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Domain decomposition methods for hyperbolic problems

PRAVIR DUTT and SUBIR SINGH LAMBA

Department of Mathematics, Indian Institute of Technology, Kanpur 208 016, India
E-mail: pravir@iitk.ac.in; subirs@gmail.com

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Abstract. In this paper a method is developed for solving hyperbolic initial boundary value problems in one space dimension using domain decomposition, which can be extended to problems in several space dimensions. We minimize a functional which is the sum of squares of the L^2 norms of the residuals and a term which is the sum of the squares of the L^2 norms of the jumps in the function across interdomain boundaries. To make the problem well posed the interdomain boundaries are made to move back and forth at alternate time steps with sufficiently high speed. We construct parallel preconditioners and obtain error estimates for the method.

The Schwarz waveform relaxation method is often employed to solve hyperbolic problems using domain decomposition but this technique faces difficulties if the system becomes characteristic at the inter-element boundaries. By making the inter-element boundaries move faster than the fastest wave speed associated with the hyperbolic system we are able to overcome this problem.

Keywords. Domain decomposition method; finite speed of propagation; pulsating boundaries; stability estimates; parallel preconditioners.

1. Introduction

In [6] a domain decomposition method for solving the wave equation using the overlapping Schwarz method is presented. The domain is divided into overlapping subdomains and the classical Schwarz waveform relaxation technique is used to construct successive approximations to the solution. However for a general hyperbolic system, the method works only if the system is noncharacteristic at the boundaries of the subdomains and this is not always possible.

In [4] a spectral method to solve hyperbolic initial boundary value problems on parallel computers in one space dimension has been proposed. In [11] these results are generalized to a domain in several dimensions which is divided into subdomains. We therefore have to consider the question of whether imposing continuity conditions across an interdomain boundary is Kreiss well posed in several dimensions. In general, however imposing continuity conditions across a boundary, even if it is noncharacteristic, is not well posed but only weakly well posed [5,11].

However, if the boundary moves sufficiently fast it can be ensured that it is noncharacteristic and then imposing continuity conditions at the moving boundary becomes well posed [11]. In fact, for symmetric hyperbolic systems with maximal dissipative boundary conditions we can prove a stability theorem for a moving element method. To prevent these moving boundaries from leaving the domain a CFL-like condition on the time step relative to the size of the subdomains has to be imposed. In the next time step the boundary moves in the opposite direction with the same speed. Thus the method can be thought of as a time stepping method as in [7]. Since the boundaries of the elements move back and forth at alternate time steps we call the proposed method ‘pulsating element method’. In this paper we restrict ourselves to one space dimension. The method can be extended to several space dimensions [11].

Domain decomposition methods for hyperbolic problems have been studied in [9,10]. The methods presented there, however, use finite differences in time unlike the approach in this paper where we use spectral elements in space-time. Moving mesh methods have been used in [2,3,13]. However, the use of moving elements for applying domain decomposition to solve hyperbolic problems has not been explored in earlier work to the best of our knowledge.

We now briefly outline the contents of this paper. In §2 stability estimates for the pulsating element method are obtained. In §3 error estimates for the method are provided. In §4 the issues of parallelization and pre-conditioning are examined. Computational results for the one dimensional case are provided in [11] and these demonstrate the effectiveness of the method.

2. Stability estimates

We shall restrict ourselves here to symmetric hyperbolic systems with maximal dissipative boundary conditions. The domain is divided into subdomains. We can prove a stability estimate provided the boundaries of the subdomains move sufficiently fast.

To keep the discussion as simple as possible a stability estimate for moving spectral elements in one space dimension is first obtained. We then state the stability estimate for moving spectral element methods in two space dimensions.

Consider therefore the following initial boundary value problem posed in the space time domain $\Omega \times (0, T)$, where $\Omega = (0, a)$:

$$\begin{aligned} \mathcal{L}u &= F \ \text{in}\, \Omega \times (0, T), \\ \mathcal{M}u &= g \ \text{in}\ \{0\} \times (0, T), \\ \mathcal{N}u &= h \ \text{in}\ \{a\} \times (0, T), \ \text{and} \\ u &= f \ \text{in}\ \Omega \times \{0\}. \end{aligned} \tag{2.1}$$

Here $\mathcal{L}u = Pu_t - Au_x - Bu$. Moreover P and A are symmetric matrices and P is positive definite. It is assumed that the boundary conditions are maximal dissipative and this is no restriction in one space dimension. We may choose $P = \mathtt{I}$ as this can always be achieved by a change of the dependent variable.

