

Smooth maps of a foliated manifold in a symplectic manifold

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Abstract. Let M be a smooth manifold with a regular foliation \mathcal{F} and a 2-form ω which induces closed forms on the leaves of \mathcal{F} in the leaf topology. A smooth map $f: (M, \mathcal{F}) \rightarrow (N, \sigma)$ in a symplectic manifold (N, σ) is called a *foliated symplectic immersion* if f restricts to an immersion on each leaf of the foliation and further, the restriction of $f^*\sigma$ is the same as the restriction of ω on each leaf of the foliation.

If f is a foliated symplectic immersion then the derivative map Df gives rise to a bundle morphism $F: TM \rightarrow TN$ which restricts to a monomorphism on $T\mathcal{F} \subseteq TM$ and satisfies the condition $F^*\sigma = \omega$ on $T\mathcal{F}$. A natural question is whether the existence of such a bundle map F ensures the existence of a foliated symplectic immersion f . As we shall see in this paper, the obstruction to the existence of such an f is only topological in nature. The result is proved using the h -principle theory of Gromov.

Keywords. Foliations; foliated immersions; foliated symplectic forms.

1. Introduction

Gromov proves in [3] that the immersions of a smooth manifold M in a symplectic manifold (N, σ) inducing a given closed form ω on M satisfy the C^0 -dense h -principle in the space of all continuous maps which pull back the de Rham cohomology class of σ onto that of ω . In this paper we prove a foliated version of this result.

Let M be a smooth manifold with a regular foliation \mathcal{F} and let ω be a 2-form on M which induces closed forms on the leaves of \mathcal{F} in the leaf topology. We shall refer to the induced form on any leaf as the ‘restriction’ of the global form on it. A smooth map $f: (M, \mathcal{F}) \rightarrow N$ is called a *foliated immersion* if f restricts to an immersion on each leaf of the foliation. Further, if the restriction of $f^*\sigma$ is the same as the restriction of ω on each leaf of the foliation then f is called a *foliated symplectic immersion*.

If f is a foliated symplectic immersion then the derivative map Df gives rise to a bundle morphism $F: TM \rightarrow TN$ which restricts to a monomorphism on $T\mathcal{F} \subseteq TM$ and satisfies the condition $F^*\sigma = \omega$ on $T\mathcal{F}$. A natural question is whether the existence of such a bundle map F ensures the existence of a foliated immersion satisfying $f^*\sigma = \omega$ on \mathcal{F} . As we shall see in this paper, the obstruction to the existence of such an f is only topological in nature.

Let $\text{Symp}^0(T\mathcal{F}, TN)$ denote the space of bundle morphisms $F: TM \rightarrow TN$ such that F restricted to $T\mathcal{F}$ is a monomorphism and $F^*\sigma = \omega$ on $T\mathcal{F}$. We endow this space with the C^0 -compact-open topology.

Theorem 1.1. *Let ω be a foliated closed 2-form (see Definition 3.1) on (M, \mathcal{F}) and let $(F_0, f_0): TM \rightarrow TN$ be a bundle homomorphism which satisfies the following conditions:*

- (1) f_0 pulls back the cohomology class of σ onto the foliated cohomology class of ω ;
- (2) F_0 is fibre-wise injective on $T\mathcal{F}$ and $F_0^*\sigma = \omega$ on $T\mathcal{F}$.

If $\dim(\mathcal{F}) < \dim(N)$, then the map f_0 admits a fine C^0 approximation by a foliated symplectic immersion $f: (M, \mathcal{F}, \omega) \rightarrow (N, \sigma)$ whose differential $Df: TM \rightarrow TN$ is homotopic to F_0 in the space $\text{Symp}^0(T\mathcal{F}, TN)$.

In other words, foliated symplectic immersions satisfy the C^0 -dense h -principle in the space of continuous maps $f: (M, \mathcal{F}) \rightarrow N$ which pull back the cohomology class of σ onto the foliated cohomology class of ω .

If we consider the trivial foliation (foliation with a single leaf) on M then as a special case we obtain the symplectic immersion theorem due to Datta [1] and Gromov [3].

For a simple application of the above theorem consider the Euclidean manifold \mathbb{R}^{2n} with the canonical symplectic form σ_0 .

COROLLARY 1.2

Let ω be a foliated closed 2-form on (M, \mathcal{F}) . Suppose $\text{rank}(\omega|_{T\mathcal{F}_x}) \geq r$ for all $x \in M$, for some $0 \leq r \leq \dim \mathcal{F}$.

Then there exists a foliated immersion $f: (M, \mathcal{F}) \rightarrow \mathbb{R}^{2n}$ such that $f^\sigma_0 = \omega$ on each leaf of the foliation provided $2n \geq \dim M + 2 \dim \mathcal{F} - r - 1$.*

The proof of the main theorem is based on the sheaf theoretic technique in h -principle and the Nash Moser implicit function theorem due to Gromov (§2.2, §2.3 of [3]). Starting with a formal solution (F_0, f_0) as in the hypothesis of the theorem, we embed the given manifold (M, \mathcal{F}) in a foliated manifold $(M', \mathcal{F}', \omega')$, where \mathcal{F}' is a foliation with $\dim \mathcal{F}' = \dim N$ and ω' is a 2-form which restricts to a symplectic form on each leaf of the foliation \mathcal{F}' . Moreover, the pullback of the form ω' on M equals ω on $T\mathcal{F}$. We observe that the foliated symplectic forms exhibit a stability property analogous to the stability of symplectic forms as in Moser’s theorem [5, 6] and this observation plays an important role in our proof. We prove through a sequence of propositions that there exists a foliated symplectic immersion $f': M' \rightarrow N$ such that the derivative of $f'|_M$ is homotopic to F_0 in $\text{Symp}^0(T\mathcal{F}, TN)$.

