{p-1} + u_i^{p-2} v_j + \dots + u_i v_j^{p-2} + v_j^{p-1}| \\ &\leq p\, |u_i - v_j|\, [\max\{\|\operatorname{supp}(U)\|, \|\operatorname{supp}(V)\|\}]^{p-1}. \end{aligned}$$

Notice that the set of admissible matrices $\mathcal{M}(U^p, V^p)$ coincides with $\mathcal{M}(U, V)$. Next, for any admissible matrix $X = (x_{ij})$,

$$\begin{aligned} d(U^p, V^p) \le H(U^p, V^p, X) &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} |u_i^p - v_j^p| \\ &\le p\,[\max\{\|\mathrm{supp}(U)\|, \|\mathrm{supp}(V)\|\}]^{p-1} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} |u_i - v_j| \\ &= p\,[\max\{\|\mathrm{supp}(U)\|, \|\mathrm{supp}(V)\|\}]^{p-1} H(U, V, X). \end{aligned}$$

Now the required inequality follows by taking the infimum over all matrices $X \in \mathcal{M}(U^p, V^p) = \mathcal{M}(U, V)$. ■

For any simple finite set $W = \{w_1, w_2, \ldots, w_k\}$ in the complex plane, the *characteristic polynomial associated to* $W$ is defined by (see [5])

$$P_W(X) = \prod_{i=1}^{k} (X - w_i) \in \mathbb{C}[X]. \tag{28}$$

In particular, if $\alpha \in \bar{\mathbb{Q}}$, and if $W$ coincides with the set of conjugates of $\alpha$ over $\mathbb{Q}$, then $P_W(X)$ becomes the minimal polynomial $P_\alpha(X)$ of $\alpha$ over $\mathbb{Q}$. The complex function

$$F(W; z) = (P_W(z))^{\frac{1}{k}}, \tag{29}$$

called *the critical function of* $W$ was introduced and studied in [5]. In particular, the critical function $F(\alpha; z)$ associated to an element $\alpha \in \bar{\mathbb{Q}}$, is

$$F(\alpha; z) = P_\alpha(z)^{\frac{1}{\deg(\alpha)}}, \tag{30}$$

where $\deg(\alpha)$ denotes the degree of $\alpha$ over $\mathbb{Q}$.

DEFINITION 5.2

For any $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$ positive and unitary divisor, we define the *critical function of* $U$ by

$$F(U; z) = \prod_{i=1}^{m} (z - u_i)^{a_i}. \tag{31}$$

Clearly, if $W = \{w_1, w_2, \ldots, w_k\}$ is a simple finite set, and if $\nu(W) = (w_1, w_2, \ldots, w_k; \frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$ is the complex divisor associated with it, then the two definitions agree,

$$F(W; z) = F(\nu(W); z).$$

Also, if $\alpha \in \bar{\mathbb{Q}}$ and $\psi(\alpha)$ is its corresponding divisor, we have

$$F(\alpha; z) = F(\psi(\alpha); z).$$

Recall that $\|\mathrm{supp}(U)\| = \max\{|u_1|, |u_2|, \ldots, |u_m|\}$. We denote the open ball (closed ball) centered at $z$ and of radius $r$ by $B(z, r)$ ($B[z, r]$, respectively).

*Lemma* 5.3. *Suppose* $U = (u_1, u_2, \ldots, u_m; a_1, a_2, \ldots, a_m)$ *and* $U \in \mathcal{D}$. *The critical function* $F(U; z)$ *of* $U$ *is analytic outside the closed disk* $B[0, \|\mathrm{supp}(U)\|]$, *and is meromorphic at* $z = \infty$, *where it has a simple pole with residue* 1. *Moreover, we have*

$$F(U; z) = z - \mathrm{Tr}(U) + \frac{C_1(U)}{z} + \frac{C_2(U)}{z^2} + \cdots,$$

*where* $C_n(U)$ *are polynomials with rational coefficients in* $\mathrm{Tr}(U), \ldots, \mathrm{Tr}(U^n)$, *for all* $n \geq 1$.

