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Meet and join matrices in the poset of exponential divisors

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Dedicated to the memory of Professor M V Subbarao

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Abstract. It is well-known that $(\mathbb{Z}_+, |) = (\mathbb{Z}_+, \mathrm{GCD}, \mathrm{LCM})$ is a lattice, where $|$ is the usual divisibility relation and GCD and LCM stand for the greatest common divisor and the least common multiple of positive integers.

The number $d = \prod_{k=1}^{r} p_k^{d^{(k)}}$ is said to be an exponential divisor or an e-divisor of $n = \prod_{k=1}^{r} p_k^{n^{(k)}}$ $(n > 1)$, written as $d \,|_e\, n$, if $d^{(k)} \mid n^{(k)}$ for all prime divisors p_k of n. It is easy to see that $(\mathbb{Z}_+\backslash\{1\}, |_e)$ is a poset under the exponential divisibility relation but not a lattice, since the greatest common exponential divisor (GCED) and the least common exponential multiple (LCEM) do not always exist.

In this paper we embed this poset in a lattice. As an application we study the GCED and LCEM matrices, analogues of GCD and LCM matrices, which are both special cases of meet and join matrices on lattices.

Keywords. Exponential divisor; lattice; meet matrix; join matrix; greatest common divisor matrix; least common multiple matrix.

1. Introduction

It is well-known that the set $\mathbb{Z}_+$ of positive integers is a poset under the usual divisibility relation $|$. It is likewise well-known that the greatest common divisor (GCD) and the least common multiple (LCM) of positive integers serve as the meet and the join on this poset. Thus $(\mathbb{Z}_+, |) = (\mathbb{Z}_+, \mathrm{GCD}, \mathrm{LCM})$ is a lattice, known as the divisor lattice. For a general account of lattices, see [1,3,14,19].

Let $n > 1$ be a positive integer and let

$$n = \prod_{k=1}^{r} p_k^{n^{(k)}} \tag{1.1}$$

be its canonical factorization, where $p_1, p_2, \ldots, p_r$ are the distinct prime divisors of n. Thus $n^{(k)} > 0$ for $k = 1, 2, \ldots, r$. The number $d = \prod_{k=1}^{r} p_k^{d^{(k)}}$ is said to be an exponential divisor or an e-divisor of n if $d^{(k)} \mid n^{(k)}$ for all $k = 1, 2, \ldots, r$. If d is an e-divisor of n, we denote $d \,|_e\, n$. Each e-divisor d of n has the same prime divisors as n. For example, the e-divisors of $2^5 3^{10}$ are $2^1 3^1$, $2^1 3^2$, $2^1 3^5$, $2^1 3^{10}$, $2^5 3^1$, $2^5 3^2$, $2^5 3^5$ and $2^5 3^{10}$, while $2^5 \nmid_e 2^5 3^{10}$ and $3^{10} \nmid_e 2^5 3^{10}$. It is clear that each e-divisor of n is also an ordinary divisor of n but the converse need not hold. For example, $2^2 3^{10} \mid 2^5 3^{10}$ but $2^2 3^{10} \nmid_e 2^5 3^{10}$.

The concept of an e-divisor originates with Subbarao [20]. For further material, see e.g. [5,16,17,21].

The greatest common e-divisor (GCED) of m and n exists if and only if $m, n > 1$ and m and n have the same prime divisors, that is, $m^{(k)}$ and $n^{(k)}$ are nonzero simultaneously. In this case the GCED is given as

$$(m, n)_e = \prod_{k=1}^{r} p_k^{(m^{(k)}, n^{(k)})}, \tag{1.2}$$

where $(m^{(k)}, n^{(k)}) = \text{GCD}(m^{(k)}, n^{(k)})$. For example, $(2^5 3^{10}, 2^6 3^{12})_e = 2^{(5,6)} 3^{(10,12)} = 2^1 3^2$, while $(2^5 3^{10}, 2^6)_e$ does not exist.

Analogously, the least common e-multiple (LCEM) of m and n exists if and only if $m, n > 1$ and m and n have the same prime divisors. In this case the LCEM is given as

$$[m, n]_e = \prod_{k=1}^{r} p_k^{[m^{(k)}, n^{(k)}]}, \tag{1.3}$$

where $[m^{(k)}, n^{(k)}] = \text{LCM}(m^{(k)}, n^{(k)})$. For example, $[2^5 3^{10}, 2^6 3^{12}]_e = 2^{[5,6]} 3^{[10,12]} = 2^{30} 3^{60}$, while $[2^5 3^{10}, 2^6]_e$ does not exist. Note that $(m, n)_e$ exists if and only if $[m, n]_e$ exists.

It is easy to see that the set $\mathbb{Z}_+ \backslash \{1\}$ is a poset under the e-divisibility relation. However, $(\mathbb{Z}_+ \backslash \{1\}, |_e)$ is not a lattice as the above examples show. We embed the poset $(\mathbb{Z}_+ \backslash \{1\}, |_e)$ in a lattice by adjoining two elements, 1 and ∞, in the poset so that for each $n \in \mathbb{Z}_+ \backslash \{1\}$ the element 1 is an e-divisor of n and analogously n is an e-divisor of ∞. Then $(m, n)_e = 1$ and $[m, n]_e = \infty$ if $(m, n)_e$ and $[m, n]_e$ does not exist in the usual sense. Furthermore, $(1, n)_e = 1$ and $[n, \infty]_e = \infty$ for all $n \in \mathbb{Z}_+^\infty$, where $\mathbb{Z}_+^\infty = \mathbb{Z}_+ \cup \{\infty\}$. Thus $(\mathbb{Z}_+^\infty, |_e)$ is a lattice, referred to as the e-divisor lattice.

Let $(P, \leq) = (P, \wedge, \vee)$ be a lattice, $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and $f \colon P \to \mathbb{C}$ be a function. The meet matrix $(S)_f$ and the join matrix $[S]_f$ on S with respect to f are defined by $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$. Rajarama Bhat [15] and Haukkanen [6] introduced meet matrices and Korkee and Haukkanen [12] introduced join matrices. Explicit formulae for the determinant and the inverse of meet and join matrices are presented in [6,11,12,15] (see also [2,8,17]). Most of these formulae are presented on meet-closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$) and join-closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$). Recently, Korkee and Haukkanen [13] introduced a method for calculating $\det(S)_f$, $(S)_f^{-1}$, $\det[S]_f$ and $[S]_f^{-1}$ on all sets S and functions f.

The GCD and LCM matrices $((S)_f)_{ij} = f((x_i, x_j))$ and $([S]_f)_{ij} = f([x_i, x_j])$ are well-known special cases of meet and join matrices (see [18] and also [4,7]). Since $(\mathbb{Z}_+^\infty, |_e)$ is a lattice, GCED matrices $((S)_{f,e})_{ij} = f((x_i, x_j)_e)$ and LCEM matrices $([S]_{f,e})_{ij} = f([x_i, x_j]_e)$ are also special cases of meet and join matrices.

