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Submanifolds weakly associated with graphs

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Abstract. We establish an interesting link between differential geometry and graph theory by defining submanifolds weakly associated with graphs. We prove that, in a local sense, every submanifold satisfies such an association, and other general results. Finally, we study submanifolds associated with graphs either in low dimensions or belonging to some special families.

Keywords. Almost Hermitian manifold; slant submanifold; weak association; graph theory.

1. Introduction

When we study a submanifold isometrically immersed in an almost Hermitian manifold $(\tilde{M}, J, g)$, it is important to take into account its behaviour with respect to the surrounding almost complex structure J. Many submanifolds presenting a homogeneous behaviour in this sense have been defined: complex submanifolds, totally real submanifolds, CR-submanifolds, and, more recently, slant, semi-slant, bi-slant or quasi-slant submanifolds. Most of them have something in common: they can be associated with graphs representing the above-mentioned algebraic behaviour.

This fact has been pointed out in the previous paper [3], in which the first two authors defined submanifolds associated with graphs, by using the graphic representation procedure introduced in [2] in order to visualize the behaviour of a submanifold with respect to J. Actually, they constructed a graph describing such behaviour at a point of the submanifold, and then they said that it is associated with the submanifold if it can be differentiably extended to any other point, in a certain way.

The idea of that association comes from the study of slant surfaces, and we can use them to show our method. Given such a surface, with slant angle θ, immersed in a 4-dimensional almost Hermitian manifold, it can be studied through a special local orthonormal frame $\{X_1, X_2, X_3, X_4\}$ such that X_1, X_2 are tangent to M, X_3, X_4 are normal to M and they satisfy the following equalities:

$$g(JX_1, X_2) = -g(JX_3, X_4) = \cos\theta, \quad g(JX_1, X_3) = g(JX_2, X_4) = \sin\theta,$$
$$g(JX_1, X_4) = g(JX_2, X_3) = 0, \tag{1.1}$$

which completely determine the behaviour of the almost complex structure J on M (see [4]).

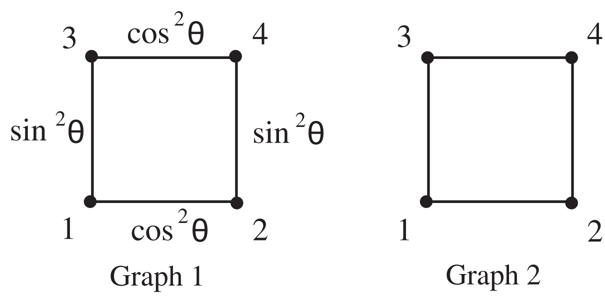

Figure 1. Graph associated with a slant surface.

Now, we can define a graph by following these steps:

1. We consider a vertex for every field of the frame, labelled with its corresponding natural index.
2. We say that the $\{i, j\}$ edge exists if and only if $g(JX_i, X_j) \neq 0$.
3. We assign on every edge the weight given by $g^2(JX_i, X_j)$.

Hence, we obtain the graph shown as Graph 1 in figure 1. If we do not follow Step 3, we just obtain Graph 2. The difference between these two graphs will be an important fact for us: the first one is a weighted graph and the second one is not (see the preliminaries section for a better explanation of these notions).

Notice that we obtain an additional visual information by putting the vertices corresponding to tangent fields at an imaginary bottom line and those which correspond to normal fields at a top line.

As we have already said, the definition of the association between submanifolds and graphs was given in [3] and we will recall it later. In that paper, some general properties and characterizations were obtained and the special case of submanifolds associated with graphs in dimension 4 was completely studied and classified.

Now, in this paper, we complete that notion by introducing the idea of a 'weak association', which basically means not to take into account the weights of the edges. We prove the important Theorem 3.1 establishing that every submanifold of an almost Hermitian manifold admits an open submanifold which is weakly associated with a graph; that is, our definition is not strange at all.

The main goal of this paper is therefore to study the graphs which can be 'weakly associated' with a submanifold. To do so, in §3, we first improve some results of [3] and we introduce some new general ones concerning how such graphs look like. By using them, we analyze these graphs in dimensions 4 and 6 in §4 and we completely determine all of them. In fact, concerning the 4-dimensional case, we extend the study done in [3] by means of the improved results presented in the previous section for the weak association. In dimension 6, we prove that only 12 of the 156 graphs with 6 vertices can be weakly associated with a submanifold (see Theorem 4.3). Moreover, we construct examples to show that there exist submanifolds weakly associated with some of those 12 graphs.

Finally, in this paper we also study submanifolds associated with some special families of graphs: forests (in §3), cycles and generalizations, and cubic graphs (in §5). For instance, we characterize the submanifolds associated with disjoint unions of cycles (see Theorems 5.5 and 5.7) and we completely determine the cubic graphs weakly associated with a submanifold in dimension 8 (see Theorem 5.9), with the corresponding examples.

2. Preliminaries

With respect to differential geometry, here we just recall the definitions of the submanifolds we will be talking about through this paper. For details and background on complex manifolds, we refer to the standard reference [10].

A submanifold M of an almost Hermitian manifold $(\tilde{M}, g, J)$ is said to be *slant* [4] if for each nonzero vector X tangent to M at p, the angle $\theta(X)$, $0 \leq \theta(X) \leq \pi/2$, between JX and T_pM is a constant, called the *slant angle* of the submanifold. In particular, complex and totally real submanifolds appear as slant submanifolds with slant angle 0 and $\pi/2$, respectively. A slant submanifold is called *proper slant* if it is neither complex nor totally real. Moreover, if the angle depends on the point, the submanifold is said to be *quasi-slant* [5].

Similarly, a differentiable distribution $\mathcal{D}$ on M is said to be a *slant distribution* if for any nonzero vector $X \in \mathcal{D}_p$, the angle between JX and the vector space $\mathcal{D}_p$ is constant, that is, it is independent of the choice of $p \in M$ and of $X \in \mathcal{D}_p$. Then, a submanifold M is said to be a *bi-slant* submanifold [2] if there exist on M two differentiable orthogonal slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ (with angles θ_1 and θ_2, respectively) such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$. It is shown in [2] that CR submanifolds [1] and semi-slant submanifolds [7] appear as particular cases of bi-slant submanifolds with $\theta_1 = 0, \theta_2 = \pi/2$ and $\theta_1 = 0, \theta_2 \neq 0$, respectively.

On the other hand, a *graph* G is a pair (V, A), where V is a finite nonempty set of vertices and A is a prescribed set of unordered pairs of distinct vertices of V, called edges. Given a pair $\{i, j\}$ in A, i and j are said to be *adjacent* vertices and $\{i, j\}$ is said to be *incident* with both i and j. The *degree* of a vertex is the number of edges incident with it. The graph consisting of just two vertices and one edge between them is called K_2. The cycle C_n $(n \geq 3)$ is the graph determined by an alternating sequence of distinct vertices and edges beginning and ending with the same point, in which each edge is incident with the two vertices immediately preceding and following it.

Throughout this paper, we are labelling graphs by distinguishing their vertices from one another by consecutive natural numbers. Therefore, we identify the vertex set of a graph with n vertices with the set $\{1, \ldots, n\}$. We will deal with weighted graphs too, i.e. graphs such that every edge has an assigned weight (a real number).

In graph theory, an isomorphism between two graphs is a one-to-one correspondence between their vertex sets which preserves adjacency. Given that we are considering labelled (and sometimes weighted) graphs, we also impose from now on that isomorphisms preserve labels (and sometimes weights). Therefore, for our purpose, an isomorphism (resp. weak isomorphism) between two such graphs with n vertices will just be the identity map from $\{1, \ldots, n\}$ into itself, preserving adjacency and edge weights (resp. adjacency). As usual, we say that two graphs are isomorphic (resp. weakly isomorphic) if there exists an isomorphism (resp. a weak isomorphism) between them. For more background on graph theory, we refer to [6].