We refer the reader to [2] for a quick review of the terminology and theory of topological sheaves which we shall extensively use in this paper. For a detailed exposition on this we refer to [3]. We organise the paper as follows. In §2 we briefly review the foliated cohomology theory. In §3 we discuss the Poincaré lemma in the context of foliated closed forms and in §4 we prove an analogue of Moser’s theorem for foliated symplectic forms. The proof of the main result of this paper is given in §§5 through 7.

2. Foliated de Rham cohomology

Let M be a smooth manifold with a regular foliation \mathcal{F} on it. Denote the space of smooth differential r -forms on M by $\Omega^r(M)$. We define as in [4], for each $r \geq 0$,

$$I^r(\mathcal{F}) = \{\omega \in \Omega^r(M) : \omega|_{\mathcal{F}} = 0\},$$

where $\omega|_{\mathcal{F}} = 0$ means that for any $x \in M$, $\omega(x)(v_1, v_2, \dots, v_r) = 0$ for all v_1, v_2, \dots, v_r in $T_x\mathcal{F}$. Clearly, $I^r(\mathcal{F})$ is a linear subspace of $\Omega^r(M)$ and $I^0(\mathcal{F}) = \Omega^0(M) = C^\infty(M)$. Moreover, $I(\mathcal{F}) = \bigoplus_{r \geq 0} I^r(\mathcal{F})$ is a graded ideal of the de Rham complex $\Omega^*(M)$. Let $\Omega^r(\mathcal{F}) = \frac{\Omega^r(M)}{I^r(\mathcal{F})}$ and $q: \Omega^r(M) \rightarrow \Omega^r(\mathcal{F})$ denote the quotient map. The exterior differential operator $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$ induces a morphism $d_{\mathcal{F}}: \Omega^r(\mathcal{F}) \rightarrow \Omega^{r+1}(\mathcal{F})$ by $d_{\mathcal{F}}(q(\omega)) = q(d\omega)$, since d maps $I^r(\mathcal{F})$ into $I^{r+1}(\mathcal{F})$. It can be easily checked that $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$ so that $(\Omega^r(\mathcal{F}), d_{\mathcal{F}})$ is a co-chain complex which is called the foliated de Rham complex of (M, \mathcal{F}) ,

$$0 \rightarrow \Omega^0(M) \rightarrow \Omega^1(\mathcal{F}) \rightarrow \Omega^2(\mathcal{F}) \rightarrow \dots \rightarrow \Omega^r(\mathcal{F}) \rightarrow \Omega^{r+1}(\mathcal{F}) \rightarrow \dots$$

Let $T\mathcal{F}$ denote the subbundle of TM consisting of all vectors which are tangent to the leaves of the foliation and let $T^*\mathcal{F}$ denote the dual of this bundle. Then it may be noted that $\Omega^k(\mathcal{F})$ is isomorphic to the space of sections of the vector bundle $\Lambda^k T^*\mathcal{F}$. With this identification we observe that q is induced by the quotient map $T^*M \rightarrow T^*\mathcal{F}$.

The commutative diagram below says that the quotient map q is a chain map between de Rham complex and foliated de Rham complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \dots \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \dots \\ 0 & \longrightarrow & \Omega^0(\mathcal{F}) & \xrightarrow{d_{\mathcal{F}}} & \Omega^1(\mathcal{F}) & \xrightarrow{d_{\mathcal{F}}} & \Omega^2(\mathcal{F}) & \xrightarrow{d_{\mathcal{F}}} & \dots \end{array}$$

The cohomology of the complex $(\Omega^*(\mathcal{F}), d_{\mathcal{F}})$ is defined as the *foliated de Rham cohomology* of (M, \mathcal{F}) and is denoted by $H^r(\mathcal{F})$. In other words,

$$H^r(\mathcal{F}) = \frac{\text{Ker}\{d_{\mathcal{F}}: \Omega^r(\mathcal{F}) \rightarrow \Omega^{r+1}(\mathcal{F})\}}{\text{Im}\{d_{\mathcal{F}}: \Omega^{r-1}(\mathcal{F}) \rightarrow \Omega^r(\mathcal{F})\}}$$

It follows directly from the above description of the foliated de Rham complex that the foliated cohomology groups vanish in dimensions $r > \dim \mathcal{F}$.

For foliated manifolds with single leaf the foliated de Rham complex is the same as the ordinary de Rham complex. Hence the foliated de Rham cohomology is the same as the ordinary de Rham cohomology.

A smooth map $f: (M, \mathcal{F}) \rightarrow (N, \mathcal{F}')$ between two foliated manifolds is called *foliation preserving* if f takes a leaf of \mathcal{F} into a leaf of \mathcal{F}' . Such a map f induces a chain map $f^\#: \Omega^*(\mathcal{F}') \rightarrow \Omega^*(\mathcal{F})$ and hence a morphism $f^*: H^*(\mathcal{F}') \rightarrow H^*(\mathcal{F})$ in the cohomology level.

3. Poincaré lemma for foliated manifolds

Throughout this section, M will denote a smooth manifold with a regular foliation \mathcal{F} .