If we attempt to apply the determinant and the inverse formulae of meet and join matrices (e.g. in [12]) to GCED and LCEM matrices, we encounter a problem. In fact, the underlying lattice in these formulae should be locally finite but the e-divisor lattice does not possess this property. In this paper we overcome this difficulty by constructing an appropriate finite sublattice of the e-divisor lattice. Then we are able to apply the general formulae to obtain formulae for the determinant and the inverse of GCED and LCEM matrices on GCED-closed and LCEM-closed sets.

This paper is organized as follows. In §2 we review the basic properties of incidence functions and meet and join matrices needed in this paper. Particular attention is paid to the Möbius function. In §3 we analyze the structure of the e-divisor lattice $(\mathbb{Z}_+^\infty, |_e)$ and evaluate the Möbius function of the meet-semilattice $(\mathbb{Z}_+, |_e)$. In §4 we construct an appropriate finite sublattice of $(\mathbb{Z}_+^\infty, |_e)$ and evaluate the Möbius function of this sublattice. These results makes it possible to apply the determinant and inverse formulae in §2 for meet and join matrices to GCED and LCEM matrices on the e-divisor lattice $(\mathbb{Z}_+^\infty, |_e)$.

In §5 we provide determinant and inverse formulae for GCED matrices. We utilize the results in §4 to obtain formulae in the case $1, \infty \notin S$, and then we apply these new formulae to obtain formulae in the two remaining cases $1 \in S$, $\infty \notin S$ and $\infty \in S$. In §6 we proceed dually for LCEM matrices.

2. On meet and join matrices on lattices

Let $(P, \leq)$ be a locally finite poset and let g be an incidence function of P, that is, g is a complex-valued function on $P \times P$ such that $g(x, y) = 0$ whenever $x \not\leq y$. If h is also an incidence function of P, the sum $g + h$ is defined by $(g + h)(x, y) = g(x, y) + h(x, y)$ and the convolution $g * h$ is defined by $(g * h)(x, y) = \sum_{x \leq z \leq y} g(x, z)h(z, y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the unity δ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The zeta function ζ of P is defined by $\zeta(x, y) = 1$ if $x \leq y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function μ of P is the inverse of ζ with respect to convolution. If necessary, we also denote these incidence functions by ζ_P and μ_P (see e.g. Proposition 2.1). Further material on incidence functions can be found in [1,14,19].

PROPOSITION 2.1

Let $(P_1, \leq_1), (P_2, \leq_2), \ldots, (P_r, \leq_r)$ be locally finite posets. Then the Cartesian product $P_1 \times P_2 \times \cdots \times P_r$ is also a locally finite poset, where the partial order is defined componentwise. Furthermore,

$$\mu_{P_1 \times P_2 \times \cdots \times P_r}((x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r)) = \prod_{k=1}^{r} \mu_{P_k}(x_k, y_k) \tag{2.1}$$

(*see Exercise* 7.12 *of* [14]).

Throughout the remainder of this section let $(P, \leq) = (P, \wedge, \vee)$ be a lattice, f be a complex-valued function on P, and S be a finite subset of P, where $S = \{x_1, x_2, \ldots, x_n\}$ with $x_i < x_j \Rightarrow i < j$.

We say that S is lower-closed if $(x_i \in S, y \in P, y \leq x_i) \Rightarrow y \in S$. We say that S is meet-closed if $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$. Analogously, we say that S is upper-closed if $(x_i \in S, y \in P, x_i \leq y) \Rightarrow y \in S$. We say that S is join-closed if $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$. It is clear that a lower-closed set is always meet-closed but the converse does not hold, and dually, an upper-closed set is always join-closed but the converse does not hold. We say that S is a sublattice of P if it is both meet-closed and join-closed. We say that S is a convex sublattice of P if $(x_i, x_j \in S, z \in P, x_i \leq z \leq x_j) \Rightarrow z \in S$. For example, if $x \leq y$, then the interval $[x, y] = \{z \in P \mid x \leq z \leq y\}$ is a convex sublattice of P.

For each $x \in P$ we define the principal order ideal of P as $\downarrow x = \{z \in P \mid z \leq x\}$. The set $\downarrow S = \{z \in P \mid \exists x_i \in S\colon z \leq x_i\}$ is said to be an order ideal of P generated by the subset S of P. Analogously, principal order filters and the order filter of P generated by S are defined as $\uparrow x = \{z \in P \mid x \leq z\}$ and $\uparrow S = \{z \in P \mid \exists x_i \in S\colon x_i \leq z\}$. We denote the least and the greatest element of the lattice P by $\hat{0}$ and $\hat{1}$ if they exist. An element $x \in P$ is said to be an atom of P if $\hat{0} < x$ and $\hat{0} \leq z < x \Rightarrow z = \hat{0}$. Analogously, an element $x \in P$ is said to be a dual atom of P if $x < \hat{1}$ and $x < z \leq \hat{1} \Rightarrow z = \hat{1}$.

DEFINITION 2.1

The $n \times n$ matrices $(S)_f$ and $[S]_f$, where $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$, are called the meet and the join matrix on S with respect to f.

We denote the restriction of an incidence function g on $S \times S$ by g_S and denote $(g_S)^{-1} = g_S^{-1}$ if $(g_S)^{-1}$ exists. Note that $(g_S)^{-1}$ need not be the same as $(g^{-1})_S$. If S is lower-closed and g^{-1} exists, then $(g_S)^{-1} = (g^{-1})_S$, which follows from a recursion formula for g^{-1} (see p. 139 of [1]). Note that if g^{-1} exists, then g_S^{-1} exists. We denote the zeta function of S by ζ_S and let $\mu_S = \zeta_S^{-1} = (\zeta_S)^{-1}$.

PROPOSITION 2.2

Let P be a locally finite lattice with the least element $\hat{0}$ and let S be a meet-closed subset of P. Then by Corollary 1 *of* [6] *we have*

$$\det(S)_f = \Delta_{S,1}\Delta_{S,2}\cdots\Delta_{S,n}, \tag{2.2}$$

where $\Delta_{S,t} = \sum_{\hat{0} \leq z \leq x_t;\ z \not\leq x_1,\ldots,x_{t-1}} \sum_{\hat{0} \leq w \leq z} f(w)\mu(w,z)$ *for* $t = 1, 2, \ldots, n$. *Furthermore, if (and only if)* $\Delta_{S,1}, \Delta_{S,2}, \ldots, \Delta_{S,n} \neq 0$, *then by Theorem* 7.1 *of* [11] *we have*

$$((S)_f^{-1})_{ij} = \sum_{x_i \leq x_t,\, x_j \leq x_t} \frac{\mu_S(x_i, x_t)\mu_S(x_j, x_t)}{\Delta_{S,t}}. \tag{2.3}$$

If S is even lower-closed, then $\Delta_{S,t}$ *simplifies to* $\Delta_{S,t} = \sum_{\hat{0} \leq w \leq x_t} f(w)\mu(w, x_t)$ *and* $\mu_S(x_s, x_t)$ *simplifies to* $\mu(x_s, x_t)$.