*Proof.* Let $\log z = \log|z| + i \arg z$ denote the principal branch of the complex logarithmic function.

For any $z$ with $|z| > \|\mathrm{supp}(U)\|$ we can write

$$\begin{aligned} F(U; z) &= \prod_{i=1}^{m} z^{a_i} \left(1 - \frac{u_i}{z}\right)^{a_i} = z \prod_{i=1}^{m} \left(1 - \frac{u_i}{z}\right)^{a_i} \\ &= z \exp\left(\sum_{i=1}^{m} a_i \log\left(1 - \frac{u_i}{z}\right)\right) = z \exp\left(-\sum_{i=1}^{m} a_i \sum_{k=1}^{\infty} \frac{u_i^k}{k z^k}\right) \\ &= z \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k z^k} \sum_{i=1}^{m} a_i u_i^k\right) = z \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{Tr}(U^k)}{k z^k}\right). \end{aligned} \tag{32}$$

We now see that $z = \infty$ is a simple pole of residue 1 for $F(U; z)$. Also,

$$\lim_{z \to \infty} (z - F(U; z)) = \mathrm{Tr}(U),$$

and the conclusion of the lemma follows. ■

We would like to point out that for any divisor $U \in \mathcal{D}$, the critical function $F(U; z)$ can be expressed in terms of the measure $\mu_U$ defined by (17). Taking (27) into account, we express $F(U; z)$ as in the proof of the above lemma to get

$$\begin{aligned} F(U; z) &= z \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k z^k} \int_{\mathrm{supp}(U)} x^k \mathrm{d}\mu_U(x)\right) \\ &= z \exp\left(-\int_{\mathrm{supp}(U)} \left(\sum_{k=1}^{\infty} \frac{x^k}{k z^k}\right) \mathrm{d}\mu_U(x)\right), \end{aligned} \tag{33}$$

after interchanging the integral with the infinite sum. Thus we have

$$F(U; z) = z \exp\left(\int_{\mathrm{supp}(U)} \log\left(1 - \frac{x}{z}\right) \mathrm{d}\mu_U(x)\right). \tag{34}$$

In light of the lemma, the function

$$G(U; z) = z - F(U; z) \tag{35}$$

is analytic in $(\mathbb{C} \cup \{\infty\}) \setminus B[0, \|\mathrm{supp}(U)\|]$. We naturally define $G(U; \infty) = \mathrm{Tr}(U)$.

The critical function $F(U; z)$ can be analytically continued in the whole $\mathbb{C}$, except for a closed continuous curve passing through the points $u_1, u_2, \ldots, u_m$ of the support of $U$.

Therefore, the function $G(U; z)$ is analytic on any open simply connected set which does not contain points from the $\mathrm{supp}(U)$, i.e., on any open simply connected set in $(\mathbb{C} \cup \{\infty\}) \setminus \mathrm{supp}(U)$.

From now on, we will restrict our attention to generalized divisors with compact support, for which we want to define a similar critical function.

Recall (see (24)) that

$$\tilde{\mathcal{D}}_\Omega = \{W \in \tilde{\mathcal{D}}\colon W \text{ has compact support in } \Omega\}.$$

For $W \in \tilde{\mathcal{D}}$, denote by $\mathcal{S}(W)$ the family of sequences $\mathcal{W} = (W_n)_{n\in\mathbb{N}}$, with

(i) $W_n \in \mathcal{D}$ for all $n \in \mathbb{N}$;
(ii) $W_n \xrightarrow{d} W$;
(iii) $\bigcup_{n\in\mathbb{N}} \mathrm{supp}(W_n)$ is bounded in $\mathbb{C}$.