3. Submanifolds (weakly) associated with graphs

Let M^m be a Riemannian manifold isometrically immersed in an almost Hermitian manifold $(\tilde{M}^n, J, g)$. Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a local orthonormal frame defined on a neighbourhood U of a point $p \in M$. Then, for any $q \in U$, we define the weighted graph

$G_{\mathcal{B},q}$ given by the set of vertices $\{1, \dots, n\}$ such that the edge $\{i, j\}$ exists if and only if $g_q(J_q X_{iq}, X_{jq}) \neq 0$, with weight $g_q^2(J_q X_{iq}, X_{jq})$. Note that this is just a generalization of the initial construction procedure.

Now we can define the association between submanifolds and graphs. Let G be a weighted graph with vertices $\{1, \dots, n\}$. Then, we say that M is *associated* (resp. *weakly associated*) with G if for any $p \in M$ there exists a neighbourhood $U(p)$ and a local orthonormal frame $\mathcal{B} = \{X_1, \dots, X_n\}$ on U satisfying the following conditions:

(a) $\{X_1, \dots, X_m\}$ are tangent to M and $\{X_{m+1}, \dots, X_n\}$ are normal to M.
(b) For any $q \in U$, the graph $G_{\mathcal{B},q}$ is isomorphic (resp. weakly isomorphic) to G.

Obviously, every submanifold associated with a graph is also weakly associated with it, since graph isomorphisms are in particular weak isomorphisms. On the other hand, it is not necessary for G to be a weighted graph to define the weak association.

Note that the above definition depends on the chosen orthonormal frame. Nevertheless, this situation is not a big obstacle because a natural equivalence relationship on the class of graphs associated with submanifolds was introduced in [3] to overcome it. In fact, two weighted labelled graphs G and G' were said to be equivalent if for any submanifold M associated with G, M is associated with G' and for any submanifold M' associated with G', M' is associated with G. Therefore, submanifolds associated with graphs can be classified through this relationship (for example, the case of dimension 4 was completely done in [3]), but this is not the objective of this paper.

First of all, let us point out how the weak association of submanifolds with graphs is not so strange at all. Indeed, the following result shows that it is a natural local fact for any submanifold.

Theorem 3.1. *Given any submanifold M of an almost Hermitian manifold, there exists an open submanifold of M which is weakly associated with a graph.*

Proof. Let M^m be a submanifold of an almost Hermitian manifold $(\tilde{M}^n, J, g)$, and $\mathcal{B} = \{X_1, \dots, X_n\}$ be a local orthonormal frame defined on an open subset U, such that $\{X_1, \dots, X_m\}$ are tangent to M and $\{X_{m+1}, \dots, X_n\}$ are normal to M. Put $f_{ij} = g(JX_i, X_j)$, $i < j$. It is clear that every f_{ij} is a diferentiable function on U.

Let us now put $\{(i, j) \,/\, i, j = 1, \dots, n,\ i < j\} = \{(i_1, j_1), \dots, (i_r, j_r)\}$. Then, we can carry through the following construction procedure. Put $U_0 = U$. For each $k = 1, \dots, r$, if $f_{i_k j_k} = 0$ on U_{k-1}, then put $U_k = U_{k-1}$. If not, by using continuity properties, we know that there exists a non-empty open subset $U_k \subseteq U_{k-1}$ such that $f_{i_k j_k}(q) \neq 0$, for any $q \in U_k$. At the end, we obtain an open subset U_r satisfying that, if there exists a point $q \in U_r$ with $f_{ij}(q) = 0$, then $f_{ij} = 0$ on U_r. Therefore, it is clear that, if we construct the graphs $G_{\mathcal{B},p}$ for any $p \in U_r$, all of them are weakly isomorphic to each other, and consequently the open submanifold U_r is weakly associated with such graphs. □

Nevertheless, we are interested in studying submanifolds which present such an association in a global (and differentiable) way. We can first give some examples:

Example 3.2. It was proved in [3] that a θ-slant submanifold is associated with the graph shown in figure 2. Similarly, it can be seen that every quasi-slant submanifold M such that $\theta_p \in (0, \pi/2)$, for any $p \in M$, is weakly associated with the same graph.

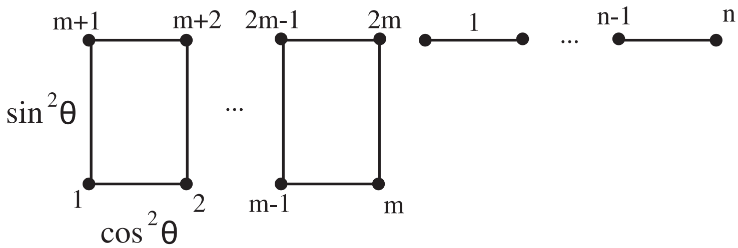

Figure 2. Graphs associated with slant submanifolds.

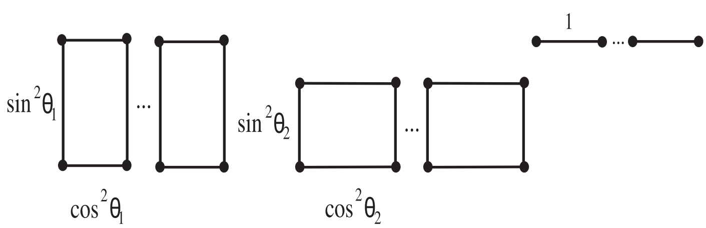

Figure 3. Graphs associated with bi-slant submanifolds.

Example 3.3. Similarly, if we have a bi-slant submanifold with slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfying $g(JX, Y) = 0$ for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, then it is associated with the graph shown in figure 3.

In [3], the first two authors completely determined and classified the submanifolds associated with graphs in dimension 4 by examining directly all the graphs with 4 vertices. To do so, it was very useful to know how a graph associated with a submanifold looked like. At this point, we want to study, for later use, what is the 'shape' of a graph weakly associated with a submanifold M of an almost-Hermitian manifold $(\tilde{M}, J, g)$. From now on, let G be such a graph. Let $p \in M$ and let $\{X_1, \ldots, X_n\}$ be the local orthonormal frame which determines the association between M and G in a neighborhood $U(p)$ of p. If A denotes the matrix

$$A = (g(JX_r, X_s)), \; 1 \leq r, s \leq n,$$

then, since $J^2 = -Id$ and g is compatible with J, it is easy to show that

$$A^2 = (a_{rs}) = -Id. \tag{3.1}$$

First, by using (3.1), we can improve some results proved in [3]. Indeed, the following lemma generalizes Lemma 3.3 of that paper.

Lemma 3.4. *Let i be a vertex of G. Then*

$$\sum_{j=1}^{n} g(JX_i, X_j)^2 = 1.$$

Proof. It follows directly from the fact of being $a_{ii} = -1$. □

Furthermore, we can also improve some other results from [3]. For example, the following ones generalize Lemma 3.4 and Proposition 3.5 of [3], respectively, with the same proofs.

Lemma 3.5. *G has no isolated vertices.*

PROPOSITION 3.6

G has no isolated triangles.

On the other hand, the following proposition improves Proposition 3.6 of [3].

PROPOSITION 3.7

Let i be a vertex in G with degree 1. *Then, the connected component containing i in G is just a K_2.*

Proof. Let j be the adjacent vertex to i. From Lemma 3.4, we have $g(JX_i, X_j)^2 = 1$. From the compatibility between g and J, it follows that $g(JX_j, X_i)^2 = 1$, which implies, by using Lemma 3.4 again, that i is the only vertex adjacent to j and so the proof is complete. □

From the above proposition, we can improve the characterization of CR-submanifolds by means of trees (connected graphs without cycles) and forests (disjoint unions of trees, see [6]) given in [3], by extending it to weak association.