PROPOSITION 2.3

Let P be a locally finite lattice with the greatest element $\hat{1}$ and let S be a join-closed subset of P. Then by Theorem 4.1 *of* [12] *we have*

$$\det[S]_f = \Gamma_{S,1}\Gamma_{S,2}\cdots\Gamma_{S,n}, \tag{2.4}$$

where $\Gamma_{S,t} = \sum_{x_t \leq z \leq \hat{1};\ x_{t+1},\ldots,x_n \not\leq z} \sum_{z \leq w \leq \hat{1}} \mu(z,w)f(w)$ *for* $t = 1, 2, \ldots, n$. *Furthermore, if (and only if)* $\Gamma_{S,1}, \Gamma_{S,2}, \ldots, \Gamma_{S,n} \neq 0$, *then by Theorem* 4.5 *of* [12] *we have*

$$([S]_f^{-1})_{ij} = \sum_{x_t \leq x_i,\, x_t \leq x_j} \frac{\mu_S(x_t, x_i)\mu_S(x_t, x_j)}{\Gamma_{S,t}}. \tag{2.5}$$

If S is even upper-closed, then $\Gamma_{S,t}$ *simplifies to* $\Gamma_{S,t} = \sum_{x_t \leq w \leq \hat{1}} \mu(x_t, w)f(w)$ *and* $\mu_S(x_t, x_s)$ *simplifies to* $\mu(x_t, x_s)$.

In Proposition 2.4 we present formulae for μ_S in terms of μ obtained by Korkee (Theorem 2 of [10]). By these formulae we obtain another representation for $(S)_f^{-1}$ on meet-closed sets and for $[S]_f^{-1}$ on join-closed sets given in Propositions 2.2 and 2.3. For the sake of brevity we do not explicitly write down the other representations.

PROPOSITION 2.4

If S is a meet-closed subset of P, then

$$\mu_S(x_i, x_j) = \sum_{x_i \le z \le x_j;\ z \not\le x_i, \ldots, x_{j-1}} \mu(x_i, z) \tag{2.6}$$

for all $x_i, x_j \in S$. Dually, if S is a join-closed subset of P, then

$$\mu_S(x_i, x_j) = \sum_{x_i \le z \le x_j;\ x_{i+1}, \ldots, x_j \not\le z} \mu(z, x_j) \tag{2.7}$$

for all $x_i, x_j \in S$. If S is a lower-closed or an upper-closed set, then $\mu_S(x_i, x_j) = \mu(x_i, x_j)$ for all $x_i, x_j \in S$.

Proof. The results clearly hold when $x_i \not\le x_j$. Let $x_i \le x_j$. Let ζ_S denote the inverse of μ_S on S and let g denote the incidence function on S, where $g(x_i, x_j)$ is the right-hand side of the equality in (2.6). Now if S is meet-closed, then

$$\begin{aligned}(g * \zeta_S)(x_i, x_j) &= \sum_{x_i \le x_t \le x_j} \ \sum_{x_i \le z \le x_t;\ z \not\le x_i, \ldots, x_{t-1}} \mu(x_i, z) \\ &\overset{(*)}{=} \sum_{x_i \le z \le x_j} \mu(x_i, z) = (\mu * \zeta)(x_i, x_j) = \begin{cases} 1 & \text{if } x_i = x_j, \\ 0 & \text{if } x_i \ne x_j. \end{cases}\end{aligned}$$

We explain the equality $(*)$. Clearly each $\mu(x_i, z)$ occurs at most once on the left-hand side and at most once on the right-hand side of $(*)$. Further, each $\mu(x_i, z)$ on the left-hand side also occurs on the right-hand side. Conversely, let $\mu(x_i, z)$ occur on the right-hand side. Since $x_i \le z \le x_j$, there must be at least one x_t, $i \le t \le j$, such that $z \le x_t$ and $z \not\le x_i, \ldots, x_{t-1}$. Now $z \le x_j$, $z \le x_t$ and S is meet-closed, so $x_i \le z \le x_t \wedge x_j = x_m$ for some m. Since $z \le x_m$, $m \le t$ and $z \not\le x_i, \ldots, x_{t-1}$, we have $m = t$ and so $x_i \le x_t \le x_j$. Thus $\mu(x_i, z)$ also occurs on the left side. Thus equality $(*)$ holds. Therefore $g = (\zeta_S)^{-1} = \mu_S$ and thus (2.6) holds. The proof of (2.7) is similar and the latter statement is obvious. □

Note that (2.6) provides a lattice theoretic explanation to the term c_{ij}, where

$$c_{ij} = \begin{cases} \displaystyle\sum_{dx_i \mid x_j;\ dx_i \nmid x_1, \ldots x_{j-1}} \mu(d), & \text{if } x_i \mid x_j, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

presented by Bourque and Ligh (Theorem 3 of [4]) in their inverse formula for GCD matrices on GCD-closed sets.

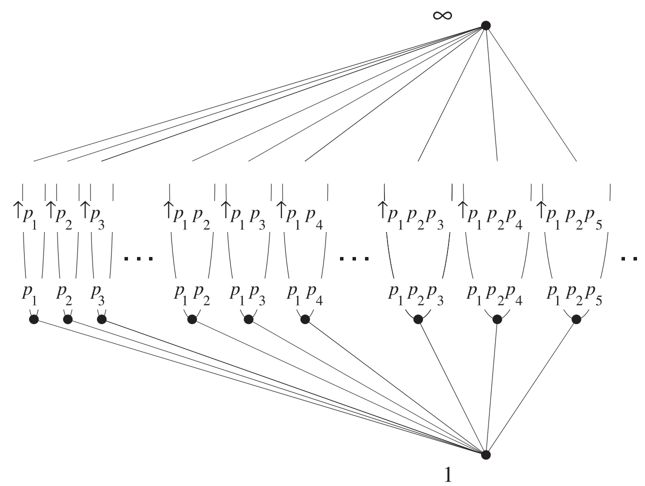

Figure 3.1.

3. The structure of the *e*-divisor lattice

In this section we describe the structure of the e-divisor lattice $(\mathbb{Z}_+^\infty, |_e)$ and calculate the Möbius function of $(\mathbb{Z}_+, |_e)$. In figure 3.1 we illustrate the e-divisor lattice. The least element and the greatest element of the lattice $(\mathbb{Z}_+^\infty, |_e)$ are 1 and ∞. The atoms are the squarefree numbers, that is, the finite products of distinct prime numbers. Each atom is covered by the principal order filter generated by this atom. These principal order filters are convex sublattices of $(\mathbb{Z}_+^\infty, |_e)$. For example, the atom $p_i p_j$ generates the filter $\uparrow p_i p_j = \{z\colon z = p_i^{z^{(i)}} p_j^{z^{(j)}};\ z^{(i)}, z^{(j)} \geq 1\}$. There does not exist any dual atoms in this lattice. The element ∞ covers each of these principal order filters.