For each $\mathcal{W} = (W_n)_{n\in\mathbb{N}} \in \mathcal{S}(W)$, we denote by $C_m(\mathcal{W})$ the topological closure of the set $\bigcup_{j\geq m} \mathrm{supp}(W_j)$ in $\mathbb{C}$. It follows that each set $C_m(\mathcal{W})$, $m \geq 1$, is bounded in $\mathbb{C}$, and thus it is compact. The decreasing sequence $(C_m(\mathcal{W}))_{m\in\mathbb{N}}$ of nonempty compact sets has a nonempty compact intersection

$$K_{\mathcal{W}} = \bigcap_{m\geq 1} C_m(\mathcal{W}). \tag{36}$$

DEFINITION 5.4

Let $W \in \tilde{\mathcal{D}}$ and let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be an open simply connected region containing the point at infinity. We call $(W, \Omega)$ an *acceptable pair* provided there exists a sequence $\mathcal{W} \in \mathcal{S}(W)$ for which the intersection $K_{\mathcal{W}} \cap \Omega$ is empty.

**Theorem 5.5.** *Let $(W, \Omega)$ be an admissible pair, and let $\mathcal{W} = (W_n)_{n\in\mathbb{N}} \in \mathcal{S}(W)$ be such that $K_{\mathcal{W}} \cap \Omega = \varnothing$. Then the sequence of analytic functions $z \longmapsto F(W_n; z) - z$ converges uniformly on compacts in $\Omega$ to an analytic function $z \longmapsto F(W; z) - z$. The function $F(W; z)$ is independent of the choice of the sequence $\mathcal{W} \in \mathcal{S}(W)$ with $K_{\mathcal{W}} \cap \Omega = \varnothing$.*

*Proof.* Let $(W, \Omega)$ be an acceptable pair, and let us fix $\mathcal{W} = (W_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(W)$, $W_n = (w_1, w_2, \ldots, w_m; a_1, a_2, \ldots, a_m)$, such that $K_{\mathcal{W}} \cap \Omega = \varnothing$.

*Step* 1. We first show that the sequence $(F(W_n; z) - z)_{n\in\mathbb{N}}$ converges uniformly to an analytic function $F(W; z) - z$ on a suitable neighborhood of the point at infinity.

Fix a number $M_\infty$ such that $(\mathbb{C}\cup\{\infty\})\setminus B(0, M_\infty) \subseteq \Omega$. Let $M_{\mathcal{W}} = \sup_{n\in\mathbb{N}} \|\mathrm{supp}(W_n)\|$, and let $M = \max\{M_\infty, M_{\mathcal{W}}\}$. We show that the sequence $(F(W_n; z))_{n\in\mathbb{N}}$ converges uniformly outside the disk $B(0, 4M)$.

For $n, r \in \mathbb{N}$ and $z \in \mathbb{C}$ with $4M \leq |z|$, and using (32), we have

$$\begin{aligned} &|F(W_n; z) - F(W_r; z)| \\ &\quad = |z| \left| \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{Tr}(W_n^k)}{kz^k}\right) - \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{Tr}(W_r^k)}{kz^k}\right) \right| \\ &\quad = |z| \left| \exp\left(-\sum_{k=1}^{\infty} \frac{\mathrm{Tr}(W_n^k)}{kz^k}\right) \right| \times \left| 1 - \exp\left(\sum_{k=1}^{\infty} \frac{\mathrm{Tr}(W_n^k) - \mathrm{Tr}(W_r^k)}{kz^k}\right) \right|. \end{aligned} \tag{37}$$

Next,

$$\left|\sum_{k=1}^{\infty}\frac{\mathrm{Tr}(W_n^k)}{kz^k}\right| \le \sum_{k=1}^{\infty}\frac{1}{k|z|^k}\sum_{i=1}^{m}a_i|w_i|^k \le \sum_{k=1}^{\infty}\frac{1}{k(4M)^k}\sum_{i=1}^{m}a_iM^k = \sum_{k=1}^{\infty}\frac{1}{k4^k} < \frac{1}{3}.$$