DEFINITION 3.1

An r -form ω on M is said to be *foliated closed* if $d_{\mathcal{F}}(q\omega) = 0$, that is, if ω restricts to a closed form on each leaf of the foliation. Similarly, an r -form ω on M is said to be *foliated exact* if there exists an $(r - 1)$ -form φ on M such that $q\omega = d_{\mathcal{F}}(q\varphi)$ which implies that ω restricts to an exact form on each leaf of the foliation.

Locally, every foliated closed form on any foliated space (M, \mathcal{F}) is foliated exact. In fact we have the following:

PROPOSITION 3.2

Let ω be a foliated closed form on (M, \mathcal{F}) such that ω vanishes on $T\mathcal{F}_x$ for some $x \in M$. Then there exists a form α of degree one less in a neighbourhood of x such that α vanishes on $T\mathcal{F}_x$ and $\omega = d\alpha$ on $T\mathcal{F}$.

The proposition is a consequence of a more general result stated below.

PROPOSITION 3.3

Let (M, \mathcal{F}) be a smooth foliated manifold and let $\pi: M' \rightarrow M$ be a vector bundle over M . Let \mathcal{F}' be the foliation on M' defined by $\mathcal{F}' = \pi^{-1}(\mathcal{F})$. Suppose ω is a foliated closed k -form on (M', \mathcal{F}') such that $i^*\omega = 0$ on \mathcal{F} , where $i: M \rightarrow M'$ embeds M as the zero section in M' . Then there exists a neighbourhood U of M in M' with a $(k - 1)$ -form β on U such that $d\beta = \omega$ on \mathcal{F}' and $\beta|_{i(M)} = 0$.

Proof. Since $\pi: M' \rightarrow M$ is a vector bundle we denote an element in M' over $x \in M$ by (x, v) , where v is in the fibre over x . Define for each $t \in [0, 1]$ a smooth map $\rho_t: M' \rightarrow M'$ by $(x, v) \mapsto (x, tv)$. Then $\rho_0(M') \subset M$ and $\rho_1 = \text{id}_{M'}$. From the definition of \mathcal{F}' it is clear that each ρ_t is foliation preserving. Let X_t denote the vector field along ρ_t defined by $X_t = \frac{d}{ds}\rho_s|_{s=t}$. Then $X_t(x, v) = v$ for all $(x, v) \in M'$ so that X_t is a foliated vector field on M' . In particular $X_t(x, 0) = 0$ for all $x \in M$.

As in [5] define an operator $I: \Omega^k(M') \rightarrow \Omega^{k-1}(M')$ by $I(\tau) = \int_0^1 \rho_t^*(X_t \cdot \tau) dt$, where τ is a k -form on M' and $X_t \cdot \tau$ denotes the interior derivative of τ with respect to X_t . First observe that $I\tau|_M = 0$ since ρ_t restricts to the identity map on M and X_t vanishes on M . Secondly, since ρ_t is foliation preserving and X_t is a foliated vector field for each t , I maps $I^k(\mathcal{F}')$ into itself.

Proceeding as in [5] we integrate the relation

$$\frac{d}{dt}(\rho_t^*\omega) = \rho_t^*(X_t \cdot d\omega) + d(\rho_t^*(X_t \cdot \omega)).$$

with respect to t in $[0, 1]$ to get

$$\rho_1^*\omega - \rho_0^*\omega = d(I\omega) + I(d\omega).$$

Noting that $i^*\omega$ vanishes on \mathcal{F} , we have $\rho_0^*\omega = \pi^*i^*\omega = 0$ on \mathcal{F}' . Further, $I(d\omega) = 0$ on \mathcal{F}' . Hence we get $\omega = d(I\omega)$ on \mathcal{F}' . The proof is completed by letting $\beta = I(\omega)$. □

COROLLARY 3.4

Let (M, \mathcal{F}) and (M', \mathcal{F}') be as in the above theorem and let ω be a section of the k -th exterior bundle $\Lambda^k(T^*M')$ defined over M . If $i^*\omega$ is a foliated closed form on (M, \mathcal{F}) then ω extends to a foliated closed form on some neighbourhood of M .

Proof. Take any extension ω' of ω on a neighbourhood of M . Defining I as in the proof of Theorem 3.3 we get

$$\omega' - \rho_0^*\omega' = I(d\omega') + dI(\omega').$$

$\rho_0^* \omega'$ is foliated closed since $\rho_0^* \omega' = \pi^* i^* \omega' = \pi^* i^* \omega$ and $i^* \omega$ is a foliated closed form on (M, \mathcal{F}) . Moreover, from the definition of I it follows that $I(d\omega')|_M = 0$. Therefore, taking exterior derivative of the above equation we get $d\omega' = dI(d\omega')$ on $T\mathcal{F}'$. Define, $\omega_1 = \omega' - I(d\omega')$. Then ω_1 is the desired extension of ω . \square

4. Stability of foliated symplectic forms

DEFINITION 4.1

A foliated closed 2-form ω on (M, \mathcal{F}) is said to be a *foliated symplectic form* if ω is nondegenerate on each leaf of \mathcal{F} . The foliated manifold (M, \mathcal{F}) together with ω is then called a *foliated symplectic manifold*.

DEFINITION 4.2

A vector field X on a foliated manifold (M, \mathcal{F}) is said to be a *foliated vector field* if X maps M into $T\mathcal{F}$. The space of all foliated vector fields on (M, \mathcal{F}) will be denoted by $\mathcal{X}_{\mathcal{F}}$.