Theorem 3.8. *A submanifold is weakly associated with a forest if and only if it is a CR-submanifold. In this case, every tree is a K_2.*

Finally, Theorem 3.8 of [3] can also be generalized by using the notion of weak association.

Theorem 3.9. *Let M^2 be a surface isometrically immersed in an almost Hermitian manifold. Then, there exists a graph G such that M is weakly associated with G if and only if M is a totally real surface, a complex surface or a quasi-slant surface such that $\theta_p \in (0, \pi/2)$ for any $p \in M$.*

Proof. Suppose that a surface M is weakly associated with a graph G. Let $1, 2$ be the tangent vertices of G. If these vertices are not adjacent, then it is clear that M is a totally real surface. Now, suppose that 1 and 2 are adjacent. If there are no other vertices adjacent with neither 1 nor 2, then M is a complex surface. If not, we have $g(JX_i, X_j) \in (0, 1)$ at any point, $i, j = 1, 2$, which implies that M is a quasi-slant surface satisfying the above condition.

The converse is a particular case of Example 3.2. □

As we have pointed out in the Introduction, one of the aims of this paper is to follow the research line introduced in [3] by studying the low-dimensional cases for a weak association. Nevertheless, we now find a new difficulty to use the same method of that paper: the graphs with 4 vertices are quickly examined, one by one, given that there are only 11 non-isomorphic (in the sense of graph theory) of them. But since there are 156 different graphs with 6 vertices (see [8]), we now need to look for new general results in order to do a similar exam with the 6-dimensional case in §4.

Next, we are going to prove two new lemmas concerning how the 'shape' of a graph weakly associated with a submanifold is restricted.

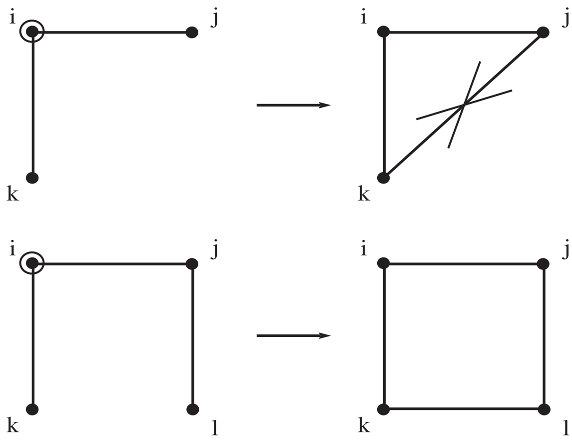

Figure 4. Graphic representation of Lemma 3.10.

Lemma 3.10. *Let i be a vertex of G with degree* 2 *and let j, k denote its adjacent vertices. Then j and k cannot be adjacent vertices. Moreover, if there is another vertex l, different from i and k, which is adjacent to j, then l is also adjacent to k (see figure* 4).

Proof. First, from (3.1), since $i \neq j$, we get

$$a_{ij} = \sum_{h=1}^{n} g(JX_i, X_h)g(JX_h, X_j) = 0. \tag{3.2}$$

Now, if $h \neq j, k$, then $g(JX_i, X_h) = 0$ and (3.2) reduces to

$$g(JX_i, X_k)g(JX_k, X_j) = 0.$$

Thus, since $g(JX_i, X_k) \neq 0$, we have that $g(JX_k, X_j) = 0$, that is, the vertices j and k are not adjacent.

On the other hand, since $i \neq l$, from (3.1) we obtain

$$a_{il} = \sum_{h=1}^{n} g(JX_i, X_h)g(JX_h, X_l) = 0. \tag{3.3}$$

By using $g(JX_i, X_h) = 0$ ($h \neq j, k$) again, from (3.3):

$$g(JX_i, X_j)g(JX_j, X_l) + g(JX_i, X_k)g(JX_k, X_l) = 0.$$

Therefore, if k and l are not adjacent vertices, then $g(JX_k, X_l) = 0$ and so, $g(JX_i, X_j)g(JX_j, X_l) = 0$, which is a contradiction since $g(JX_i, X_j) \neq 0$ and $g(JX_j, X_l) \neq 0$. □

Note that the above lemma generalizes Proposition 3.6, that is, if there is a triangle in a graph weakly associated with a submanifold, then all its vertices should have degree greater than or equal to 3.

Lemma 3.11. *Let i be a vertex of G with degree* 3 *and let j, k, l denote its adjacent vertices. The following properties are satisfied*:

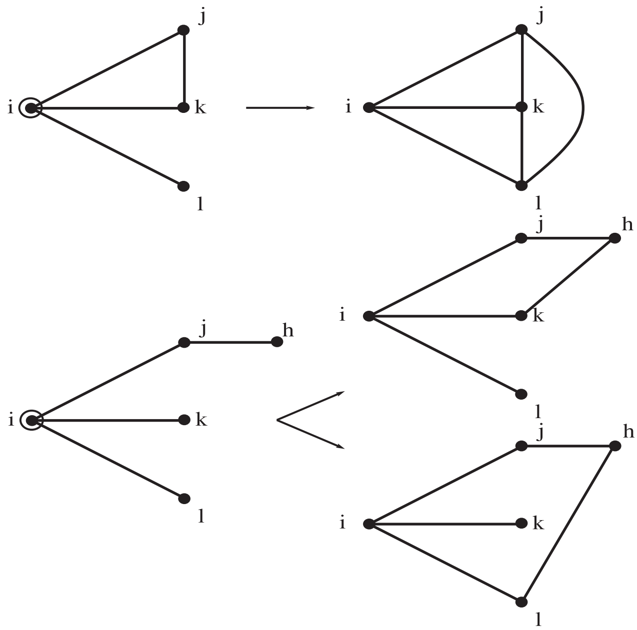

Figure 5. Graphic representation of Lemma 3.11.

(i) *If j and k are adjacent, then l is adjacent to both j and k.*
(ii) *If there is another vertex h, different from i, k, l which is adjacent to j, then, h is also adjacent to either k or l (see figure 5).*

Proof. First, from (3.1), since $i \neq j$, we have

$$a_{ij} = \sum_{h=1}^{n} g(JX_i, X_h)g(JX_h, X_j) = 0. \tag{3.4}$$

But we know that $g(JX_i, X_h) = 0$ if $h \neq j, k, l$ and from (3.4),

$$g(JX_i, X_k)g(JX_k, X_j) + g(JX_i, X_l)g(JX_l, X_j) = 0.$$

Consequently, we deduce that the vertices j and l should be adjacent because, if this is not the case, then $g(JX_l, X_j) = 0$ and so, $g(JX_i, X_k)g(JX_k, X_j) = 0$, which is a contradiction. Similarly, we obtain that the vertices k and l are also adjacent by using, from (3.1), that

$$a_{ik} = \sum_{h=1}^{n} g(JX_i, X_h)g(JX_h, X_k) = 0.$$

On the other hand, to prove statement (ii), from (3.1) again and since $i \neq h$, we get

$$a_{ih} = \sum_{s=1}^{n} g(JX_i, X_s)g(JX_s, X_h) = 0. \tag{3.5}$$

Next, since i has degree 3 and j, k, l are its adjacent vertices, from (3.5) we see that

$$g(JX_i, X_j)g(JX_j, X_h) + g(JX_i, X_k)g(JX_k, X_h) + g(JX_i, X_l)g(JX_l, X_h) = 0.$$

If we suppose that $g(JX_k, X_h) = 0$ and $g(JX_l, X_h) = 0$, then

$$g(JX_i, X_j)g(JX_j, X_h) = 0,$$

which is a contradiction. □

We observe that statement (i) of the above lemma implies that if there is a triangle in a graph associated with a submanifold and one of its vertices has degree 3, then the triangle lies in a tetrahedron.