Let $\rho(E)$ denote the spectral radius of the matrix E. Then we define

$$c = \max_{(x,t)\in\Omega\times(0,T)} \rho(A) \tag{2.2}$$

We now divide the space time domain $\Omega \times (0, T)$ into elements. Let $h = a/N$ and $x_i = ih$ for $i = 0, \ldots, N$. Choose the time interval $\tau \leq h/3c$ so that $T/\tau = M$, where M is an integer. We choose τ as large as possible subject to these constraints so that τ is approximately $h/3c$.

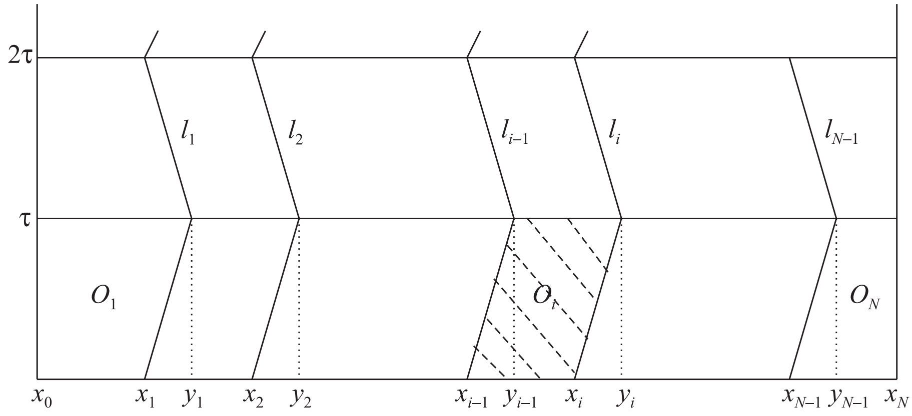

Figure 1.

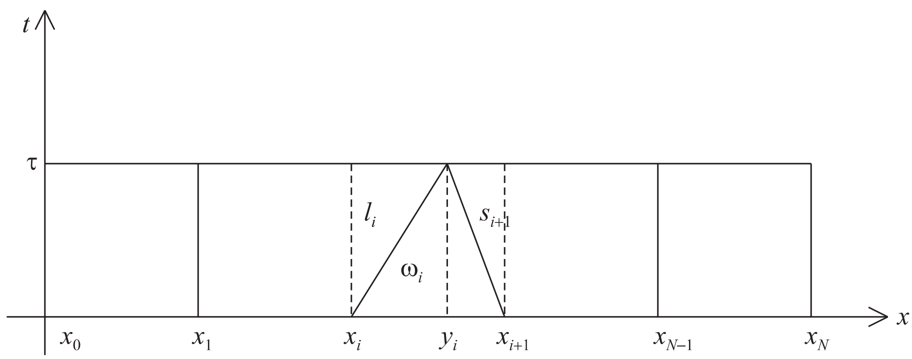

Figure 2.

Let the point $(x_i, 0)$ move with velocity $\mathrm{v} = 2c$ to the point (y_i, τ) for $i = 1, \ldots, N-1$. In the next time interval the point (y_i, τ) moves with velocity $-\mathrm{v}$ to the point $(x_i, 2\tau)$. This process is then repeated and the point $(x_i, 0)$ traces out the curve l_i as shown in figure 1.

Consider the triangle ω_i in figure 2 with vertices $(x_i, 0)$ and (y_i, τ). The sides of the triangle are l_i and s_{i+1} with slopes $1/\mathrm{v}$ and $-1/c$ and a portion of the X-axis. Here by l_i is meant the restriction of l_i to $\{(x, t){:}\ 0 < t < \tau\}$. We can now prove the following lemma.

Lemma 2.1. *Consider the IBVP*

$$\begin{aligned} \mathcal{L}\mathcal{X} &= \nu \ \textit{in}\ \Omega \times (0, \tau), \\ \mathcal{M}\mathcal{X} &= 0 \ \textit{in}\ \{0\} \times (0, \tau), \\ \mathcal{N}\mathcal{X} &= 0 \ \textit{in}\ \{a\} \times (0, \tau), \quad \textit{and} \\ \mathcal{X} &= 0 \ \textit{in}\ \Omega \times \{0\}. \end{aligned} \tag{2.3}$$

We assume that ν *is a smooth function of its arguments and vanishes in the neighbourhood of the space-time corners* $(0, 0)$ *and* $(a, 0)$. *Then there exists a constant* η_0 *such that* *for $\eta \geq \eta_0$ the estimate*

$$\sum_{i=1}^{N-1} k \int_{l_i \cap \{0 \leq t \leq \tau\}} |Q^{\frac{1}{2}} \mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}s \leq \frac{K}{\eta} \int_{\Omega \times (0,\tau)} |\mathcal{L}\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t \tag{2.4}$$

holds. Here K is a constant and $Q = \frac{\mathtt{v}\mathtt{I}+A}{\sqrt{1+\mathtt{v}^2}}$.