Note that each $(\uparrow p_i, |_e)$ is lattice-isomorphic to the lattice $(\mathbb{Z}_+, |)$ with the natural isomorphism $\Phi(p_i^{x^{(i)}}) = x^{(i)}$. We denote this isomorphism as $(\uparrow p_i, |_e) \cong (\mathbb{Z}_+, |)$. Further, denote $\mathbb{Z}_+^r = \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$, where $\mathbb{Z}_+$ appears r times on the right-hand side. Then for each $\uparrow(p_1 p_2 \cdots p_r)$ we have $(\uparrow(p_1 p_2 \cdots p_r), |_e) \cong (\mathbb{Z}_+^r, |)$, where the divisibility in $\mathbb{Z}_+^r$ is defined componentwise. Clearly $\Phi(\prod_{k=1}^r p_k^{x^{(k)}}) = (x^{(1)}, x^{(2)}, \ldots, x^{(r)})$ serves as the natural isomorphism between these lattices.

Now we are in a position to calculate the Möbius incidence function μ_e of the e-divisor meet-semilattice $(\mathbb{Z}_+, |_e)$. Note that $(\mathbb{Z}_+^\infty, |_e)$ is not locally finite and therefore we are not able to define μ_e there. At first, by the definition $\mu * \zeta = \delta$ of the Möbius function we have

$$\begin{cases} \mu_e(x,x) = 1, & \text{for all } x \in \mathbb{Z}_+, \\ \mu_e(1,x) = -1, & \text{if } x \text{ is an atom of } (\mathbb{Z}_+, |_e), \\ \mu_e(1,x) = 0, & \text{if } 1 < x < \infty \text{ is not an atom of } (\mathbb{Z}_+, |_e), \\ \mu_e(x,y) = 0, & \text{if } x \text{ and } y \text{ do not have the same prime divisors.} \end{cases} \tag{3.1}$$

Second, let x and y have the same prime divisors $p_1, p_2, \ldots, p_r$, that is, let $x, y \in \uparrow(p_1 p_2 \cdots p_r)$. Since $\uparrow(p_1 p_2 \cdots p_r)$ is a convex sublattice of $(\mathbb{Z}_+^\infty, |_e)$, we can replace

$\mu_e(x, y)$ with $\mu_{(\uparrow(p_1\cdots p_r),|_e)}(x, y)$. Further, since isomorphic lattices have the same Möbius incidence functions, by Proposition 2.1 we have

$$\begin{aligned}\mu_e(x, y) &= \mu_{(\uparrow(p_1\cdots p_r),|_e)}\left(\prod_{k=1}^{r} p_k^{x^{(k)}}, \prod_{k=1}^{r} p_k^{y^{(k)}}\right)\\ &= \mu_{(\mathbb{Z}_+^r,|)}((x^{(1)}, x^{(2)}, \dots, x^{(r)}), (y^{(1)}, y^{(2)}, \dots, y^{(r)}))\\ &= \prod_{k=1}^{r} \mu_{(\mathbb{Z}_+,|)}(x^{(k)}, y^{(k)}) = \prod_{k=1}^{r} \mu(y^{(k)}/x^{(k)}),\end{aligned} \tag{3.2}$$

where the last μ is the usual number-theoretic Möbius function $\mu(n)$. For a connection between the Möbius incidence function in the divisor lattice and the usual number-theoretic Möbius function $\mu(n)$, see [1,14].

4. An appropriate sublattice of the *e*-divisor lattice

It is clear that the e-divisor lattice $(\mathbb{Z}_+^\infty, |_e)$ is not finite. It is not even locally finite, since for each $n < \infty$ the interval $[n, \infty] = \{m \in \mathbb{Z}_+^\infty : n \mid_e m \mid_e \infty\}$ is infinite. In order to use the formulae (2.2)–(2.5) simultaneously for GCED and LCEM matrices and for a fixed S, the underlying lattice must be finite. We overcome this difficulty by adopting an appropriate finite sublattice of the e-divisor lattice $(\mathbb{Z}_+^\infty, |_e)$. We denote this sublattice as $\mathcal{E}_S$ and we build it as follows (the subscript S in $\mathcal{E}_S$ indicates that $\mathcal{E}_S$ depends on S).

Let $S = \{x_1, x_2, \dots, x_n\}$ be a finite subset of $\mathbb{Z}_+^\infty$. First assume that $\infty \notin S$. For all $x_i \in S$, let

$$x_i = \prod_{k=1}^{r} p_k^{x_i^{(k)}}, \tag{4.1}$$

where $p_1, p_2, \dots, p_r$ are the distinct prime divisors of the elements of S. Thus it is possible that $x_i^{(k)} = 0$ for some pairs i, k. Further, let

$$u_k = \mathrm{LCM}\,\{x_i^{(k)} \geq 1 : x_i \in S\} \tag{4.2}$$

for $k = 1, 2, \dots, r$. Now, let $\mathcal{E}_S$ be the unique lattice possessing the following properties. Its least element is 1 and its greatest element is ∞. The atoms of $\mathcal{E}_S$ are

$$\underbrace{p_1, p_2, \dots, p_r}_{\text{1 prime divisor}},\ \underbrace{p_1p_2, \dots, p_{r-1}p_r}_{\text{2 prime divisors}}, \dots, \underbrace{p_1p_2\cdots p_r}_{r\text{ prime divisors}}, \tag{4.3}$$

the dual atoms are

$$\underbrace{p_1^{u_1}, p_2^{u_2}, \dots, p_r^{u_r}}_{\text{1 prime divisor}},\ \underbrace{p_1^{u_1}p_2^{u_2}, \dots, p_{r-1}^{u_{r-1}}p_r^{u_r}}_{\text{2 prime divisors}}, \dots, \underbrace{p_1^{u_1}p_2^{u_2}\cdots p_r^{u_r}}_{r\text{ prime divisors}}, \tag{4.4}$$

and between any pair of an atom $p_{i1}p_{i2}\cdots p_{im}$ and the corresponding dual atom $p_{i1}^{u_{i1}}p_{i2}^{u_{i2}}\cdots p_{im}^{u_{im}}$ we construct the interval $[p_{i1}p_{i2}\cdots p_{im}, p_{i1}^{u_{i1}}p_{i2}^{u_{i2}}\cdots p_{im}^{u_{im}}]$ of $(\mathbb{Z}_+^\infty, |_e)$. We illustrate the lattice $\mathcal{E}_S$ in figure 4.1.