Now, we can generalize Lemma 3.10 and Lemma 3.11 by considering a vertex with any degree.

PROPOSITION 3.12

Let i denote a vertex of G with degree $t \geq 3$ and $j_1, \ldots, j_t$ its adjacent vertices. If two of these vertices, j_r and j_s, $1 \leq r, s \leq t$, are adjacent, then j_r (resp. j_s) is also adjacent, at least, to $j_{r'}$ (resp. $j_{s'}$), with $1 \leq r', s' \leq t$, $r', s' \neq r, s$ (see figure 6).

Proof. Since $i \neq j_r$, from (3.1) we get

$$a_{ij_r} = \sum_{h=1}^{n} g(JX_i, X_h)g(JX_h, X_{j_r}) = 0. \tag{3.6}$$

If we suppose that, except for the vertex j_s, none of the vertices j_k, $1 \leq k \leq t$, is adjacent to j_r, then (3.6) reduces to

$$g(JX_i, X_{j_s})g(JX_{j_s}, X_{j_r}) = 0,$$

which is a contradiction. Similarly, by using the element a_{ij_s}, we deduce the same result for the vertex j_s. □

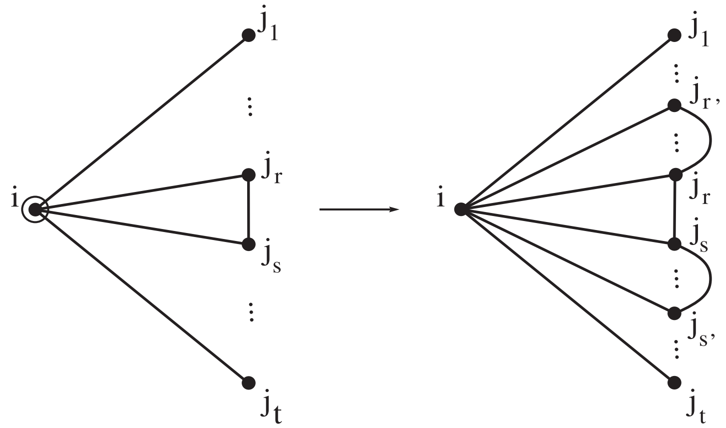

Figure 6. Graphic representation of Proposition 3.12.

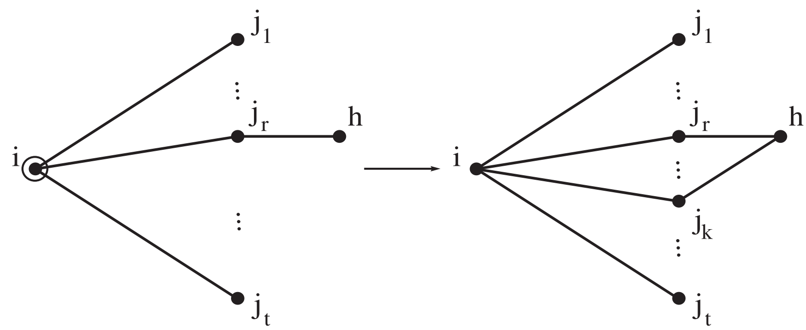

Figure 7. Graphic representation of Proposition 3.13.

Consequently, we have obtained that, in the conditions of the above proposition, the vertices j_r and j_s lie in, at least, two triangles with vertices i, j_r, j_s and either i, j_r, $j_{r'}$, $1 \leq r' \leq t$ and $r' \neq r, s$ or i, j_s, $j_{s'}$, $1 \leq s' \leq t$ and $s' \neq r, s$, respectively. If, moreover, $r' = s'$, then we see that the triangle with vertices i, j_r, j_s lies in the tetrahedron determined by the vertices i, j_r, j_s, $j_{r'}(= j_{s'})$.

PROPOSITION 3.13

Let i denote a vertex of G with degree $t \geq 2$ and $j_1, \dots, j_t$ its adjacent vertices. If there is another vertex h, different from them, which is adjacent to any of the vertices j_r, $1 \leq r \leq t$, then h is adjacent to, at least, another of the vertices j_k, $1 \leq k \leq t$ and $k \neq r$ (see figure 7).

Proof. Since $i \neq h$, from (3.1), we have

$$a_{ih} = \sum_{s=1}^{n} g(JX_i, X_s)g(JX_s, X_h) = 0. \tag{3.7}$$

Suppose that none of the adjacent vertices to i, except for the vertex j_r, is adjacent to h, that is, $g(JX_{j_l}, X_h) = 0$, $1 \leq l \leq t$, $l \neq r$. Then,

$$g(JX_i, X_{j_r})g(JX_{j_r}, X_h) \neq 0,$$

which is a contradiction with (3.7). □

4. Low-dimensional cases

Our goal in this section is to determine the graphs which can be weakly associated with a submanifold of an almost Hermitian manifold of dimension either 4 or 6.

In the first case, the work was basically done in [3]. Actually, in that paper the authors did not consider weak association, but, by using their arguments with the improved results presented in the above section, it can be easily seen that the only graphs with 4 vertices which can be weakly associated with a submanifold are those of figure 8.

Now, we can focus on the 6-dimensional case, that is, on graphs with 6 vertices. As we have already pointed out, there are 156 graphs. They can be seen in pages 9–11 of [8]. By using the general results obtained in the above section, a general glance at these graphs allows us to reject all of them, except possibly 15, for being associated with a submanifold (see figure 9).

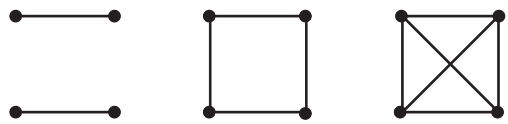

Figure 8. Graphs possibly associated with a submanifold in dimension 4.

However, we need to check if those 15 graphs are really weakly associated with a submanifold. Actually, the graphs $G_{13}-G_{15}$ are not. We will show this fact in two propositions, because the proofs for G_{13} and G_{15} are similar.

PROPOSITION 4.1

The graphs G_{13} and G_{15} of figure 9 can not be weakly associated with a submanifold.

Proof. Let $\tilde{M}$ be a 6-dimensional almost Hermitian manifold and suppose that there exists a submanifold M of $\tilde{M}$ and a local orthonormal frame of vector fields $\mathcal{B} = \{X_1, \ldots, X_6\}$ of $\tilde{M}$ such that M is weakly associated with G_{13} via $\mathcal{B}$, where we can assume that its vertices are labelled by $i_1, \ldots, i_6$ from top to bottom and from left to right. Then, since $g(JX_{i_1}, X_{i_6}) = 0$, from Lemma 3.4, we have

$$\sum_{k=2}^{5} g(JX_{i_1}, X_{i_k})^2 = 1 \tag{4.1}$$

and

$$\sum_{k=2}^{5} g(JX_{i_6}, X_{i_k})^2 = 1. \tag{4.2}$$

Moreover, since the vertices i_2, i_3, i_4, i_5 have degree 2, by using the same lemma we obtain

$$g(JX_{i_k}, X_{i_1})^2 = 1 - g(JX_{i_k}, X_{i_6})^2,\ k = 2, \ldots, 5. \tag{4.3}$$

But (4.1)–(4.3) are contradictory due to the compatibility of g and J.