Observe that a foliated symplectic form ω on (M, \mathcal{F}) defines a bundle isomorphism $I_{\omega}: T\mathcal{F} \rightarrow T^*\mathcal{F}$ which is given by the correspondence $v \mapsto v \cdot \omega|_{\mathcal{F}}$. I_{ω} induces a bijection $\mathcal{X}_{\mathcal{F}} \rightarrow \Omega^1(\mathcal{F})$ which takes a foliated vector field X onto the foliated 1-form $q(X \cdot \omega)$.

PROPOSITION 4.3

Suppose ω_0 and ω_1 are two local foliated symplectic forms on (M, \mathcal{F}) such that $\omega_1 = \omega_0$ on $T\mathcal{F}_x$ for some $x \in M$. Then there exist open neighbourhoods U and V of x in M and a foliation preserving isotopy $\delta_t: U \rightarrow V$, $0 \leq t \leq 1$, such that $d\delta_t(x)$ is the identity map of $T_x M$ for all t , and $\delta_1^* \omega_1 = \omega_0$ on \mathcal{F} .

PROPOSITION 4.4

Let $M, (M', \mathcal{F}')$ be as in Proposition 3.3. Suppose ω_0 and ω_1 are two foliated symplectic forms on M' such that $\omega_1 = \omega_0$ on $T\mathcal{F}'|_M$. Then there exist open neighbourhoods U and V of M in M' and a foliation preserving isotopy $\delta_t: U \rightarrow V$ such that $d\delta_t = id$ on $TM'|_M$ and $\delta_1^* \omega_1 = \omega_0$ on \mathcal{F}' .

Proof. It follows from the hypothesis that $\omega_1 - \omega_0$ is a foliated closed form which vanishes on $T\mathcal{F}'|_M$. Therefore, by Proposition 3.3, $\omega_1 - \omega_0 = d\alpha$ on \mathcal{F}' , for some 1-form α satisfying $\alpha|_M = 0$. For $0 \leq t \leq 1$ we define a family of foliated closed forms ω_t by $\omega_t = \omega_0 + t d\alpha$. Since $\omega_1 = \omega_0$ on $T\mathcal{F}'|_M$, each ω_t restricts to a foliated symplectic form on some neighbourhood U of M in M' . For each $t \in [0, 1]$, define a foliated vector field X_t by $X_t = I_{\omega_t}^{-1}(-q(\alpha))$ so that $X_t \cdot \omega_t = -\alpha$ on $T\mathcal{F}'$. Let δ_t , $t \in [0, 1]$ be the one-parameter family of diffeomorphisms defined on some open neighbourhood of M such that $\delta_0 = id_M$ and $d\delta_t/dt = X_t$. Since $\alpha = 0$ on $T\mathcal{F}'|_M$, it follows that $\delta_t|_M = id$. Moreover, since X_t is a foliated vector field, $\{\delta_t\}$ is a foliation preserving diffeotopy.

Now, consider the identity

$$\frac{d}{dt}(\delta_t^* \omega_t) = \delta_t^* \left(\frac{d}{dt} \omega_t + X_t \cdot d\omega_t + d(X_t \cdot \omega_t) \right).$$

Since ω_t is a foliated closed form, $X_t \cdot d\omega_t = 0$ on $T\mathcal{F}'$ we obtain from the above relation $\frac{d}{dt}q(\delta_t^*\omega_t) = 0$. Hence $\delta_t^*\omega_t = \delta_0^*\omega_0 = \omega_0$ on $T\mathcal{F}'$. In particular $\delta_1^*\omega_1 = \omega_0$ on $T\mathcal{F}'$. Also, it is easy to see that δ_t fixes M point-wise and therefore $\frac{d}{dt}(d\delta_t)(x) = d\frac{d}{dt}\delta_t(x) = d(X_t(x)) = 0$ for all $x \in M$. \square

We end this section with the following result.

Theorem 4.5. *Let M be a closed manifold and let \mathcal{F} be a regular foliation on M . Let $\{\omega_t\}_{t \in [0,1]}$ be a smooth family of foliated symplectic forms on M such that $\omega_t = \omega_0 + d\alpha_t$ on \mathcal{F} , where α_t is a smooth family of 1-forms. Then there exists a smooth foliated diffeotopy $\delta_t: M \rightarrow M$, $\delta_0 = id$, such that $\delta_t^*(\omega_t) = \omega_0$ on \mathcal{F} (that is, $\delta_t^*(\omega_t) = \omega_0$ in $\Omega^2(\mathcal{F})$).*

5. Construction of an extension

In the subsequent discussion, (N, σ) will denote a symplectic manifold and (M, \mathcal{F}) will denote a foliated manifold with a foliated closed 2-form ω .

Let $(F_0, f_0): TM \rightarrow TN$ be a bundle homomorphism which satisfies the hypothesis of Theorem 1.1:

- (1) $f_0: M \rightarrow N$ is a continuous map such that the foliated cohomology class of $f_0^*\sigma$ is the same as that of ω ;
- (2) $F_0|_{T\mathcal{F}}: T\mathcal{F} \rightarrow TN$ is a monomorphism and $F_0^*\sigma = \omega$ on $T\mathcal{F}$.