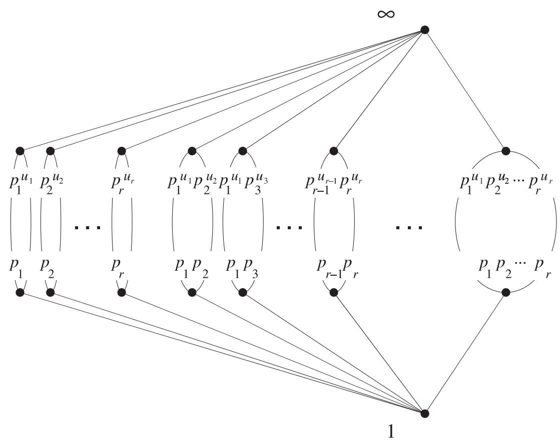

Figure 4.1.

Note that for each $[p_i, p_i^{u_i}]$ we have $([p_i, p_i^{u_i}], |_e) \cong (D_{u_i}, |)$, where D_{u_i} is the set of the usual positive divisors of u_i. More generally, for each interval $[p_{i1}p_{i2}\cdots p_{im}, p_{i1}^{u_{i1}}p_{i2}^{u_{i2}}\cdots p_{im}^{u_{im}}]$ we have $([p_{i1}p_{i2}\cdots p_{im}, p_{i1}^{u_{i1}}p_{i2}^{u_{i2}}\cdots p_{im}^{u_{im}}], |_e) \cong (D_{u_{i1}} \times D_{u_{i2}} \times \cdots \times D_{u_{im}}, |)$, where the divisibility in $D_{u_{i1}} \times \cdots \times D_{u_{im}}$ is defined componentwise.

Second, if $\infty \in S$, then $\mathcal{E}_S = \mathcal{E}_{S'}$, where $S' = S\backslash\{\infty\}$ and $\mathcal{E}_{S'}$ is constructed as described above.

Now, $(\mathcal{E}_S, |_e)$ is a finite sublattice of $(\mathbb{Z}_+, |_e)$ and therefore Propositions 2.2–2.3 are available in $(\mathcal{E}_S, |_e)$ and further for GCED and LCEM matrices.

Example 4.1. Let $S = \{q^5, p^2q^5, p^4s^7, q^3s^7, p^2q^3s\}$, where $p_1 = p$, $p_2 = q$, $p_3 = s$ are three distinct prime numbers (see the circled dots in figure 4.2). Then $u_1 = [2, 4] = 4$, $u_2 = [3, 5] = 15$ and $u_3 = [1, 7] = 7$. The atoms of the lattice $\mathcal{E}_S$ are p, q, s, pq, ps, qs, pqs and the dual atoms are $p^4, q^{15}, s^7, p^4q^{15}, p^4s^7, q^{15}s^7, p^4q^{15}s^7$. For brevity, we do not write down all the expressions of the elements of $\mathcal{E}_S$ in figure 4.2. For example, between

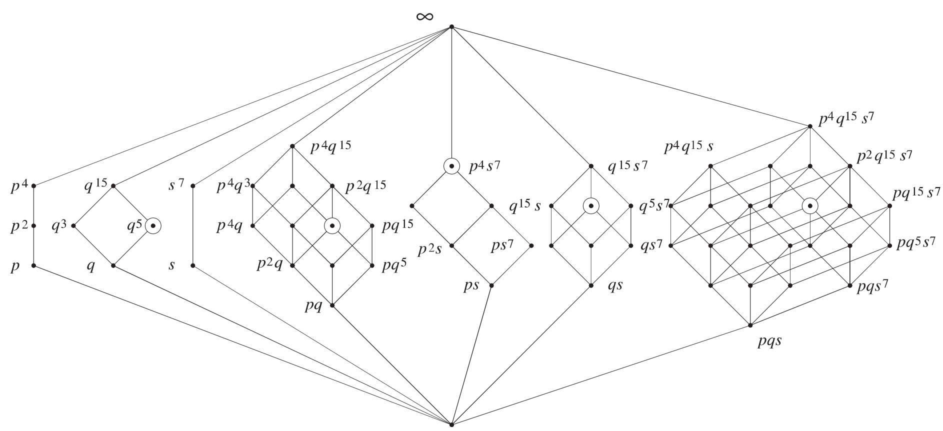

Figure 4.2.

the atom pq and the dual atom p^4q^{15} we find a sublattice of $\mathcal{E}_S$, which is isomorphic to $D_4 \times D_{15}$.

Note that if we use $\mathcal{E}_S$ in Propositions 2.2–2.3, we must use $\mu_{\mathcal{E}_S}$ instead of μ_e. Fortunately, $\mathcal{E}_S \backslash \{\infty\}$ is a lower-closed subset of $\mathbb{Z}_+$ and thus we have $\mu_{\mathcal{E}_S}(x, y) = \mu_e(x, y)$ for all $x, y \in \mathcal{E}_S \backslash \{\infty\}$. In $(\mathbb{Z}_+^\infty, |_e)$ we cannot define $\mu_e(x, \infty)$, $x < \infty$, since the interval $[x, \infty]$ is not finite. In $(\mathcal{E}_S, |_e)$ also $\mu_{\mathcal{E}_S}(x, \infty)$, $x < \infty$, is defined. It follows from the definition $\mu * \zeta = \delta$ of the Möbius function that

$$\begin{cases} \mu_{\mathcal{E}_S}(x, x) = 1, & \text{for all } x \in \mathcal{E}_S, \\ \mu_{\mathcal{E}_S}(1, x) = -1, & \text{if } x \text{ is an atom of } \mathcal{E}_S, \\ \mu_{\mathcal{E}_S}(x, \infty) = -1, & \text{if } x \text{ is a dual atom of } \mathcal{E}_S, \\ \mu_{\mathcal{E}_S}(1, x) = 0, & \text{if } 1 < x \text{ is not an atom of } \mathcal{E}_S, \\ \mu_{\mathcal{E}_S}(x, \infty) = 0, & \text{if } x < \infty \text{ is not a dual atom of } \mathcal{E}_S, \\ \mu_{\mathcal{E}_S}(x, y) = 0, & \text{if } x \text{ and } y \text{ do not have the same prime divisors}, \\ \mu_{\mathcal{E}_S}(x, y) = \prod_{k=1}^{m} \mu\left(\frac{y^{(k)}}{x^{(k)}}\right), & \text{if } p_1, \ldots, p_m \text{ are the prime divisors of } x, y. \end{cases} \tag{4.5}$$

5. On GCED matrices on the *e*-divisor lattice

In this section we provide determinant and inverse formulae for GCED matrices $(S)_{f,e}$, which are defined by $((S)_{f,e})_{ij} = f((x_i, x_j)_e)$. We proceed as follows.

Let $S = \{x_1, x_2, \ldots, x_n\}$ with $x_i \mid_e x_j \Rightarrow i < j$ be a finite GCED-closed set of $\mathbb{Z}_+^\infty$. We distinguish three cases. In Case 1, we assume that $1, \infty \notin S$ and we write Proposition 2.2 for the GCED matrix $(S)_{f,e}$ using the sublattice $\mathcal{E}_S$ constructed in §4 and formula (4.5) for the Möbius function of $\mathcal{E}_S$. We obtain expressions for the determinant and the inverse of $(S)_{f,e}$ in terms of the number-theoretic Möbius function. In Case 2, we assume that $1 \in S$ and $\infty \notin S$. We obtain determinant and inverse formulae in terms of the expressions in Case 1. In Case 3, we assume that $\infty \in S$. We obtain determinant and inverse formulae in terms of the expressions in Case 1 or Case 2.