Now $\mathcal{X}$ is a smooth function. Let ω_i be the domain as shown in figure 2. Then integrating by parts gives

$$\begin{aligned}
&\int_{\omega_i} \mathcal{X}^T \mathcal{L}\mathcal{X} \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t \\
&= \frac{1}{2} \int_{s_{i+1}} \left(\frac{(\mathcal{X}^T \mathtt{I} \mathcal{X})c - \mathcal{X}^T A \mathcal{X}}{\sqrt{1+c^2}} \right) \mathrm{e}^{-2\eta t} \mathrm{d}s + \frac{1}{2} \int_{l_i} \left(\frac{(\mathcal{X}^T \mathtt{I} \mathcal{X})\mathtt{v} + \mathcal{X}^T A \mathcal{X}}{\sqrt{1+\mathtt{v}^2}} \right) \mathrm{e}^{-2\eta t} \mathrm{d}s \\
&\quad - \frac{1}{2} \int_{\omega_i} \mathcal{X}^T \left(-2\eta \mathtt{I} - A_x + 2B\right) \mathcal{X} \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t.
\end{aligned} \tag{2.5}$$

Now for $(x, t) \in s_{i+1}$,

$$\mathcal{X}^T (c\mathtt{I} - A)\mathcal{X} \geq 0$$

using (2.2). Hence from (2.5) we conclude that

$$\begin{aligned}
&\int_{l_i} |Q^{\frac{1}{2}} \mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}s \\
&\quad \leq K \left(\frac{1}{\eta} \int_{\omega_i} |\mathcal{L}\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t + \eta \int_{\omega_i} |\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t \right).
\end{aligned}$$

Summing the above relation over i yields

$$\begin{aligned}
&\sum_{i=1}^{N-1} \int_{l_i} |Q^{\frac{1}{2}} \mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}s \\
&\quad \leq K \left(\frac{1}{\eta} \int_{\Omega \times (0,\tau)} |\mathcal{L}\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t + \eta \int_{\Omega \times (0,\tau)} |\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t \right).
\end{aligned} \tag{2.6}$$

Since $\mathcal{X}$ is a solution of the IBVP (2.3)

$$\eta \int_{\Omega \times (0,\tau)} |\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t \leq \frac{K}{\eta} \int_{\Omega \times (0,\tau)} |\mathcal{L}\mathcal{X}|^2 \mathrm{e}^{-2\eta t} \mathrm{d}x\, \mathrm{d}t. \tag{2.7}$$

Here K denotes a generic constant.

Combining (2.6) and (2.7) we obtain the result. ■

Let O_i be the domain as shown in figure 1, and w_i be a continuously differentiable function defined on $\bar{O}_i$. Let w be the function defined on $\Omega \times (0, \tau)$ such that its restriction to O_i is w_i. Then w will, in general, be discontinuous across the lines l_i for $i = 1, \ldots, N-1$.

Let $\{w_i\}_{1 \le i \le N}$ satisfy

$$\begin{aligned}
\mathcal{L}w_i &= \mathcal{F}_i \text{ in } O_i, \text{ for } i = 1, \dots, N, \\
\mathcal{M}w_1 &= g \text{ in } \{0\} \times (0, \tau), \\
\mathcal{N}w_N &= h \text{ in } \{a\} \times (0, \tau), \\
[w] &= w_{i+1} - w_i = g_i \text{ on } l_i \text{ for } i = 1, \dots, N-1, \quad \text{and} \\
w_i &= f_i \text{ in } \Omega_i \times \{0\}.
\end{aligned} \tag{2.8}$$

Here Ω_i denotes the interval (x_i, x_{i+1}). We now prove the following result using a duality argument.

Lemma 2.2. *If* $\{w_i\}_{1 \le i \le N}$ *are continuously differentiable then the estimate*

$$\begin{aligned}
&K\eta \| w \mathrm{e}^{-\eta t} \|^2_{\Omega \times (0,\tau)} \\
&\quad \le \sum_{i=1}^{N} \frac{1}{\eta} \| \mathcal{L}w_i \mathrm{e}^{-\eta t} \|^2_{O_i} + \sum_{i=1}^{N} \| w_i \mathrm{e}^{-\eta t} \|^2_{\Omega_i \times \{0\}} + \sum_{i=1}^{N-1} \| Q^{\frac{1}{2}} [w] \mathrm{e}^{-\eta t} \|^2_{l_i} \\
&\qquad + C(\| \mathcal{M}w_1 \mathrm{e}^{-\eta t} \|^2_{\{0\} \times (0,\tau)} + \| \mathcal{N}w_N \mathrm{e}^{-\eta t} \|^2_{\{a\} \times (0,\tau)})
\end{aligned} \tag{2.9}$$

holds. Here $Q = \frac{\mathrm{vI} + A}{\sqrt{1+\mathrm{v}^2}}$.