PROPOSITION 5.1

There is a foliated symplectic manifold $(M', \mathcal{F}', \omega')$ and a foliation preserving embedding $i: (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ such that $i^\omega' = \omega$ on $T\mathcal{F}$, where i is the embedding of M in M' .*

Further, (F_0, f_0) extends to a bundle homomorphism $(F'_0, f'_0): TM' \rightarrow TN$ which satisfies the following properties:

- (1) $f'_0: M' \rightarrow N$ is a continuous map such that the foliated cohomology class of $f'^*_0\sigma$ is the same as that of ω' ;
- (2) $F'_0|_{T\mathcal{F}'}: T\mathcal{F}' \rightarrow TN$ is a monomorphism and $F'^*_0\sigma = \omega'$ on $T\mathcal{F}'$.

Proof. Consider the quotient bundle $\pi: f_0^*TN/T\mathcal{F} \rightarrow M$ and denote the total space of the bundle by X . Then $\dim X = \dim M + (\dim N - \dim \mathcal{F}) = \text{codim } \mathcal{F} + \dim N$. Let \mathcal{F}' be the foliation on X defined by the map π , that is $\mathcal{F}' = \pi^{-1}(\mathcal{F})$. Clearly, $\text{codim } \mathcal{F}' = \text{codim } \mathcal{F}$. Hence $\dim \mathcal{F}' = \dim N$.

M is embedded in X as the zero-section of π ; consequently TM is canonically embedded in $TX|_M = TM \oplus X$. Let \tilde{F} be any extension of F_0 to a bundle morphism $TX|_M \rightarrow f_0^*TN$ such that it maps X isomorphically onto a complementary subbundle of $d f_0(T\mathcal{F})$ in f_0^*TN . Define $\tilde{F}_0 = q \circ \tilde{F}$, where $q: f_0^*TN \rightarrow TN$ is the canonical bundle morphism. By the above construction $\tilde{F}_0|_{T\mathcal{F}'}$ is a bundle isomorphism over M . Therefore, $\tilde{F}_0^*\sigma$ is non-degenerate on $T\mathcal{F}'|_M$.

Since $\tilde{F}_0^*\sigma$ restricts to a foliated closed form on M , by Corollary 3.4 it extends to a foliated closed (and hence a foliated symplectic) form ω' on a neighbourhood of M in (X, \mathcal{F}') . Then, following Proposition 4.4, we can show that \tilde{F}_0 extends to a bundle morphism F'_0 on an open neighbourhood M' of M such that $F'^*_0\sigma = \omega'$ on $T\mathcal{F}'$.

The inclusion $i: M \hookrightarrow M'$ induces a morphism $\Omega^2(\mathcal{F}') \longrightarrow \Omega^2(\mathcal{F})$ which takes the class of ω' in $\Omega^2(\mathcal{F}')$ onto the class of ω in $\Omega^2(\mathcal{F})$. The induced map $i^*: H^2(\mathcal{F}') \longrightarrow H^2(\mathcal{F})$ in the cohomology level, therefore, maps the foliated cohomology class of ω' onto that of ω .

If f'_0 denotes the underlying map of F'_0 then it is an extension of f_0 . Since i^* is an isomorphism, it follows that f'_0 pulls back the de Rham cohomology class of σ onto the foliated cohomology class of ω' . \square

6. Sheaf of foliated symplectic immersions

DEFINITION 6.1

A foliated immersion $f: (M, \mathcal{F}, \omega) \longrightarrow (N, \sigma)$ is said to be a *foliated symplectic immersion* if the restriction of $f^*\sigma$ is same as the restriction of ω on each leaf of the foliation.

Let M', \mathcal{F}' and ω' be as defined in the previous section. Let $\mathcal{S}_{\mathcal{F}'}$ denote the sheaf of foliated immersions $f: (M', \mathcal{F}') \longrightarrow N$ which satisfy $f^*\sigma = \omega'$ on \mathcal{F}' . The topology of the sheaf comes from the C^∞ compact open topology on $C^\infty(M, N)$.

Let $\mathcal{R}_{\mathcal{F}'} \subset J^1(M', N)$ be the space of 1-jets of foliated symplectic immersions of $(M', \mathcal{F}', \omega')$ in (N, σ) . We endow the sheaf of sections of $J^1(M', N)$ with the C^0 compact open topology. $\Psi_{\mathcal{F}'}$ will denote the subsheaf of sections of the 1-jet bundle whose images lie in $\mathcal{R}_{\mathcal{F}'}$. Note that $\Psi_{\mathcal{F}'}(M')$ can be identified with the space $\text{Symp}^0(T\mathcal{F}', TN)$ defined in the introduction.

The sheaf $\Psi_{\mathcal{F}'}|_M$ is flexible (§1.4.2(A') of [3]). Moreover, we have the following.

PROPOSITION 6.2

The 1-jet map $j^1: \mathcal{S}_{\mathcal{F}'}(x) \rightarrow \Psi_{\mathcal{F}'}(x)$ is a weak homotopy equivalence for all $x \in M'$.

Proof. In view of the discussion in §2.3.2(D), (D') of [3] it is enough to show that an infinitesimal solution of $\mathcal{R}_{\mathcal{F}'}$ can be homotoped to a local solution. Let $j_f^1(x) \in \mathcal{R}_{\mathcal{F}'}$ so that $df_x|T\mathcal{F}'_x$ is an injective linear map and $f^*\sigma = \omega'$ on $T\mathcal{F}'_x$.