Since the underlying lattice must be locally finite, we do not consider $(\mathbb{Z}_+^\infty, |_e)$ but its sublattice $\mathcal{E}_S$. This is possible, since the GCED matrix $(S)_{f,e}$ under $\mathcal{E}_S$ is the same as that under $(\mathbb{Z}_+^\infty, |_e)$. In this context meet-closed and lower-closed sets become GCED-closed and ED-closed (*e*-divisor-closed) sets.

Case 1. $1, \infty \notin S$. Assume that S is GCED-closed such that $1, \infty \notin S$. Then S must be a subset of an interval $[p_{i1}p_{i2}\cdots p_{im}, p_{i1}^{u_{i1}}p_{i2}^{u_{i2}}\cdots p_{im}^{u_{im}}]$. For the sake of brevity we assume that S is a subset of the interval $[p_1p_2\cdots p_m, p_1^{u_1}p_2^{u_2}\cdots p_m^{u_m}]$.

Theorem 5.1. *Let S be given as above and let f be an arithmetical function. Then*

$$\det(S)_{f,e} = \Delta_{S,1}\Delta_{S,2}\cdots\Delta_{S,n}, \tag{5.1}$$

where

$$\Delta_{S,t} = \sum_{p_1\cdots p_m \,|_e\, z \,|_e\, x_t;\; z \nmid_e x_1,\ldots,x_{t-1}} \; \sum_{p_1\cdots p_m \,|_e\, w \,|_e\, z} f(w) \prod_{k=1}^{m} \mu(z^{(k)}/w^{(k)}) \tag{5.2}$$

for $t = 1, 2, \ldots, n$. *Furthermore, if (and only if)* $\Delta_{S,1}, \Delta_{S,2}, \ldots, \Delta_{S,n} \neq 0$, *then we have*

$$((S)^{-1}_{f,e})_{ij} = \sum_{x_i \,|_e\, x_t,\ x_j \,|_e\, x_t} \frac{\mu_S(x_i, x_t)\mu_S(x_j, x_t)}{\Delta_{S,t}}, \tag{5.3}$$

where

$$\mu_S(x_s, x_t) = \sum_{x_s \,|_e\, z \,|_e\, x_t;\ z \nmid_e x_s, \ldots, x_{t-1}} \prod_{k=1}^{m} \mu(x_t^{(k)}/z^{(k)}). \tag{5.4}$$

If S is even ED-closed (with respect to $[p_1 p_2 \cdots p_m, p_1^{u_1} p_2^{u_2} \cdots p_m^{u_m}]$*), then* $\Delta_{S,t}$ *simplifies to* $\Delta_{S,t} = \sum_{p_1 \cdots p_m |_e w |_e x_t} f(w) \prod_{k=1}^{m} \mu(x_t^{(k)}/w^{(k)})$ *and* $\mu_S(x_s, x_t)$ *simplifies to* $\mu_{\mathcal{E}_S}(x_s, x_t) = \prod_{k=1}^{m} \mu(x_t^{(k)}/x_s^{(k)})$.

Proof. We apply Proposition 2.2 with $P = \mathcal{E}_S$. Then the Möbius function of P is $\mu_{\mathcal{E}_S}$, which is given in (4.5). □

Case 2. $1 \in S$ and $\infty \notin S$. Assume that S is GCED-closed such that $1 \in S$ and $\infty \notin S$. Then S can be written in the form $S = S_1 \cup S_2 \cup \cdots \cup S_m$, where $S_1 = \{x_{11}\} = \{1\}$, $S_2 = \{x_{21}, \ldots, x_{2t_2}\}, \ldots, S_m = \{x_{m1}, \ldots, x_{mt_m}\}$, for each $i = 2, \ldots, m$ the elements of S_i have the same prime divisors and for all $2 \leq i < j \leq m$ the set of the distinct prime divisors of the elements of S_i is not the same as that of S_j.

Theorem 5.2. *Let S be given as above. Let f be an arithmetical function and let* $g(x) \equiv f(x) - f(1)$. *Then*

$$\det(S)_{f,e} = f(1)\det(S_2)_{g,e}\det(S_3)_{g,e}\cdots\det(S_m)_{g,e}. \tag{5.5}$$

Furthermore, if (and only if) $f(1), \det(S_2)_{g,e}, \det(S_3)_{g,e}, \ldots, \det(S_m)_{g,e} \neq 0$, *then we have*

$$(S)^{-1}_{f,e} = \begin{bmatrix} s_{11} & -\mathbf{1}_{t_2}(S_2)^{-1}_{g,e} & -\mathbf{1}_{t_3}(S_3)^{-1}_{g,e} & \cdots & -\mathbf{1}_{t_m}(S_m)^{-1}_{g,e} \\ -(S_2)^{-1}_{g,e}\mathbf{1}^{\mathrm{T}}_{t_2} & (S_2)^{-1}_{g,e} & O_{t_2\times t_3} & \cdots & O_{t_2\times t_m} \\ -(S_2)^{-1}_{g,e}\mathbf{1}^{\mathrm{T}}_{t_3} & O_{t_3\times t_2} & (S_3)^{-1}_{g,e} & \cdots & O_{t_3\times t_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(S_m)^{-1}_{g,e}\mathbf{1}^{\mathrm{T}}_{t_m} & O_{t_m\times t_2} & O_{t_m\times t_3} & \cdots & (S_m)^{-1}_{g,e} \end{bmatrix}, \tag{5.6}$$

where

$$s_{11} = 1/f(1) + \mathbf{1}_{t_2}(S_2)^{-1}_{g,e}\mathbf{1}^{\mathrm{T}}_{t_2} + \cdots + \mathbf{1}_{t_m}(S_m)^{-1}_{g,e}\mathbf{1}^{\mathrm{T}}_{t_m}, \tag{5.7}$$

$\mathbf{1}_{t_i}$ *denotes the* $1 \times t_i$ *row vector* $\mathbf{1}_{t_i} = (1, 1, \ldots, 1)$, $\mathbf{1}^{\mathrm{T}}_{t_i}$ *denotes its transpose and* $O_{t_i \times t_j}$ *denotes the* $t_i \times t_j$ *zero matrix. Finally, each of* $\det(S_i)_{g,e}$ *and* $(S_i)^{-1}_{g,e}$ *can be calculated by using Theorem* 5.1.