Hence

$$\left|\exp\left(-\sum_{k=1}^{\infty}\frac{\mathrm{Tr}(W_n^k)}{kz^k}\right)\right| \le \mathrm{e}^{\frac{1}{3}} < 2. \tag{38}$$

Also, by Lemma 5.1 we know that for each $k \ge 1$,

$$\begin{aligned}|\mathrm{Tr}(W_n^k) - \mathrm{Tr}(W_r^k)| &\le k[\max\{\,\|\mathrm{supp}(W_n)\|, \|\mathrm{supp}(W_r)\|\}]^{k-1}\, d(W_n, W_r)\\ &\le kM^{k-1}d(W_n, W_r).\end{aligned}$$

We derive that

$$\begin{aligned}\left|\sum_{k=1}^{\infty}\frac{\mathrm{Tr}(W_n^k)-\mathrm{Tr}(W_r^k)}{kz^k}\right| &\le \sum_{k-1}^{\infty}\frac{M^{k-1}}{|z|^k}\, d(W_n, W_r)\\ &\le \frac{1}{|z|}\, d(W_n, W_r)\sum_{k-1}^{\infty}\frac{1}{4^{k-1}}\\ &= \frac{4}{3|z|}\, d(W_n, W_r).\end{aligned}$$

For $n$ and $r$ large enough, we have that $\frac{4}{3|z|}\, d(W_n, W_r) \le 1$, uniformly for all $|z| \ge 4M$. Also note that for all $\zeta \in \mathbb{C}$ with $|\zeta| \le 1$ one has

$$|1-\exp(\zeta)| = \left|\sum_{j=1}^{\infty}\frac{\zeta^j}{j!}\right| \le |\zeta|\sum_{j=1}^{\infty}\frac{1}{j!} < 2|\zeta|.$$

It follows that for $n, r$ large enough,

$$\left|1-\exp\left(\sum_{k=1}^{\infty}\frac{\mathrm{Tr}(W_n^k)-\mathrm{Tr}(W_r^k)}{kz^k}\right)\right| \le \frac{8}{3|z|}\, d(W_n, W_r). \tag{39}$$

By combining (37), (38) and (39), we find that for $n, r$ sufficiently large, and $|z| \ge 4M$,

$$|F(W_n; z) - F(W_r; z)| \le \frac{16}{3}\, d(W_n, W_r).$$

This proves that the sequence of analytic functions $(F(W_n; z)-z)_{n\in\mathbb{N}}$ converges uniformly on $(\mathbb{C}\cup\{\infty\})\backslash B(0, 4M)$. Thus its limit, which we denote by $F(W; z) - z$, is analytic on $(\mathbb{C}\cup\{\infty\})\backslash B[0, 4M]$. Moreover, since each $F(W_n; z) - z$ vanishes at the point of infinity, so does $F(W; z) - z$.

*Step* 2. The next stage of the proof is to show that $F(W; z) - z$ defined as above on $(\mathbb{C}\cup\{\infty\})\backslash B[0, 4M]$ can be analytically continued to the entire region $\Omega$. To this end, with $\mathcal{W}$ and $M$ fixed as above, let us choose an arbitrary simply connected open region $O$ which has nonempty intersection with $\mathbb{C}\backslash B[0, 4M]$, and such that its closure $K = \bar{O}$ is compact and contained in $\Omega$. Since $\Omega \cap K_{\mathcal{W}} = \varnothing$, it follows that $K \cap K_{\mathcal{W}} = \varnothing$.