Now, if the graph G_{15} is weakly associated with a submanifold of a 6-dimensional almost Hermitian manifold $(\tilde{M}, J, g)$ via a local orthonormal frame of vector fields $\mathcal{B} = \{X_1, \ldots, X_6\}$, where we can assume that its vertices are labelled by $i_1, \ldots, i_6$ from top to bottom and from left to right as above, then, by using Lemma 3.4 as always, we get

$$\begin{aligned} 1 = {} & g(JX_{i_1}, X_{i_2})^2 + g(JX_{i_1}, X_{i_3})^2 + g(JX_{i_1}, X_{i_4})^2 \\ & + g(JX_{i_1}, X_{i_5})^2 + g(JX_{i_1}, X_{i_6})^2; \end{aligned} \tag{4.4}$$

$$1 = g(JX_{i_2}, X_{i_1})^2 + g(JX_{i_2}, X_{i_5})^2 + g(JX_{i_2}, X_{i_6})^2, \tag{4.5}$$

$$1 = g(JX_{i_3}, X_{i_1})^2 + g(JX_{i_3}, X_{i_5})^2 + g(JX_{i_3}, X_{i_6})^2, \tag{4.6}$$

$$1 = g(JX_{i_4}, X_{i_1})^2 + g(JX_{i_4}, X_{i_5})^2 + g(JX_{i_4}, X_{i_6})^2, \tag{4.7}$$

$$\begin{aligned} 1 = {} & g(JX_{i_5}, X_{i_1})^2 + g(JX_{i_5}, X_{i_2})^2 + g(JX_{i_5}, X_{i_3})^2 \\ & + g(JX_{i_5}, X_{i_4})^2 + g(JX_{i_5}, X_{i_6})^2, \end{aligned} \tag{4.8}$$

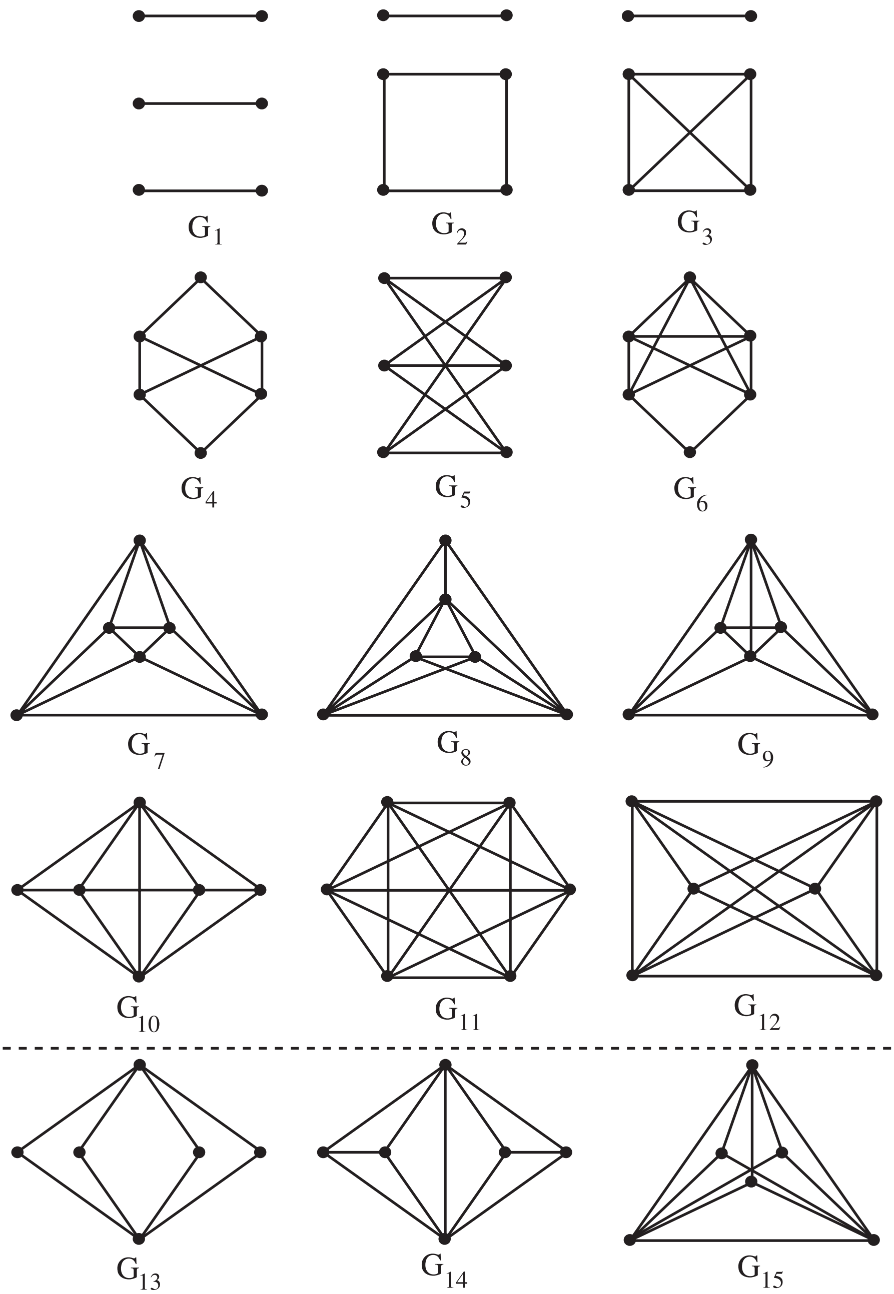

Figure 9. Graphs possibly associated with a submanifold in dimension 6.

$$
\begin{aligned}
1 = {} & g(JX_{i_6}, X_{i_1})^2 + g(JX_{i_6}, X_{i_2})^2 + g(JX_{i_6}, X_{i_3})^2 \\
& + g(JX_{i_6}, X_{i_4})^2 + g(JX_{i_6}, X_{i_5})^2.
\end{aligned}
\tag{4.9}
$$

Now, if we sum (4.5)–(4.7) and take into account (4.4), (4.8) and (4.9), we obtain

$$g(JX_{i_1}, X_{i_5})^2 + g(JX_{i_1}, X_{i_6})^2 + g(JX_{i_5}, X_{i_6})^2 = 0,$$

which is a contradiction. □

PROPOSITION 4.2

The graph G_{14} *of figure* 9 *can not be weakly associated with a submanifold.*

Proof. Let $\tilde{M}$ be a 6-dimensional almost Hermitian manifold and suppose that there exists a submanifold M of $\tilde{M}$ and a local orthonormal frame of vector fields $\mathcal{B} = \{X_1, \ldots, X_6\}$ of $\tilde{M}$ such that M is weakly associated with G_{14} via $\mathcal{B}$, where we can assume that its vertices are labelled by $i_1, \ldots, i_6$ from top to bottom and from left to right. Then, from (3.1) we have

$$\begin{aligned} 0 = a_{i_2 i_4} &= \sum_{k=1}^{6} g(JX_{i_2}, X_{i_k}) g(JX_{i_k}, X_{i_4}) \\ &= g(JX_{i_2}, X_{i_1}) g(JX_{i_1}, X_{i_4}) + g(JX_{i_2}, X_{i_6}) g(JX_{i_6}, X_{i_4}) \end{aligned}$$

and

$$\begin{aligned} 0 = a_{i_3 i_4} &= \sum_{k=1}^{6} g(JX_{i_3}, X_{i_k}) g(JX_{i_k}, X_{i_4}) \\ &= g(JX_{i_3}, X_{i_1}) g(JX_{i_1}, X_{i_4}) + g(JX_{i_3}, X_{i_6}) g(JX_{i_6}, X_{i_4}), \end{aligned}$$

that is, at each point, $(g(JX_{i_1}, X_{i_4}), g(JX_{i_6}, X_{i_4}))$ is a non-null solution of the system:

$$\begin{cases} g(JX_{i_2}, X_{i_1})x_1 + g(JX_{i_2}, X_{i_6})x_2 = 0, \\ g(JX_{i_3}, X_{i_1})x_1 + g(JX_{i_3}, X_{i_6})x_2 = 0. \end{cases}$$

This implies that

$$g(JX_{i_2}, X_{i_1}) g(JX_{i_3}, X_{i_6}) - g(JX_{i_2}, X_{i_6}) g(JX_{i_3}, X_{i_1}) = 0. \tag{4.10}$$