Proof. Using the notation above, $(S)_{f,e}$ can be partitioned as

$$(S)_{f,e} = \begin{bmatrix} f(1) & f(1)\mathbf{1}_{t_2} & \cdots & f(1)\mathbf{1}_{t_m} \\ f(1)\mathbf{1}^{\mathrm{T}}_{t_2} & (S_2)_{f,e} & \cdots & f(1)\mathbf{1}^{\mathrm{T}}_{t_2}\mathbf{1}_{t_m} \\ \vdots & \vdots & \ddots & \vdots \\ f(1)\mathbf{1}^{\mathrm{T}}_{t_m} & f(1)\mathbf{1}^{\mathrm{T}}_{t_m}\mathbf{1}_{t_2} & \cdots & (S_m)_{f,e} \end{bmatrix}. \tag{5.8}$$

By subtracting the first row from the rest of the rows of $(S)_{f,e}$ and by denoting $g(x) \equiv f(x) - f(1)$ we have

$$(S)_{f,e} \sim \begin{bmatrix} f(1) & f(1)\mathbf{1}_{t_2} & \cdots & f(1)\mathbf{1}_{t_m} \\ \mathbf{0}_{t_2}^{\mathrm{T}} & (S_2)_{g,e} & \cdots & O_{t_2\times t_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{t_m}^{\mathrm{T}} & O_{t_m\times t_2} & \cdots & (S_m)_{g,e} \end{bmatrix} = M, \tag{5.9}$$

where $\mathbf{0}_{t_i}$ denotes the $1 \times t_i$ row vector $\mathbf{0}_{t_i} = (0, 0, \ldots, 0)$ and $\mathbf{0}_i^{\mathrm{T}}$ denotes its transpose. Since these elementary row operations do not change the value of the determinant, we see that (5.5) holds.

The statement for invertibility of $(S)_{f,e}$ in Theorem 5.2 is obvious. Consider now the row-equivalent matrices $(S)_{f,e}$ and M. By using the appropriate elementary matrix E we have $M = E(S)_{f,e}$. Thus $M^{-1} = (S)_{f,e}^{-1}E^{-1}$ and further $(S)_{f,e}^{-1} = M^{-1}E$. Note that we can calculate M^{-1} easily. If we now subtract the first column from the rest of the columns of M^{-1}, then we obtain (5.6) and (5.7). Since the sets $S_2, \ldots, S_m$ are all GCED-closed and 1 does not belong to them, each $\det(S_i)_{g,e}$ in (5.5) and $(S_i)_{g,e}^{-1}$ in (5.6) and (5.7) can be calculated by Theorem 5.1. □

Note that the matrices $\mathbf{1}_{t_i}(S_i)_{g,e}^{-1}$ and $(S_i)_{g,e}^{-1}\mathbf{1}_{t_i}^{\mathrm{T}}$ in (5.6) consist of the column sums and the row sums of $(S_i)_{g,e}^{-1}$ respectively. Further, each $\mathbf{1}_{t_i}(S_i)_{g,e}^{-1}\mathbf{1}_{t_i}^{\mathrm{T}}$ in (5.7) is the sum of the elements of $(S_i)_{g,e}^{-1}$.

Case 3. $\infty \in S$. Assume that S is GCED-closed such that $\infty \in S$ and denote $T = S\backslash\{\infty\}$. Note that the case $\det(T)_{f,e} = 0$ is trivial, since then $\det(S)_{f,e} = 0$, see the details in Theorem 3.3 of [13]. For more general information on partitioned matrices, used in the proof of Theorem 5.3, see [9,22].

Theorem 5.3. *Let S be given as above and let f be a complex-valued function on $\mathcal{E}_S$. Assume that* $\det(T)_{f,e} \neq 0$. *Then*

$$\det(S)_{f,e} = c \cdot \det(T)_{f,e}, \tag{5.10}$$

where

$$c = f(\infty) - \mathbf{f}(T)_{f,e}^{-1}\mathbf{f}^{\mathrm{T}} \tag{5.11}$$

and $\mathbf{f}$ *denotes the row vector* $\mathbf{f} = (f(x_1), f(x_2), \ldots, f(x_{n-1}))$. *Furthermore, if (and only if)* $c \neq 0$, *then we have*

$$(S)_{f,e}^{-1} = \begin{bmatrix} (T)_{f,e}^{-1}[I + \mathbf{f}^{\mathrm{T}}\mathbf{f}(T)_{f,e}^{-1}/c] & -(T)_{f,e}^{-1}\mathbf{f}^{\mathrm{T}}/c \\ -\mathbf{f}(T)_{f,e}^{-1}/c & 1/c \end{bmatrix}, \tag{5.12}$$

where I denotes the $(n-1) \times (n-1)$ *identity matrix. Finally,* $\det(T)_{f,e}$ *and* $(T)_{f,e}^{-1}$ *can be calculated by using Theorem* 5.1 *or Theorem* 5.2.

Proof. Using the notation above, $(S)_{f,e}$ can be partitioned as

$$\begin{aligned}(S)_{f,e} &= \begin{bmatrix} (T)_{f,e} & \mathbf{f}^{\mathrm{T}} \\ \mathbf{f} & f(\infty) \end{bmatrix} = \begin{bmatrix} I & \mathbf{0}_{n-1}^{\mathrm{T}} \\ \mathbf{f}(T)_{f,e}^{-1} & 1 \end{bmatrix} \\ &\times \begin{bmatrix} (T)_{f,e} & \mathbf{0}_{n-1}^{\mathrm{T}} \\ \mathbf{0}_{n-1} & f(\infty) - \mathbf{f}(T)_{f,e}^{-1}\mathbf{f}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & (T)_{f,e}^{-1}\mathbf{f}^{\mathrm{T}} \\ \mathbf{0}_{n-1} & 1 \end{bmatrix}. \end{aligned} \tag{5.13}$$

By calculating the determinant and the inverse of the product of the three matrices on the right-hand side of (5.13) we easily obtain (5.10)–(5.12). Since ∞ is the greatest element of $\mathcal{E}_S$, T is also GCED-closed and ∞ does not appear in $(T)_{f,e}$. Thus each $\det(T)_{f,e}$ and $(T)_{f,e}^{-1}$ can be calculated by using Theorems 5.1 or 5.2. □

6. On LCED matrices on the *e*-divisor lattice

In this section we obtain determinant and inverse formulae for LCEM matrices $[S]_{f,e}$, which are defined by $([S]_{f,e})_{ij} = f([x_i, x_j]_e)$. We proceed dually to §5 and therefore we do not present all details. We apply Proposition 2.3 in $\mathcal{E}_S$. Then join-closed set becomes LCEM-closed set. Note that upper-closed sets in $(\mathbb{Z}_+^\infty, |_e)$ are infinite. Thus we define the associated concept of EM-closed set (e-multiple-closed) technically differently. We say that S is EM-closed if it is LCEM-closed and $x_i \mid_e z \mid_e x_n \Rightarrow z \in S$ holds for all $x_i \in S$.