Since $\dim \mathcal{F}' = \dim N$ and $df(x)$ is injective on $T\mathcal{F}'(x)$ it follows that f is a foliated immersion on a neighbourhood of x and $f^*\sigma$ is non-degenerate on $T\mathcal{F}'$. Let $\bar{\omega} = f^*\sigma$. $\bar{\omega}$ is a foliated symplectic form on a neighbourhood of x such that $\bar{\omega}_x = \omega'_x$ on $T\mathcal{F}'_x$. By applying Proposition 4.3 we get a local diffeotopy δ_t on $\text{Op}(x)$ in M' such that $d\delta_t = \text{id}$ at x and $\delta_t^* f^* \sigma = \delta_t^* \bar{\omega} = \omega'$ on $T\mathcal{F}'_x$. Let $f_t = f \circ \delta_t$, $0 \leq t \leq 1$. f_1 is a local solution of $\mathcal{R}_{\mathcal{F}'}$ and $j_{f_1}^1(x) = j_f^1(x) \in \mathcal{R}_{\mathcal{F}'}$ for all t . Thus we have proved that an infinitesimal solution can be homotoped to a local solution of $\mathcal{R}_{\mathcal{F}'}$ and this completes the proof. \square

It is important to note that $\mathcal{S}_{\mathcal{F}'}$ is not microflexible (§3.4.1(B) of [3]) and therefore we can not apply the sheaf theoretic technique directly to it. But it is possible to find an associated sheaf which has this desired property apart from having the same local weak homotopy type as $\mathcal{S}_{\mathcal{F}'}$.

To define the associated sheaf we start with the bundle map $(F'_0, f'_0): TM' \longrightarrow TN$ that we constructed in Proposition 5.1.

Consider the product manifold $M' \times N$ with the product form $\hat{\omega} = p_2^* \sigma - p_1^* \omega'$, where p_1 and p_2 are the projection maps from $M' \times N$ onto the first and the second factors respectively. Let $\hat{\mathcal{F}}$ denote the foliation on $M' \times N$ induced from \mathcal{F}' by the first projection map p_1 . It is easy to see that $\hat{\omega}$ is a foliated symplectic form on $(M' \times N, \hat{\mathcal{F}})$.

If f'_0 is as above then $g_0 = (1, f'_0): (M', \mathcal{F}') \longrightarrow (M' \times N, \hat{\mathcal{F}})$ is a foliation preserving embedding and therefore it induces a map $g_0^*: H^2(\hat{\mathcal{F}}) \longrightarrow H^2(\mathcal{F}')$ in the foliated cohomology level. It is easily seen that $g_0^*([\hat{\omega}]_{\hat{\mathcal{F}}}) = 0$ in the foliated cohomology group $H^2(\mathcal{F}')$. Now, there exists a neighbourhood Y of Image g_0 such that Image g_0 is a strong deformation retract of Y ; further the deformation retraction is foliation preserving. Consequently, $\hat{\omega}$ is a foliated exact form on $(Y, \hat{\mathcal{F}})$. Hence there exists a 1-form τ on Y such that $\hat{\omega} = d\tau$ on $T\hat{\mathcal{F}}$.

Suppose, $f: (M', \mathcal{F}') \longrightarrow N$ is a foliated immersion such that $f^*\sigma$ and ω' define the same foliated form in $\Omega^2(\mathcal{F}')$, and suppose the graph map $g = (1, f)$ has its image contained in Y . Then observe that g is foliation preserving and $g^*\tau$ is a foliated closed form on (M, \mathcal{F}') .

Denote by $\Gamma^\infty(Y)$, the space of sections $M' \longrightarrow M' \times N$ whose images lie in Y . Let $\mathcal{E}_{\mathcal{F}'}$ denote the sheaf of all those pairs (g, φ) in $\Gamma^\infty(Y) \times C^\infty(M')$ which satisfy the following conditions:

- (1) $p_2g: (M', \mathcal{F}') \longrightarrow N$ is a foliated immersion, so that g is foliation preserving, and
- (2) $g^*\tau + d\varphi = 0$ on $T\mathcal{F}'$.

Observe that there is a canonical map $\pi: \mathcal{E}_{\mathcal{F}'} \longrightarrow \mathcal{S}_{\mathcal{F}'}$ which maps (g, φ) onto p_2g .

PROPOSITION 6.3

The topological sheaves $\mathcal{E}_{\mathcal{F}'}$ and $\mathcal{S}_{\mathcal{F}'}$ have the same local weak homotopy type.

Consequently, $j^1 \circ \pi: \mathcal{E}_{\mathcal{F}'}(x) \rightarrow \Psi_{\mathcal{F}'}(x)$ is a weak homotopy equivalence for all $x \in M'$.

Proof. Take any two elements (g, φ_1) and (g, φ_2) in $\mathcal{E}_{\mathcal{F}'}(x)$ over some $f \in \mathcal{S}_{\mathcal{F}'}(x)$, then $d(\varphi_1 - \varphi_2) = 0$ on \mathcal{F}' in some foliated neighbourhood of $x \in M'$ which implies that $(\varphi_1 - \varphi_2) = \text{constant}$ on each plaque of \mathcal{F}' . If \mathcal{F}' is of codimension k , we get $\mathcal{E}_{\mathcal{F}'}(x) = \mathcal{S}_{\mathcal{F}'}(x) \times C^\infty(0)$, where $C^\infty(0)$ is the space of germs of real-valued functions on \mathbb{R}^k at 0. Since $C^\infty(0)$ is contractible, this completes the proof. □

Lemma 6.4. The sheaf $\mathcal{E}_{\mathcal{F}'}$ is microflexible.