On the other hand, from (3.1) again, we obtain that

$$0 = a_{i_2 i_3} = g(JX_{i_2}, X_{i_1}) g(JX_{i_1}, X_{i_3}) + g(JX_{i_2}, X_{i_6}) g(JX_{i_6}, X_{i_3})$$

and, since $g(JX_{i_j}, X_{i_k}) = -g(JX_{i_k}, X_{i_j})$, we observe, together with (4.10), that at each point, $(g(JX_{i_2}, X_{i_1}), g(JX_{i_2}, X_{i_6}))$ is a non-null solution of the system:

$$\begin{cases} g(JX_{i_3}, X_{i_6})x_1 - g(JX_{i_3}, X_{i_1})x_2 = 0, \\ g(JX_{i_3}, X_{i_1})x_1 + g(JX_{i_3}, X_{i_6})x_2 = 0. \end{cases}$$

Then, $g(JX_{i_3}, X_{i_6})^2 + g(JX_{i_3}, X_{i_1})^2 = 0$, which is a contradiction. □

Consequently, we have proved the following theorem:

Theorem 4.3. *If a submanifold of a 6-dimensional almost Hermitian manifold is weakly associated with a graph, then this graph must be one of the* $G_1 - G_{12}$ *of figure* 9.

Now, the natural question is: are really all of them weakly associated with a submanifold? For graphs $G_1 - G_{11}$, the answer is positive. In fact, we are going to present an example of submanifold associated (not only weakly) with each one of them.

We consider $\mathbf{R}^6$ with Cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and endowed with its standard almost Hermitian structure given by the tensor fields g and J defined by

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij},\ g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 0,\ g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \delta_{ij},$$

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

The structure of the examples we are going to present is the following: for each graph G_i, $i = 1, \ldots, 11$, labelled, as above, from top to bottom and from left to right, we shall give an orthonormal frame $\mathcal{B} = \{X_1, \ldots, X_6\}$ on $\mathbf{R}^6$ such that the corresponding graphs $G_{\mathcal{B},q}$ are isomorphic to G_i, and satisfying that all their brackets products vanish. Therefore, any distribution spanned by some of these vector fields is involutive and the corresponding integral submanifold is the desired one.

The orthonormal frames corresponding to graphs $G_1 - G_{11}$ appear in tables 1–3.

The question for graph G_{12} remains still open.

5. Some relevant families of graphs

Since the number of graphs grows very quickly with respect to the number of their vertices, at this moment the task of analyzing graphs weakly associated with submanifolds in dimensions greater than 6 seems to be unapproachable. Therefore, we think that it is more interesting to deal with some particular families of graphs.

As we have shown above, submanifolds weakly associated with graphs without cycles (forests) are completely determined. Thus, in this section we study submanifolds weakly associated with graphs containing cycles. As a first step, we begin by considering this question when the graphs are just cycles.

The first examples of such submanifolds are just particular cases of those from Example 3.2.

Example 5.1. Proper θ-slant surfaces immersed in a 4-dimensional almost Hermitian manifold are submanifolds associated with Graph 1 shown in figure 1. On the other hand, if M is a quasi-slant submanifold of $\tilde{M}^4$ such that $\theta_p \in (0, \pi/2)$ for any $p \in M$, then M is a submanifold weakly associated with Graph 2 shown in figure 1.

We can also construct some new examples.

Example 5.2. Let us consider $\mathbf{R}^4$ with its usual cartesian coordinates (x_1, x_2, y_1, y_2) and put $X_1 = \partial/\partial_{x_1}$, $X_2 = \sqrt{2}/2((\partial/\partial_{y_1}) + (\partial/\partial_{y_2}))$, $X_3 = \partial/\partial_{x_2}$ and $X_4 = \sqrt{2}/2((\partial/\partial_{y_1}) - (\partial/\partial_{y_2}))$. Then, it is clear that the distribution $\mathcal{D}$ spanned by X_1, X_2, X_3 is integrable and so, it determines a foliation of submanifolds associated with the cyclic graph $C_{4,5}$ shown in figure 10, with $a^2 = b^2 = 1/2$.

Table 1. Orthonormal frames corresponding to graphs $G_1 - G_5$.

Graph	Corresponding orthonormal frame
G_1	$X_1 = \frac{\partial}{\partial x_1}; X_2 = \frac{\partial}{\partial y_1}; X_3 = \frac{\partial}{\partial x_2};$ $X_4 = \frac{\partial}{\partial y_2}; X_5 = \frac{\partial}{\partial x_3}; X_6 = \frac{\partial}{\partial y_3}.$
G_2	$X_1 = \frac{\partial}{\partial x_1}; X_2 = \frac{\partial}{\partial y_1}; X_3 = \frac{\partial}{\partial x_3};$ $X_4 = \sin\alpha \frac{\partial}{\partial x_2} - \cos\alpha \frac{\partial}{\partial y_3}; X_5 = \cos\alpha \frac{\partial}{\partial x_2} + \sin\alpha \frac{\partial}{\partial y_3}; X_6 = \frac{\partial}{\partial y_2};$ $\alpha \in \left(0, \frac{\pi}{2}\right).$
G_3	$X_1 = \frac{\partial}{\partial x_1}; X_2 = \frac{\partial}{\partial y_1}; X_3 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3} - \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}\right);$ $X_4 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3} + \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}\right); X_5 = \frac{\partial}{\partial y_2}; X_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_3} - \frac{\partial}{\partial x_2}\right).$
G_4	$X_1 = \frac{\partial}{\partial x_1}; X_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right); X_3 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_1}\right);$ $X_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}\right); X_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_3} - \frac{\partial}{\partial x_2}\right); X_6 = \frac{\partial}{\partial x_3}.$
G_5	$X_1 = \frac{1}{\sqrt{2}}\left(\sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right); X_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}\right);$ $X_3 = \cos\alpha \frac{\partial}{\partial x_1} - \sin\alpha \frac{\partial}{\partial x_2}; X_4 = \frac{\partial}{\partial y_2};$ $X_5 = \frac{1}{\sqrt{2}}\left(\sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right); X_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_3} - \frac{\partial}{\partial y_1}\right);$ $\alpha \in \left(0, \frac{\pi}{2}\right).$

Example 5.3. Every regular curve in a 4-dimensional almost Hermitian manifold is a submanifold associated with the cyclic graph $C_{4,1}$ shown in figure 10, for any a, b such that $a^2 + b^2 = 1$. To prove this, it is enough to consider the unit tangent vector field X along the curve and a vector field Y such that $\{X, JX, Y, JY\}$ is a local orthonormal frame. Then, we just define the association frame by $X_1 = X$, $X_2 = aJX + bJY$, $X_3 = Y$ and $X_4 = bJX - aJY$.

Table 2. Orthonormal frames corresponding to graphs G_6-G_9.