Case 1. $1, \infty \notin S$. We here use the same notations as in §5. Assume that S is LCEM-closed such that $1, \infty \notin S$. Then S must be a subset of an interval $[p_1 p_2 \cdots p_m, p_1^{u_1} p_2^{u_2} \cdots p_m^{u_m}]$.

Theorem 6.1. *Let S be given as above and let f be an arithmetical function. Then*

$$\det[S]_{f,e} = \Gamma_{S,1}\Gamma_{S,2}\cdots\Gamma_{S,n}, \tag{6.1}$$

where

$$\Gamma_{S,t} = \sum_{x_t \mid_e z \mid_e p_1^{u_1}\cdots p_m^{u_m};\ x_{t+1},\ldots,x_n \nmid_e z} \ \sum_{z \mid_e w \mid_e p_1^{u_1}\cdots p_m^{u_m}} f(w) \prod_{k=1}^{m} \mu(w^{(k)}/z^{(k)}) \tag{6.2}$$

for $t = 1, 2, \ldots, n$. *Furthermore, if (and only if)* $\Gamma_{S,1}, \Gamma_{S,2}, \ldots, \Gamma_{S,n} \neq 0$, *then we have*

$$([S]_{f,e}^{-1})_{ij} = \sum_{x_t \mid_e x_i,\ x_t \mid_e x_j} \frac{\mu_S(x_t, x_i)\mu_S(x_t, x_j)}{\Gamma_{S,t}}, \tag{6.3}$$

where

$$\mu_S(x_t, x_s) = \sum_{x_t \mid_e z \mid_e x_s;\ x_{t+1},\ldots,x_s \nmid_e z} \ \prod_{k=1}^{m} \mu(z^{(k)}/x_s^{(k)}).$$

If S is even EM-closed (with respect to $[p_1 p_2 \cdots p_m, p_1^{u_1} p_2^{u_2} \cdots p_m^{u_m}]$*), then* $\Delta_{S,t}$ *simplifies to* $\Delta_{S,t} = \sum_{x_t \mid_e w \mid_e p_1^{u_1}\cdots p_m^{u_m}} f(w) \prod_{k=1}^{m} \mu(w^{(k)}/x_t^{(k)})$ *and* $\mu_S(x_t, x_s)$ *simplifies to* $\mu_{\mathcal{E}_S}(x_t, x_s) = \prod_{k=1}^{m} \mu(x_s^{(k)}/x_t^{(k)})$.

Case 2. $1 \notin S$ and $\infty \in S$. Assume that S is LCEM-closed such that $1 \notin S$ and $\infty \in S$. Then dually to §5, S can be written in the form $S = S_1 \cup S_2 \cup \cdots \cup S_m$, where $S_1 = \{x_{11}, \ldots, x_{1t_1}\}, \ldots, S_{m-1} = \{x_{(m-1)1}, \ldots, x_{(m-1)t_{(m-1)}}\}$, $S_m = \{x_{m1}\} = \{\infty\}$, for each $i = 1, \ldots, m-1$ the elements of S_i have the same prime divisors and for all $1 \leq i < j \leq m-1$ the set of the distinct prime divisors of the elements of S_i is not the same as that of S_j.

Theorem 6.2. *Let S be given as above. Let f be a complex-valued function on $\mathcal{E}_S$ and let $h(x) \equiv f(x) - f(\infty)$. Then*

$$\det[S]_{f,e} = \det[S_1]_{h,e} \det[S_2]_{h,e} \cdots \det[S_{m-1}]_{h,e} f(\infty). \tag{6.4}$$

Furthermore, if (and only if) $\det[S_1]_{h,e}, \det[S_2]_{h,e}, \ldots, \det[S_{m-1}]_{h,e}, f(\infty) \neq 0$, *then we have*

$$[S]_{f,e}^{-1} = \begin{bmatrix} [S_1]_{h,e}^{-1} & \cdots & O_{t_1 \times t_{m-1}} & -[S_1]_{h,e}^{-1}\mathbf{1}_{t_1}^{\mathrm{T}} \\ \vdots & \ddots & \vdots & \vdots \\ O_{t_{m-1} \times t_1} & \cdots & [S_{m-1}]_{h,e}^{-1} & -[S_1]_{h,e}^{-1}\mathbf{1}_{t_{m-1}}^{\mathrm{T}} \\ -\mathbf{1}_{t_1}[S_1]_{h,e}^{-1} & \cdots & -\mathbf{1}_{t_{m-1}}[S_{m-1}]_{h,e}^{-1} & s_{nn} \end{bmatrix}, \tag{6.5}$$

where

$$s_{nn} = \mathbf{1}_{t_1}[S_1]_{h,e}^{-1}\mathbf{1}_{t_1}^{\mathrm{T}} + \cdots + \mathbf{1}_{t_{m-1}}[S_{m-1}]_{h,e}^{-1}\mathbf{1}_{t_{m-1}}^{\mathrm{T}} + 1/f(\infty). \tag{6.6}$$

Finally, each of $\det[S_i]_{h,e}$ *and* $[S_i]_{h,e}^{-1}$ *can be calculated by using Theorem* 6.1.

Note that the matrices $\mathbf{1}_{t_i}[S_i]_{h,e}^{-1}$ and $[S_i]_{h,e}^{-1}\mathbf{1}_{t_i}^{\mathrm{T}}$ consist of the column sums and the row sums of $[S_i]_{h,e}^{-1}$ respectively. Further, each $\mathbf{1}_{t_i}[S_i]_{h,e}^{-1}\mathbf{1}_{t_i}^{\mathrm{T}}$ is the sum of the elements of $[S_i]_{h,e}^{-1}$.

Case 3. $1 \in S$. Assume that S is LCEM-closed such that $1 \in S$ and denote $T = S\backslash\{1\}$.

Theorem 6.3. *Let S be given as above and let f be a complex-valued function on $\mathcal{E}_S$. Let* $\det[T]_{f,e} \neq 0$. *Then*

$$\det[S]_{f,e} = d \cdot \det[T]_{f,e}, \tag{6.7}$$

where

$$d = f(1) - \mathbf{g}[T]_{f,e}^{-1}\mathbf{g}^{\mathrm{T}} \tag{6.8}$$

and $\mathbf{g}$ *denotes the row vector* $\mathbf{g} = (f(x_2), f(x_3), \ldots, f(x_n))$. *Furthermore, if (and only if)* $d \neq 0$, *then we have*

$$[S]_{f,e}^{-1} = \begin{bmatrix} 1/d & -\mathbf{g}[T]_{f,e}^{-1}/d \\ -[T]_{f,e}^{-1}\mathbf{g}^{\mathrm{T}}/d & [T]_{f,e}^{-1}\left(I + \mathbf{g}^{\mathrm{T}}\mathbf{g}[T]_{f,e}^{-1}/d\right) \end{bmatrix}. \tag{6.9}$$

Finally, $\det[T]_{f,e}$ *and* $[T]_{f,e}^{-1}$ *can be calculated by using Theorem* 6.1 *or Theorem* 6.2.

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