Proof. Define a first order differential operator

$$\mathcal{D}: \Gamma^\infty(Y) \times C^\infty(M') \longrightarrow \Omega^1(\mathcal{F}')$$

by

$$(g, \phi) \mapsto (g^*\tau + d\phi)|_{T\mathcal{F}'}$$

Let L denote the linearisation of \mathcal{D} at (g, ϕ) . Then

$$L(\partial, \tilde{\phi}) = g^*(\partial \cdot \hat{\omega} + d(\partial \cdot \tau) + d\tilde{\phi})|_{T\mathcal{F}'},$$

where ∂ is a smooth vector field on N along g and $\tilde{\phi}$ is a smooth function on M .

L is right invertible if given any 1-form $\tilde{\omega} \in \Omega^1(\mathcal{F}')$ we can solve the following system of equations:

$$\begin{aligned} g^*(\partial \cdot \hat{\omega})|_{T\mathcal{F}'} &= \tilde{\omega}, \\ \partial \cdot \tau + \tilde{\phi} &= 0. \end{aligned}$$

Note that $\hat{\omega}$ is a foliated symplectic form on $(\hat{M}, \hat{\mathcal{F}})$; hence, if $g: M' \rightarrow Y \subset M' \times N$ is a foliation preserving section, then the map $\partial \mapsto g^*(\partial \cdot \hat{\omega})|_{T\mathcal{F}'}$ is an epimorphism. Consequently, for such a g we can solve the first equation for ∂ , and then take $\tilde{\phi} = -\partial \cdot \tau$.

Thus, the operator \mathcal{D} is infinitesimally invertible on all those (g, ϕ) for which $p_2g: (M', \mathcal{F}') \rightarrow (Y, \hat{\mathcal{F}})$ is a foliation preserving immersion.

Further, the foliation preserving immersions are solutions to some first order open differential relation.

Hence $\mathcal{E}_{\mathcal{F}'}$ is a microflexible sheaf (§2.3.2 of [3]). □

7. Proof of Theorem 1.1

We shall first prove that $\mathcal{E}_{\mathcal{F}'|_M}$ is flexible. Once we show this we can conclude with the sheaf homomorphism theorem that $j^1 \circ \pi: \mathcal{E}_{\mathcal{F}'|_M} \rightarrow \Psi_{\mathcal{F}'|_M}$ is a weak homotopy equivalence.

Recall a result on continuous sheaves from §2.2.3 of [3].

Theorem 7.1. *Let Φ be a microflexible sheaf over a manifold V and let a submanifold $V_0 \subset V$ of positive codimension be sharply movable by acting diffeotopies. Then the sheaf $\Phi_0 = \Phi|_{V_0}$ is a flexible sheaf.*

Consider the pseudogroup \mathcal{D} consisting of local diffeotopies δ_t on M' that preserve the foliation \mathcal{F}' and at the same time preserve the class of ω' in $\Omega^2(\mathcal{F}')$. Clearly \mathcal{D} acts on the sheaf $\mathcal{S}'_{\mathcal{F}'}$. Such diffeotopies may be obtained by integrating a foliated vector field ∂ such that $\partial \cdot \omega'$ is a foliated closed form on (M', \mathcal{F}') . Indeed, if ∂ is a foliated vector field then its restriction to each leaf is a vector field on the leaf. Thus it integrates to a foliation preserving diffeomorphism on M' . Also, observe that $\mathcal{L}_{\partial}\omega' = \partial \cdot d\omega' + d(\partial \cdot \omega') = 0$ on $T\mathcal{F}'$. Hence $\delta_t^*\omega' = \omega'$ in $\Omega^2(\mathcal{F}')$.

We shall show that there exists a pseudosubgroup of \mathcal{D} which acts on $\mathcal{E}_{\mathcal{F}'}$ and sharply moves M in M' . As we have already proved microflexibility of $\mathcal{E}_{\mathcal{F}'}$, the flexibility of $\mathcal{E}_{\mathcal{F}'|_M}$ will follow from Theorem 7.1.

DEFINITION 7.2

A foliation preserving diffeotopy δ_t of (M', \mathcal{F}') is called *exact foliation preserving diffeotopy* if there exists a 1-parameter family of 0-forms α_t on M' such that $\delta'_t \cdot \omega' = d\alpha_t$ on \mathcal{F}' . In particular, $\delta'_t \cdot \omega'$ is a foliated exact form for each t .

If α_t can be chosen to be identically zero on the maximal open subset where δ_t is constant then such a diffeotopy is called a *strictly exact foliation preserving diffeotopy*.

Lemma 7.3. *The strictly exact foliation preserving diffeotopies of M' act on $\mathcal{E}_{\mathcal{F}'}$.*

Proof. Let δ_t be a strictly exact foliation preserving diffeotopy on M' . We define a diffeotopy $\bar{\delta}_t$ on $M' \times N$ by $\bar{\delta}_t(x, y) = (\delta_t(x), y)$, where $x \in M'$ and $y \in N$. Recall that $M' \times N$ has the product foliation $\hat{\mathcal{F}}$ and $\bar{\delta}_t(\hat{\mathcal{F}}) \subseteq \hat{\mathcal{F}}$. Moreover, $\bar{\delta}'_t \cdot (\sigma - \omega')$ is foliated exact (since $\delta'_t \cdot \omega'$ is foliated exact).