Recall the definition (36) of $K_{\mathcal{W}}$. Then we see that

$$\varnothing = K \cap K_{\mathcal{W}} = K \cap \left( \bigcap_{n=1}^{\infty} C_n \right) = \bigcap_{n=1}^{\infty} (K \cap C_n),$$

and since $K \cap C_n$ is a decreasing sequence of compact sets, there exists an $n_0$ such that $K \cap C_{n_0} = \varnothing$. Therefore the distance between the compacts $K$ and $C_{n_0}$ is strictly positive. Let

$$\delta := \inf_{\substack{x \in K \\ y \in C_{n_0}}} |x - y| > 0.$$

Since $\mathrm{supp}(W_n) \subseteq C_{n_0}$ for all $n \geq n_0$, we have that $|x - y| \geq \delta$ for all $x \in K$, all $y \in \mathrm{supp}(W_n)$, and for all $n \geq n_0$.

Fix $n$ with $n \geq n_0$ and write $W_n = (w_1, w_2, \ldots, w_m; a_1, a_2, \ldots, a_m)$. Then $|x - w_j| \geq \delta$ for all $j \in \{1, \ldots, m\}$ and all $x \in K$. One can choose a polygonal line $\gamma_n$ joining the points $w_1, \ldots, w_m \in \mathrm{supp}(W_n)$ such that $\gamma_n$ lies at a distance at least $\frac{\delta}{2}$ from $K$. Note that $\frac{\delta}{2}$ is independent of $n \geq n_0$.

Now, each $F(W_n; z) - z$ is well-defined and analytic on $\mathbb{C} \cup \{\infty\}\backslash\gamma_n$, so it has a well-defined restriction to $K_{\delta/4}$. Here $K_{\delta/4} = \{z \in \mathbb{C}\colon B(z, \delta/4) \cap K \neq \varnothing\}$ is the $\delta/4$ neighborhood of $K$.

We now show that the sequence of analytic functions $(F(W_n; z))_{n\in\mathbb{N}}$ is uniformly bounded on $K_{\delta/4}$. Recall that all the weights $a_j$ are positive real numbers with $\sum_{j=1}^{m} a_j = 1$. For any $z \in K_{\delta/4}$, we have

$$|F(W_n; z)| = \left| \prod_{j=1}^{m} (z - w_j)^{a_j} \right| = \prod_{j=1}^{m} |z - w_j|^{a_j}.$$

Since $|w_j| \leq M$ for $1 \leq j \leq m$, and $|z| \leq \delta/4 + \sup_{\zeta\in K} |\zeta|$, we deduce that

$$|F(W_n; z)| \leq \prod_{j=1}^{m} \left( M + \frac{\delta}{4} + \sup_{\zeta\in K} |\zeta| \right)^{a_j} = M + \frac{\delta}{4} + \sup_{\zeta\in K} |\zeta|, \tag{40}$$

uniformly for all $z \in K_{\delta/4}$. Since the right-hand side of (40) is independent of $n$, we conclude that the sequence of analytic functions $(F(W_n; z))_{n\in\mathbb{N}}$ is uniformly bounded on $K_{\delta/4}$.

By the Arzela–Ascoli theorem (see Chapter 5, §4.4 of [1]) it follows that the family $(F(W_n; z))_{n\geq n_0}$ is normal on $K_{\delta/4}$. This implies that there exists a subsequence $(F(W_{n_k}; z))_{k\in\mathbb{N}}$ of $(F(W_n; z))_{n\geq n_0}$ that converges uniformly on compact subsets of $K_{\delta/4}$ to a function $F(W; z)$.

It remains to show that the entire sequence $(F(W_n; z))_{n\geq n_0}$ converges uniformly on compact subsets of $K_{\delta/4}$ to $F(W; z)$. If not, there exist a compact $E \subset K_{\delta/4}$, a number $\varepsilon_0 > 0$ and a subsequence $(W_{n_j})_{j\in\mathbb{N}}$ such that, for all $j$,