Consider the adjoint IBVP

$$\begin{aligned}
\mathcal{L}^t \phi &= \psi \text{ in } \Omega \times (0, \tau), \\
\mathcal{M}^t \phi &= 0 \text{ in } \{0\} \times (0, \tau), \\
\mathcal{N}^t \phi &= 0 \text{ in } \{a\} \times (0, \tau), \quad \text{and} \\
\phi &= 0 \text{ in } \Omega \times \{\tau\}.
\end{aligned} \tag{2.10}$$

We assume that ψ is a smooth function which vanishes in the neighbourhood of the space time corners $(0, \tau)$ and (a, τ).

Now the adjoint IBVP is well posed since the original IBVP is well posed. Hence the estimate

$$\begin{aligned}
&\eta \| \phi \mathrm{e}^{\eta t} \|^2_{\Omega \times (0,\tau)} + \| \phi \mathrm{e}^{\eta t} \|^2_{\{0\} \times (0,\tau)} + \| \phi \mathrm{e}^{\eta t} \|^2_{\{a\} \times (0,\tau)} + \| \phi \mathrm{e}^{\eta t} \|^2_{\Omega \times \{0\}} \\
&\quad \le \frac{K}{\eta} \| \psi \mathrm{e}^{\eta t} \|^2_{\Omega \times (0,\tau)}
\end{aligned} \tag{2.11}$$

holds. Moreover ϕ is smooth. Hence

$$\begin{aligned}
&\sum_{i=1}^{N} (\mathcal{L}w_i, \phi)_{O_i} \\
&\quad = (w, \psi)_{\Omega \times (0,\tau)} - (w, \phi)_{\Omega \times \{0\}} + (g, A^{\mathrm{I}} \phi^{\mathrm{I}})_{\{0\} \times (0,\tau)} \\
&\qquad - (h, A^{\mathrm{II}} \phi^{\mathrm{II}})_{\{a\} \times (0,\tau)} + \sum_{i=1}^{N-1} \left([w], \frac{(\mathrm{vI} + A)\phi}{\sqrt{1+\mathrm{v}^2}} \right)_{l_i}.
\end{aligned} \tag{2.12}$$

In the above $A^{\mathtt{I}}$, $\phi^{\mathtt{I}}$ represent the inflow components of A and ϕ at $x=0$ and $A^{\mathtt{II}}$ and $\phi^{\mathtt{II}}$ represent the inflow component at $x=a$. Moreover $Q=\frac{\mathtt{v}\mathtt{I}+A}{\sqrt{1+\mathtt{v}^2}}$.

Now we can show just as we did in Lemma 2.1 that there exists a constant η_0 such that for $\eta>\eta_0$,

$$\sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}\phi \mathrm{e}^{\eta t}\|_{l_i}^2 \leq \frac{K}{\eta}\|\psi \mathrm{e}^{\eta t}\|^2_{\Omega\times(0,\tau)}. \tag{2.13}$$

Hence using (2.11) and (2.13) gives

$$\begin{aligned}
&|(w,\psi)_{\Omega\times(0,\tau)}| \\
&\quad\leq \sum_{i=1}^{N}\|\mathcal{L}w_i \mathrm{e}^{-\eta t}\|_{O_i}\|\phi \mathrm{e}^{\eta t}\|_{O_i} + \sum_{i=1}^{N}\|w_i \mathrm{e}^{-\eta t}\|_{\Omega_i\times\{0\}}\|\phi \mathrm{e}^{\eta t}\|_{\Omega_i\times\{0\}} \\
&\quad\quad + K(\|g \mathrm{e}^{-\eta t}\|_{\{0\}\times(0,\tau)}\|\phi \mathrm{e}^{\eta t}\|_{\{0\}\times(0,\tau)} + \|h \mathrm{e}^{-\eta t}\|_{\{a\}\times(0,\tau)}\|\phi \mathrm{e}^{\eta t}\|_{\{a\}\times(0,\tau)}) \\
&\quad\quad + \sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}[w]\mathrm{e}^{-\eta t}\|_{l_i}\|Q^{\frac{1}{2}}\phi \mathrm{e}^{\eta t}\|_{l_i}.
\end{aligned} \tag{2.14}$$