Proceeding exactly as in §3.4.1 of [3], let α_t be a smooth family of 0-forms on $M' \times N$ satisfying $\bar{\delta}'_t \cdot (\sigma - \omega') = d\alpha_t$ on $\hat{\mathcal{F}}$. Since,

$$\frac{d}{dt}(\bar{\delta}_t^*\tau) = \bar{\delta}_t^*(d(\bar{\delta}'_t \cdot \tau) + \bar{\delta}'_t \cdot d\tau) = \bar{\delta}_t^*(d(\bar{\delta}'_t \cdot \tau) + \bar{\delta}'_t \cdot (\sigma - \omega'))$$

we have

$$\frac{d}{dt}(\bar{\delta}_t^* \tau) = d\bar{\delta}_t^*(\bar{\delta}_t' \cdot \tau + \alpha_t) \quad \text{on } \hat{\mathcal{F}}. \tag{1}$$

Integrating this equation with respect to t we obtain $\bar{\delta}_t^* \tau = \tau + d\varphi_t$ on $\hat{\mathcal{F}}$, where φ_t is a family of functions on M' .

Let (g, φ) be a section in $\mathcal{E}_{\mathcal{F}'}$ and let δ_t be as above. Define the action as in [3] by $\delta_t \cdot (g, \varphi) = (g', \varphi')$, where $g' = \bar{\delta}_t g \delta_t^{-1}$ and $\varphi' = (\delta_t^{-1})^*(\varphi + g^* \varphi_t)$. \square

Lemma 7.4. The strictly exact foliation preserving diffeotopies of $(M', \omega', \mathcal{F}')$ sharply move M .

Proof. Suppose that S is a closed hypersurface which lies in a small open set U of M . Take a vector $\partial_0 \in T_{x_0} \mathcal{F}'$ transversal to U in M and extend it to a foliated vector field $\partial = I_{\omega'}^{-1}(q(dH))$ which is transversal to U (provided we take U sufficiently small), where H is a real-valued function on U . The isotopy $\{\delta_t\}$ defined by ∂ is foliation preserving. Take $S_\epsilon = \bigcup_{t \in [0, \epsilon]} \delta_t(S) \subseteq M'$ and $a \in C^\infty(M')$ so that a vanishes outside a neighbourhood of $\mathcal{OP}(S_\epsilon)$ and $a = 1$ on a smaller neighbourhood of S_ϵ . The diffeotopy corresponding to $I_{\omega'}^{-1}(q(d(aH)))$ can move S (along the foliation) as sharply as we want. \square

Combining the results obtained in Lemma 6.4, Lemma 7.3 and Lemma 7.4 we conclude by Theorem 7.1 as follows.

PROPOSITION 7.5

The sheaf $\mathcal{E}_{\mathcal{F}'}|_M$ is flexible.

Proof of Theorem 1.1. It now follows from Proposition 6.3 and Proposition 7.5 (as a consequence of sheaf homomorphism theorem (§2.2.1 of [3])) that the composition $\mathcal{E}_{\mathcal{F}'}|_M \rightarrow \mathcal{S}_{\mathcal{F}'}|_M \xrightarrow{d} \Psi_{\mathcal{F}'}|_M$ is a weak homotopy equivalence. This implies that there is an $f' \in \mathcal{S}_{\mathcal{F}'}|_M$ such that Df' is homotopic to F'_0 in $\text{Symp}^0(T\mathcal{F}', TN)$. Moreover, the homotopy between f'_0 and f' can be made to lie in an arbitrary C^0 neighbourhood of f'_0 . If f denotes the restriction of f' to M , then $f \in \mathcal{S}_{\mathcal{F}}$ and Df is homotopic to F_0 in the space $\text{Symp}^0(T\mathcal{F}, TN)$. This proves Theorem 1.1. \square

Proof of Corollary 1.2. Let ω_0 be a linear 2-form on \mathbb{R}^m of rank r . Then r is an even integer, say $r = 2k$, such that $\omega_0^k \neq 0$ and $\omega_0^{k+1} = 0$. ω_0 can be extended to a symplectic form $\bar{\omega}_0$ on $\mathbb{R}^m \times \mathbb{R}^{m-2k} = \mathbb{R}^{2(m-k)}$. Observe that an injective linear map $\bar{F}: \mathbb{R}^{2(m-k)} \rightarrow \mathbb{R}^{2n}$ satisfying $\bar{F}^* \sigma_0 = \bar{\omega}_0$ restricts to an injective linear map $F: \mathbb{R}^m \rightarrow \mathbb{R}^{2n}$ satisfying $F^* \sigma_0 = \omega_0$. Thus, the space of injective linear maps $\bar{F}: \mathbb{R}^{2(m-k)} \rightarrow \mathbb{R}^{2n}$ satisfying $\bar{F}^* \sigma_0 = \bar{\omega}_0$ has the same homotopy type as the Stieffel manifold $V_{m-k}(\mathbb{C}^n)$ and it is known that $V_{m-k}(\mathbb{C}^n)$ is $2n - 2(m - k) = 2n - 2m + r$ connected.

Now let ω be a foliated closed 2-form on (M, \mathcal{F}) , and suppose that $\text{rank}(\omega_x|_{T\mathcal{F}_x}) \geq r$ for all $x \in M$ for some $0 \leq r \leq m$, where $m = \dim M$. In view of Theorem 1.1, it is enough to show the existence a monomorphism $F: T\mathcal{F} \rightarrow \mathbb{R}^{2n}$ such that $F^* \sigma_0 = \omega$. It follows from the above discussion that such a bundle map exists if $2n - 2 \dim \mathcal{F} + r \geq \dim M - 1$. This completes the proof of the corollary. \square

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