Graph	Corresponding orthonormal frame
G_6	$X_1=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}\right)$; $X_2=\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3}-\frac{1}{2}\left(\frac{\partial}{\partial y_3}-\frac{\partial}{\partial x_2}\right)$; $X_3=\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3}+\frac{1}{2}\left(\frac{\partial}{\partial y_3}-\frac{\partial}{\partial x_2}\right)$; $X_4=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)$; $X_5=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_2}-\frac{\partial}{\partial y_1}\right)$; $X_6=\frac{\partial}{\partial x_1}$.
G_7	$X_1=\sin\alpha\frac{\partial}{\partial x_1}+\cos\alpha\frac{\partial}{\partial x_2}$; $X_2=\sin\gamma\frac{\partial}{\partial y_1}+\cos\gamma\frac{\partial}{\partial y_3}$; $X_3=\sin\beta\frac{\partial}{\partial x_3}+\cos\beta\frac{\partial}{\partial y_2}$; $X_4=\sin\alpha\frac{\partial}{\partial x_2}-\cos\alpha\frac{\partial}{\partial x_1}$; $X_5=\sin\beta\frac{\partial}{\partial y_2}-\cos\beta\frac{\partial}{\partial x_3}$; $X_6=\sin\gamma\frac{\partial}{\partial y_3}-\cos\gamma\frac{\partial}{\partial y_1}$; $\alpha,\beta,\gamma\in\left(0,\frac{\pi}{2}\right)$.
G_8	$X_1=\frac{\partial}{\partial y_3}$; $X_2=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_3}-\frac{\partial}{\partial y_2}\right)$; $X_3=\frac{1}{2}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)-\frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1}$; $X_4=\frac{1}{2}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)+\frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1}$; $X_5=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}+\frac{\partial}{\partial y_2}\right)$; $X_6=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_3}-\frac{\partial}{\partial y_2}\right)$.
G_9	$X_1=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}+\frac{\partial}{\partial y_2}\right)$; $X_2=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)$; $X_3=\frac{1}{2}\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial y_2}\right)+\frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1}$; $X_4=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_3}-\frac{\partial}{\partial y_2}\right)$; $X_5=\frac{1}{2}\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial y_2}\right)-\frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1}$; $X_6=\frac{\partial}{\partial y_3}$.

Similarly, we have the following.

Example 5.4. Every orientable hypersurface M of a 4-dimensional almost Hermitian manifold is a submanifold associated with the cyclic graph $C_{4,5}$ shown in figure 10, for any a, b such that $a^2+b^2=1$. In this case, we just choose a local orthonormal frame $\{JC, X, JX, C\}$ where C is the normal to M and X is tangent to M, and we define $X_1=aJX+bJC$, $X_2=X$, $X_3=bJX-aJC$ and $X_4=C$.

Table 3. Orthonormal frames corresponding to graphs G_{10} and G_{11}.

Graph	Corresponding orthonormal frame
G_{10}	$X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right); X_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}\right);$
	$X_3 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_2}\right);$
	$X_4 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_2}\right);$
	$X_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_3}\right); X_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_2}\right).$
G_{11}	$X_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_2}\right);$
	$X_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1} + \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right); X_3 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_3} - \frac{1}{2}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_2}\right);$
	$X_4 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) - \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_1}; X_5 = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_2}\right) + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_3};$
	$X_6 = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_2}\right).$

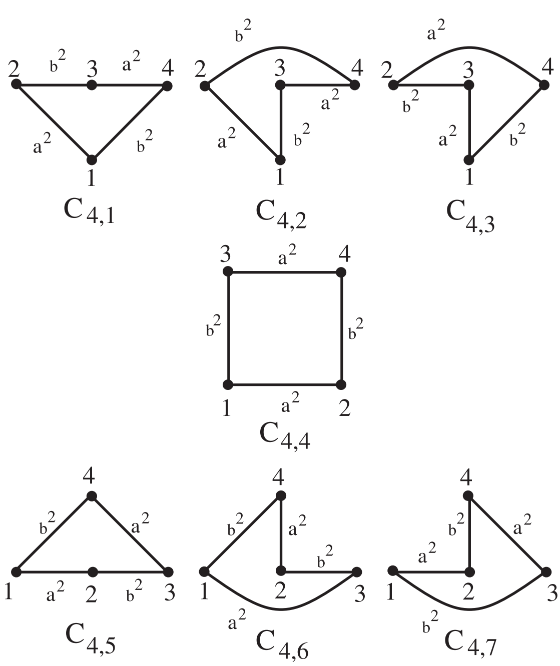

Figure 10. Cycles weakly associated with submanifolds.

Therefore, we see that there are ample examples of submanifolds associated with a cycle with 4 vertices (C_4). In fact, the following results show that this is the only possibility for the number of vertices of a cycle weakly associated with a submanifold:

Theorem 5.5. *Let $(\tilde{M}^n, J, g)$ be an almost Hermitian manifold. Suppose that there exist a point $p \in \tilde{M}$ and a local orthonormal frame of vector fields $\mathcal{B} = \{X_1, \ldots, X_n\}$ on a neighbourhood of p such that an isolated cycle C_r $(r \leq n)$ appears in the graph $G_{\mathcal{B},p}$. Then, $r = 4$.*

Proof. We denote by $i_1, \ldots, i_r$ the vertices of C_r, consecutively and we put $e_k = X_k(p)$. If $r > 4$, then from (3.1), we have

$$0 = a_{i_1 i_3} = \sum_{k=1}^{n} g(Je_{i_1}, e_{i_k}) g(Je_{i_k}, e_{i_3}) = g(Je_{i_1}, e_{i_2}) g(Je_{i_2}, e_{i_3}),$$

which is a contradiction. On the other hand, if $r = 3$, then from (3.1) again, we get

$$0 = a_{i_1 i_2} = \sum_{k=1}^{n} g(Je_{i_1}, e_{i_k}) g(Je_{i_k}, e_{i_2}) = g(Je_{i_1}, e_{i_3}) g(Je_{i_3}, e_{i_2}),$$

which is also a contradiction. □

We then obtain the following direct corollary.

COROLLARY 5.6

Let $(\tilde{M}^n, J, g)$ be an almost Hermitian manifold. If there exists a submanifold weakly associated with a cycle in $\tilde{M}^n$, then $n = 4$. In particular, this cycle is just C_4.

Hence, every cycle weakly associated with a submanifold must be one of those drawn in figure 10 (actually, all of them are isomorphic, in a general sense, to C_4). Notice that the first three cycles only differ on the labelling of vertices. The same fact is true for the last three ones. On the other hand, it is clear that $C_{4,1}$, $C_{4,2}$ and $C_{4,3}$ correspond to curves, $C_{4,4}$ is weakly associated with surfaces and $C_{4,5}$, $C_{4,6}$ and $C_{4,7}$ represent hypersurfaces.

If we want to study submanifolds associated with these graphs, we need to assign weights a^2, b^2 as shown in the figure, such that $a^2 + b^2 = 1$. This follows from Lemma 3.3 of [3], and we refer to that paper for a detailed analysis of submanifolds associated with graphs in dimension 4.

Now, we can study graphs consisting of finite disjoint unions of C_4's and K_2's. For example, the graph shown in figure 3 is such a graph. Actually, the following result shows that this kind of graphs characterize a natural generalization of bi-slant submanifolds appearing in Example 3.3.

Theorem 5.7. *A submanifold M of an almost Hermitian manifold $(\tilde{M}^n, J, g)$ is associated with a graph consisting of a finite disjoint union of some C_4's and K_2's if and only if its tangent bundle admits a direct orthogonal decomposition*

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r$$

such that $\mathcal{D}$ is a complex distribution, $\mathcal{D}^\perp$ is a totally real distribution and, for any $i = 1 \ldots r$, $\mathcal{D}_i$ is a θ_i-slant distribution satisfying $g(JX, Y) = 0$ for any $X \in \mathcal{D}_i$ and any $Y \in \mathcal{D}_j$, $i \neq j$.

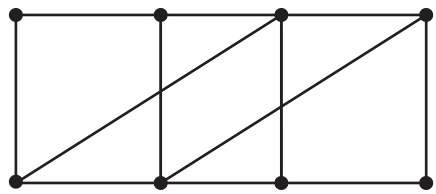

Figure 11. Non-disjoint union of C_4's referred in text.

Proof. This proof can be made directly by reasoning through association frames, just by taking into account that the vertices corresponding to tangent horizontal K_2's define the complex distribution, the tangent vertices appearing in vertical K_2's define the totally real distribution, and those corresponding to each group of C_4's with the same horizontal weight determine a distribution which can be proved to be slant. Moreover, the definition of these distribution does not depend on the chosen association frame. The converse follows by choosing suitable adapted frames. □

Notice that these submanifolds are a particular case of skew CR submanifolds (see [9]).