Now using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&|(w,\psi)_{\Omega\times(0,\tau)}|^2 \\
&\quad\leq \Bigg[\sum_{i=1}^{N}\frac{1}{\eta}\|\mathcal{L}w_i \mathrm{e}^{-\eta t}\|^2_{O_i} + \sum_{i=1}^{N}\|w_i \mathrm{e}^{-\eta t}\|^2_{\Omega_i\times\{0\}} \\
&\quad\quad + C(\|g \mathrm{e}^{-\eta t}\|^2_{\{0\}\times(0,\tau)} + \|h \mathrm{e}^{-\eta t}\|^2_{\{a\}\times(0,\tau)}) + \sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}[w]\mathrm{e}^{-\eta t}\|^2_{l_i}\Bigg] \\
&\quad\quad \times \Bigg[\eta\|\phi \mathrm{e}^{\eta t}\|^2_{\Omega\times(0,\tau)} + \|\phi \mathrm{e}^{\eta t}\|^2_{\Omega\times\{0\}} + \|\phi \mathrm{e}^{\eta t}\|^2_{\{0\}\times(0,\tau)} \\
&\quad\quad + \|\phi \mathrm{e}^{\eta t}\|^2_{\{a\}\times(0,\tau)} + \sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}\phi \mathrm{e}^{\eta t}\|^2_{l_i}\Bigg].
\end{aligned} \tag{2.15}$$

Hence using (2.11) and (2.13) yields the inequality

$$\begin{aligned}
&K\eta \sup \frac{|(w,\psi)_{\Omega\times(0,\tau)}|^2}{\|\psi \mathrm{e}^{\eta t}\|^2_{\Omega\times(0,\tau)}} \\
&\quad\leq \sum_{i=1}^{N}\frac{1}{\eta}\|\mathcal{L}w_i \mathrm{e}^{-\eta t}\|^2_{O_i} + \sum_{i=1}^{N}\|w_i \mathrm{e}^{-\eta t}\|^2_{\Omega_i\times\{0\}} \\
&\quad\quad + C(\|\mathcal{M}w_1 \mathrm{e}^{-\eta t}\|^2_{\{0\}\times(0,\tau)} + \|\mathcal{N}w_N \mathrm{e}^{-\eta t}\|^2_{\{a\}\times(0,\tau)}) \\
&\quad\quad + \sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}[w]\mathrm{e}^{-\eta t}\|^2_{l_i}.
\end{aligned}$$

And this gives the result. ■

Using the above result we can prove a generalized version of Lemma 2.1 which is stated below.

Lemma 2.3. *Let* $\{w_i\}_{1\le i\le N}$ *satisfy* (2.8). *Then there exists a constant L such that the estimate*

$$\sum_{i=1}^{N-1}\int_{l_i}|Q^{\frac{1}{2}}w_{i+1}\mathrm{e}^{-\eta t}|^2\mathrm{d}s$$
$$\le L\left[\sum_{i=1}^{N}\frac{1}{\eta}\|\mathcal{L}w_i\mathrm{e}^{-\eta t}\|^2_{O_i}+\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|^2_{\Omega_i\times\{0\}}+\|\mathcal{M}w_1\mathrm{e}^{-\eta t}\|^2_{\{0\}\times(0,\tau)}\right.$$
$$\left.+\|\mathcal{N}w_N\mathrm{e}^{-\eta t}\|^2_{\{a\}\times(0,\tau)}+\sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}[w]\mathrm{e}^{-\eta t}\|^2_{l_i}\right] \tag{2.16}$$

holds for all $\eta\ge\eta_0$.

The proof is just the same as in Lemma 2.1. In the last step we use Lemma 2.2 to obtain the result. ■

We can now prove the following stability theorem.

Theorem 2.4. *There exist constants* f *and* η_1 *such that for all* $\eta\ge\eta_1$ *the estimate*

$$f\left(\eta\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|^2_{O_i}+\|w_1\mathrm{e}^{-\eta t}\|^2_{\{0\}\times(0,\tau)}+\|w_N\mathrm{e}^{-\eta t}\|^2_{\{a\}\times(0,\tau)}\right.$$
$$\left.+\frac{1}{N}\left(\sum_{i=1}^{N-1}(\|w_i\mathrm{e}^{-\eta t}\|^2_{l_i}+\|w_{i+1}\mathrm{e}^{-\eta t}\|^2_{l_i})\right)\right)+\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|^2_{\bar{O}_i\bigcap\{t=\tau\}}$$
$$\le\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|^2_{\bar{O}_i\bigcap\{t=0\}}+\frac{1}{\eta}\sum_{i=1}^{N}\|\mathcal{L}w_i\mathrm{e}^{-\eta t}\|^2_{O_i}$$
$$+N\left(\|\mathcal{M}w_1\mathrm{e}^{-\eta t}\|^2_{\{0\}\times(0,\tau)}+\|\mathcal{N}w_N\mathrm{e}^{-\eta t}\|^2_{\{a\}\times(0,\tau)}+2\sum_{i=1}^{N-1}\|[w]\mathrm{e}^{-\eta t}\|^2_{l_i}\right) \tag{2.17}$$

holds for N large enough. Here $\eta_1>\eta_0$.