$$\sup_{z\in E} |F(W_{n_j}; z) - F(W; z)| \geq \varepsilon_0. \tag{41}$$

On the other hand, applying again the Arzela–Ascoli theorem, there exists a subsequence $(W_{n_{j_k}})_{k\in\mathbb{N}}$ of $(W_{n_j})_{j\in\mathbb{N}}$ such that $(F(W_{n_{j_k}}; z))_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $K_{\delta/4}$ (in particular, on $E$ and on $K$) to an analytic function $H(z)$. Note that by relation (41) the functions $H(z)$ and $F(W; z)$ are not identical. Recall that the entire sequence $(F(W_n; z))_{n\geq n_0}$ converges uniformly on $K \cap (\mathbb{C}\backslash B(0, 4M))$ to $F(W; z)$, and that $K = \bar{O}$. Therefore $H(z)$ and $F(W; z)$ have the same restriction to $K \cap (\mathbb{C}\backslash B(0, 4M))$, and hence to the nonempty open set $O \cap (\mathbb{C}\backslash B[0, 4M])$. So $H(z)$ coincides with $F(W; z)$ on the entire region $K_{\delta/4}$, and we obtain a contradiction. This completes the proof that the function $F(W; z) - z$ has an analytic continuation from $(\mathbb{C} \cup \{\infty\})\backslash B[0, 4M]$ to $((\mathbb{C} \cup \{\infty\})\backslash B[0, 4M]) \cup K_{\frac{\delta}{2}}$, and that the sequence $(F(W_n; z) - z)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of this region to $F(W; z) - z$.

This further implies analytic continuation to the entire $\Omega$ by applying the above procedure to an increasing sequence $O = O_1 \subset \overline{O_1} = K_1 \subset O_2 \subset \overline{O_2} = K_2 \subset \ldots$ with $\bigcup_{n=1}^{\infty} O_n = \Omega$.

*Step* 3. Finally, we need to show that $F(W; z)$ is independent of the choice of the sequence $\mathcal{W} = (W_n)_{n\in\mathbb{N}} \in \mathcal{S}(W)$ with $K_{\mathcal{W}} \cap \Omega = \varnothing$. To this end, let us choose two arbitrary sequences $\mathcal{W}_1 = (W_{n,1})_{n\in\mathbb{N}}$ and $\mathcal{W}_2 = (W_{n,2})_{n\in\mathbb{N}}$ in $\mathcal{S}(W)$ with $K_{\mathcal{W}_1} \cap \Omega = \varnothing$ and $K_{\mathcal{W}_2} \cap \Omega = \varnothing$. We know that the corresponding sequences of analytic functions $(F(W_{n,1}; z) - z)_{n\in\mathbb{N}}$ and $\left(F(W_{n,2}; z) - z\right)_{n\in\mathbb{N}}$ converge uniformly on compact subsets of $\Omega$ to the analytic functions $F_1(W; z) - z$ and $F_2(W; z) - z$, respectively. Consider a new sequence $\mathcal{W} = (W_n)_{n\in\mathbb{N}}$ defined by $W_n = \begin{cases} W_{n,1}, & \text{if } n \text{ is odd} \\ W_{n,2}, & \text{if } n \text{ is even} \end{cases}$. It is easy to see that $\mathcal{W} \in \mathcal{S}(W)$ and that $K_{\mathcal{W}} \cap \Omega = \varnothing$. Therefore the sequence $(F(W_n; z) - z)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\Omega$ to an analytic function $F(W; z) - z$, and so do its two subsequences $(F(W_{n,1}; z) - z)_{n\in\mathbb{N}}$ and $(F(W_{n,2}; z) - z)_{n\in\mathbb{N}}$. It follows that $F_1(W; z) - z = F_2(W; z) - z = F(W; z) - z$.

This completes the proof of the theorem. ■

*Remark* 5.6. Based on Remark 4.4 one might think that Theorem 5.5 is weaker than Theorem 1.11 in [5]. However, Theorem 5.5 above has a significantly larger area of applicability than Theorem 1.11 in [5], as one can see from the relationship (5) between the two topologies.

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