The next step in our study could be to deal with graphs consisting of non-disjoint union of cycles, i.e., union of cycles with some edges connecting them. An idea to determine the shape of such a graph is to choose a vertex and try to fix the necessary edges incident on it, working at every step with the smallest possible degrees. In this sense, we have obtained the following proposition.

PROPOSITION 5.8

Let G be a graph consisting of the non-disjoint union of two C_4's such that one of its vertices has degree 2, *and with the smallest degrees on the other vertices. If G is weakly associated with a submanifold*, *then it must be the graph of figure* 11.

Proof. Let i_1 be one vertex of G with degree 2, and denote by i_2, i_3 its adjacent vertices (which lie, of course, in the same C_4 as i_1). If we suppose that both i_2 and i_3 also have degree 2, then the remaining vertex of the cycle, say i_4, should be adjacent to at least one vertex of the other cycle, say i_5. But, if we apply Lemma 3.10 to vertices i_1, i_2, i_4, i_5 (with i_2 as the settled vertex with degree 2), it follows that i_5 must be adjacent to i_1, which is impossible. Therefore, either i_2 or i_3 must have degree at least 3. Actually, let us show that both of them have degree greater than or equal to 3:

From Lemma 3.10 we know that they can not be adjacent to each other. Hence, if i_2 has degree $d \geq 3$, then it is adjacent to i_1 (of course), to the remaining vertex of the cycle i_4, and to $d - 2$ vertices of the other cycle. But, if we now apply Lemma 3.10 to vertices i_1, i_2, i_3, i_5 (with i_1 as the settled vertex with degree 2), i_5 being any of those $d-2$ vertices adjacent to i_2, then we deduce that i_5 is also adjacent to i_3, and so the degree of this vertex is also greater than or equal to 3. A similar argument can be followed if we suppose that i_3 has degree $d \geq 3$ to prove the same property for i_2.

Let us go back and suppose that i_2 has just degree 3. As we have pointed out above, we already know that it is adjacent to i_1, the remaining vertex of the cycle i_4 and to one vertex of the other cycle, say i_5, which is adjacent, in its turn, to i_3. Let us denote by i_6, i_7 the vertices adjacent to i_5 in the second cycle. We can now apply Lemma 3.11(ii) to vertices i_1, i_2, i_4, i_5, i_6 (with i_2 as the settled vertex with degree 3) and we have that i_6 must be adjacent to i_4. Moreover, by applying the same result to i_1, i_2, i_4, i_5, i_7, we also have that

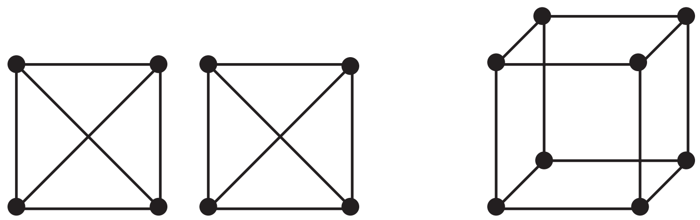

Figure 12. Cubic graphs associated with a submanifold in dimension 8.

i_7 is adjacent to i_4. At this point, we have already obtained a graph isomorphic (in the general sense) to that of figure 11. □

Actually, some different examples of submanifolds associated with the graph of figure 11 can be obtained from Example 4.2 of [2].

Finally, let us offer one result concerning another relevant family of graphs: cubic graphs. A graph is said to be a *cubic graph* if all its vertices have degree 3. The first two authors proved in [3] that the only cubic graph with 4 vertices (namely, the tetrahedron, called the complete graph K_4 in graph theory) is associated with some submanifolds. With respect to cubic graphs with 6 vertices, there are two of them (see [8]), and we have seen in §4 that only one can be associated with submanifolds (namely, graph G_5 of figure 9, better known as $K_{3,3}$ in graph theory). We have also given some examples of such an association. Now, we obtain the following theorem concerning cubic graphs with 8 vertices.

Theorem 5.9. *Let G be a cubic graph weakly associated with a submanifold of an almost Hermitian manifold of dimension* 8. *Then, G must be either the disjoint union of two tetrahedrons or a cube* (*see figure* 12).

Proof. Let i_1 denote any of the vertices of G whose degree must be 3 since G is a cubic graph. Then, let i_2, i_3, i_4 denote its adjacent vertices and let H be the subgraph of G induced by its non-adjacent vertices. There are two cases to take into account:

Case I. There are at least two adjacent vertices among i_2, i_3 *and* i_4. Thus, from Lemma 3.11(i), i_1, i_2, i_3 and i_4 induce a tetrahedron. Moreover, since the vertices of H are of degree 3 and not adjacent to $i_1 - i_4$, we get that H is another tetrahedron.

Case II. There are no adjacent vertices among i_2, i_3 *and* i_4. Then, since all vertices of H are of degree 3, we easily show that H has three edges.

In this situation, if any vertex of H is of degree 2 in H, it has to be adjacent to exactly one of $i_2 - i_4$ since it is of degree 3 in G, but this fact contradicts Lemma 3.11(ii). Consequently, there is one vertex in H, say i_5, which is adjacent to the other three vertices of H, all of them of degree 1 in H because H has only three edges, and so, adjacent exactly to two vertices of $i_2 - i_4$.

Let us denote the vertices adjacent to i_2 and i_5 by i_6 and i_7 and the vertex adjacent to i_5 but not to i_2 by i_8. If i_3 is adjacent to i_6 and i_7, then the vertex i_4 can not be adjacent to any of them and thus, it is adjacent only to i_1 and maybe to i_8, which is a contradiction because i_4 is of degree 3. Therefore, i_3 is adjacent to i_8 and one among i_6 and i_7, say i_6. Finally, i_4 should be adjacent to i_7 and i_8 since all vertices are of degree 3. Thus, G is isomorphic to a cube and this completes the proof. □

Actually, we do have some examples of submanifolds associated with graphs of figure 12. With respect to the graph $K_4 \sqcup K_4$ consisting of the disjoint union of two tetrahedrons, we have shown in Example 3.3 that a bi-slant submanifold M with slant angles $\theta_1, \theta_2 \in (0, \pi/2)$ and slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfying $g(JX, Y) = 0$ for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, is associated with the disjoint union of two C_4's (labelled, as in figure 3, from bottom to top and from left to right). Then, if we denote by $\{X_1, \ldots, X_8\}$ the corresponding association frame, we just have to take into account the changes

$$Y_1 = \lambda_1 X_1 + \mu_1 X_2,\ Y_2 = \mu_1 X_1 - \lambda_1 X_2,\ Y_3 = \lambda_2 X_3 + \mu_2 X_4,$$
$$Y_4 = \mu_2 X_3 - \lambda_2 X_4,$$
$$Y_j = X_j,\ j = 5, \ldots, 8,$$

with $\lambda_i^2 = c_i^2/\sin^2\theta_i$, $\mu_i^2 = d_i^2/\sin^2\theta_i$, c_i, d_i being real numbers such that $c_i^2 + d_i^2 = \sin^2\theta_i$, $i = 1, 2$. Therefore, M is associated with $K_4 \sqcup K_4$ through $\{Y_1, \ldots, Y_8\}$, with the appropriate weights on the graph ($\cos^2\theta_i$ on the horizontal edges, c_i^2 on the vertical ones and d_i^2 on the diagonals, $i = 1, 2$).

On the other hand, some different examples of submanifolds associated with a cube were given by the first author in [2]. But we can also understand the cube as the non-disjoint union of two C_4's (for example, its top and bottom faces). Thus, we have the following obvious corollary from Theorem 5.9.

COROLLARY 5.10

Let G be a graph consisting of the non-disjoint union of two C_4's such that all its vertices have degree 3. *If G is weakly associated with a submanifold*, *then it must be a cube.*

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