Using integeration by parts give

$$2\sum_{i=1}^{N}(\mathcal{L}w_i,w_i\mathrm{e}^{-2\eta t})_{O_i}$$
$$=\sum_{i=1}^{N}(w_i,w_i\mathrm{e}^{-2\eta t})_{\bar{O}_i\bigcap\{t=\tau\}}-\sum_{i=1}^{N}(w_i,w_i\mathrm{e}^{-2\eta t})_{\bar{O}_i\bigcap\{t=0\}}$$
$$+\sum_{i=1}^{N-1}\left[\left(w,\frac{(\mathrm{v}\mathrm{I}+A)}{\sqrt{1+\mathrm{v}^2}}w\mathrm{e}^{-2\eta t}\right)_{l_i}\right]-(w_N,Aw_N\mathrm{e}^{-2\eta t})_{\{a\}\times(0,\tau)}$$
$$+(w_1,Aw_1\mathrm{e}^{-2\eta t})_{\{0\}\times(0,\tau)}+\sum_{i=1}^{N}(w_i,Dw_i\mathrm{e}^{-2\eta t})_{O_i}, \tag{2.18}$$

where $D=2\eta\mathrm{I}+A_x-2B$.

In the above

$$\left[\left(w,\, \frac{(\mathtt{v}\mathtt{I}+A)}{\sqrt{1+\mathtt{v}^2}} w \mathrm{e}^{-2\eta t}\right)_{l_i}\right] = \left(w_{i+1},\, \frac{(\mathtt{v}\mathtt{I}+A)}{\sqrt{1+\mathtt{v}^2}}[w]\mathrm{e}^{-2\eta t}\right)_{l_i} + \left(w_i,\, \frac{(\mathtt{v}\mathtt{I}+A)}{\sqrt{1+\mathtt{v}^2}}[w]\mathrm{e}^{-2\eta t}\right)_{l_i}.$$

Taking η sufficiently large we can make $D > 3\eta\mathtt{I}/2$. Hence there exist constants f, c and η_0 such that for $\eta > \eta_0$ the estimate

$$\begin{aligned}
f&\left(\eta\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|_{O_i}^2 + \|w_1\mathrm{e}^{-\eta t}\|_{\{0\}\times(0,\tau)}^2 + \|w_N\mathrm{e}^{-\eta t}\|_{\{a\}\times(0,\tau)}^2\right)\\
&+\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|_{\bar{O}_i\bigcap\{t=\tau\}}^2\\
&\le \sum_{i=1}^{N}\|w_i\|_{\bar{O}_i\bigcap\{t=0\}}^2 + N\sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}[w]\mathrm{e}^{-\eta t}\|_{l_i}^2\\
&+\frac{1}{2N}\left(\sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}w_i\mathrm{e}^{-\eta t}\|_{l_i}^2 + \sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}w_{i+1}\mathrm{e}^{-\eta t}\|_{l_i}^2\right)\\
&+C(\|\mathcal{M}w_1\mathrm{e}^{-\eta t}\|_{\{0\}\times(0,\tau)}^2 + \|\mathcal{N}w_N\mathrm{e}^{-\eta t}\|_{\{a\}\times(0,\tau)}^2)\\
&+\frac{1}{\eta}\sum_{i=1}^{N}\|\mathcal{L}w_i\mathrm{e}^{-\eta t}\|_{O_i}^2
\end{aligned}\tag{2.19}$$

holds. Here we have used the condition that the boundary conditions are maximal dissipative. Now from (2.16) it follows that

$$\begin{aligned}
&\frac{1}{N}\left(\sum_{i=1}^{N-1}(\|Q^{\frac{1}{2}}w_i\mathrm{e}^{-\eta t}\|_{l_i}^2 + \|Q^{\frac{1}{2}}w_{i+1}\mathrm{e}^{-\eta t}\|_{l_i}^2)\right)\\
&\quad\le \frac{3L}{N}\left(\sum_{i=1}^{N}\frac{1}{\eta}\|\mathcal{L}w_i\mathrm{e}^{-\eta t}\|_{O_i}^2 + \sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|_{\bar{O}_i\bigcap\{t=0\}}^2 + \|\mathcal{M}w_1\mathrm{e}^{-\eta t}\|_{\{0\}\times(0,\tau)}^2\right.\\
&\qquad\left.+\|\mathcal{N}w_N\mathrm{e}^{-\eta t}\|_{\{a\}\times(0,\tau)}^2\right) + \frac{2}{N}\sum_{i=1}^{N-1}\|Q^{\frac{1}{2}}[w]\mathrm{e}^{-\eta t}\|_{l_i}^2.
\end{aligned}\tag{2.20}$$

Adding (2.19) and (2.20) and choosing f small enough gives the estimate

$$\begin{aligned}
f&\left(\eta\sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|_{O_i}^2 + \|w_1\mathrm{e}^{-\eta t}\|_{\{0\}\times(0,\tau)}^2 + \|w_N\mathrm{e}^{-\eta t}\|_{\{a\}\times(0,\tau)}^2\right.\\
&\left.+\frac{1}{N}\left(\sum_{i=1}^{N-1}(\|w_i\mathrm{e}^{-\eta t}\|_{l_i}^2 + \|w_{i+1}\mathrm{e}^{-\eta t}\|_{l_i}^2)\right)\right) + \sum_{i=1}^{N}\|w_i\mathrm{e}^{-\eta t}\|_{\bar{O}_i\bigcap\{t=\tau\}}^2
\end{aligned}$$

$$\leq \left(1+\frac{b}{N}\right)\left(\sum_{i=1}^{N}\|w_i\|^2_{\bar{O}_i\bigcap\{t=0\}}+\frac{4}{5\eta}\sum_{i=1}^{N}\|\mathcal{L}w_i \mathrm{e}^{-\eta t}\|^2_{O_i}\right.$$
$$\left.+C(\|\mathcal{M}w_1\mathrm{e}^{-\eta t}\|^2_{\{0\}\times(0,\tau)}+\|\mathcal{N}w_N\mathrm{e}^{-\eta t}\|^2_{\{a\}\times(0,\tau)})+\frac{3N}{2}\sum_{i=1}^{N-1}\|[w]\mathrm{e}^{-\eta t}\|^2_{l_i}\right) \tag{2.21}$$

holds, where b is a positive constant. In the above we have used the relation

$$\frac{c\mathtt{I}}{\sqrt{1+4c^2}} \leq Q \leq \frac{3c\mathtt{I}}{\sqrt{1+4c^2}}.$$

Now there exists a positive constant α such that

$$1+\frac{b}{N}=\mathrm{e}^{\alpha\tau}.$$

Choosing $\eta_1 = \eta_0 + \frac{\alpha}{2}$ the result follows from (2.21). ■

Recollect that $x_i = ih$ and $y_i = x_i + \mathrm{v}\tau$ for $0 \leq i \leq N-1$. Let $x_0(t) = 0$, and $x_N(t) = a$ for $0 \leq t \leq T$. Define $x_i(t) = x_i + \mathrm{v}(t-(j-1)\tau)$ for $(j-1)\tau \leq t \leq j\tau$, if j is odd and, $x_i(t) = y_i - \mathrm{v}(t-(j-1)\tau)$ for $(j-1)\tau \leq t \leq j\tau$, if j is even for $1 \leq j \leq M$. Let l_i denote the curve consisting of piecewise straight line segments given by

$$l_i = \{(x,t) : x = x_i(t),\ 0 \leq t \leq T\},\ \text{ for } 1 \leq i \leq N-1.$$

By W_i we denote the open set

$$W_i = \{(x,\ t) : x_{i-1}(t) < x < x_i(t),\ 0 < t < T\}.$$

Let $w_{i,j}$ denote a function continuously differentiable on $\bar{W}_{i,j}$, where

$$W_{i,j} = W_i \cap \{(x,t) : (j-1)\tau < t < j\tau\} \text{ for } 1 \leq j \leq M,$$

and w_i denote the function whose restriction to $W_{i,j}$ is $w_{i,j}$ and which may be discontinuous at $t = j\tau$ for $j = 1,\ldots,Q-1$. Then the following theorem follows immediately from Theorem 2.4.

Theorem 2.5. *There exist constants f and η_1 such that for all $\eta \geq \eta_1$ the estimate*

$$f\left(\eta\sum_{i=1}^{N}\|w_{i,j}\mathrm{e}^{-\eta t}\|^2_{W_i\cap\{(j-1)\tau<t<j\tau\}}+\|w_{1,j}\mathrm{e}^{-\eta t}\|^2_{\{0\}\times((j-1)\tau,j\tau)}\right.$$
$$+\|w_{N,j}\mathrm{e}^{-\eta t}\|^2_{\{a\}\times((j-1)\tau,j\tau)}+\frac{1}{N}\left(\sum_{i=1}^{N-1}\|w_{i,j}\mathrm{e}^{-\eta t}\|^2_{l_i\cap\{(j-1)\tau<t<j\tau\}}\right.$$
$$\left.\left.+\sum_{i=1}^{N-1}\|w_{i+1,j}\mathrm{e}^{-\eta t}\|^2_{l_i\cap\{(j-1)\tau<t<j\tau\}}\right)\right)+\sum_{i=1}^{N}\|w_{i,j}\mathrm{e}^{-\eta t}\|^2_{\bar{W}_i\cap\{t=j\tau\}}$$

$$\leq \sum_{i=1}^{N} \|w_{i,j} e^{-\eta t}\|^2_{\bar{W}_i \cap \{t=(j-1)\tau\}} + \sum_{i=1}^{N} \frac{1}{\eta} \|\mathcal{L} w_{i,j} e^{-\eta t}\|^2_{W_i \cap \{(j-1)\tau < t < j\tau\}}$$
$$+ 2N \Bigg(\|\mathcal{M} w_{1,j} e^{-\eta t}\|^2_{\{0\}\times((j-1)\tau, j\tau)} + \|\mathcal{N} w_{N,j} e^{-\eta t}\|^2_{\{a\}\times((j-1)\tau, j\tau)}$$
$$+ \sum_{i=1}^{N-1} \|[w] e^{-\eta t}\|^2_{l_i \cap \{(j-1)<t<j\tau\}} \Bigg) \tag{2.22}$$

holds for $1 \leq j \leq M$.

3. The numerical scheme and error estimates

Here we briefly describe the numerical scheme and obtain error estimates in one space dimension as they can be easily extended to multi-dimensions.

Let $W_{i,,j}$ denote the trapezoid $W_i \cap \{(x,t)\colon t_{j-1} < t < t_j\}$ as shown in figure 3. We map $W_{i,j}$ to the square $S = (-1,1)^2$ and let $M_{i,j}(\xi,\eta) = (X_{i,j}(\xi,\eta), T_{i,j}(\xi,\eta))$ denote the inverse mapping from S to $W_{i,j}$. Thus $M_{2,j}, \ldots, M_{N-1,j}$ are affine maps. The maps $M_{1,j}$ and $M_{N,j}$ are however bilinear. On each $W_{i,j}$ we shall define the approximate representation $u^p_{i,j}$ of u to be a polynomial of degree p in ξ and η i.e.

$$u^p_{i,j}(\xi,\eta) = \sum_{k=0}^{p} \sum_{l=0}^{p} a_{k,l} \xi^k \eta^l.$$

Now we examine the case when the mapping, say $M_{1,1}$ is bilinear. We then have [1]

$$X_{1,1}(\xi,\eta) = \left(\frac{1+\xi}{2}\right)\left(h + 2c\tau\left(\frac{1+\eta}{2}\right)\right)$$
$$T_{1,1}(\xi,\eta) = \tau\left(\frac{1+\eta}{2}\right). \tag{3.1}$$

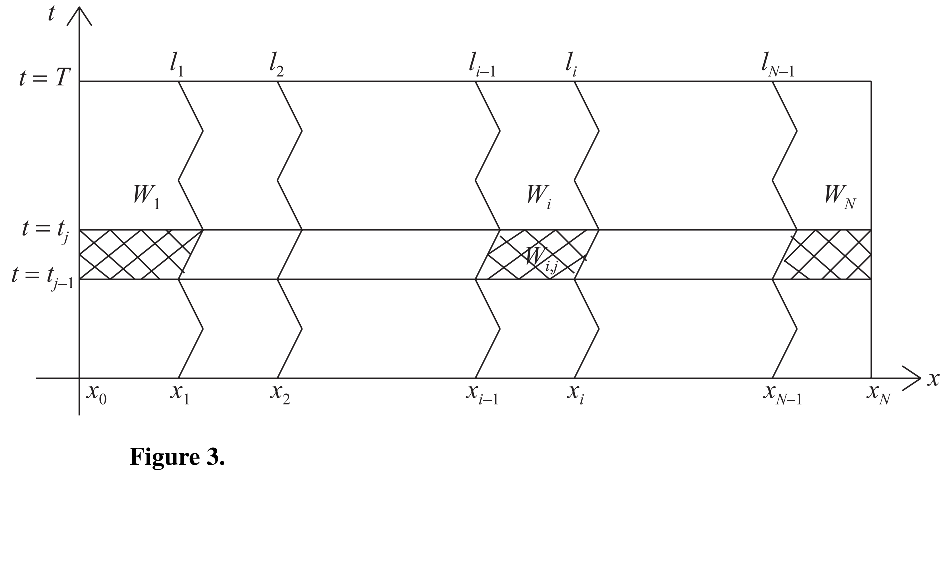

Figure 3.

Moreover the Jacobian $J_{1,1}(\xi\ \eta)$ satisfies

$$J_{1,1}(\xi,\eta) = \left(h + 2c\tau\frac{(1+\eta)}{2}\right)\frac{\tau}{4} = \Theta(h\tau). \tag{3.2}$$

For $i \neq 1$ or N the mapping $M_{i,1}$ assumes the form

$$\begin{aligned} X_{i,1}(\xi,\eta) &= (i-1)h\frac{(1-\xi)}{2} + ih\frac{(1+\xi)}{2} + 2c\tau\frac{(1+\eta)}{2} \\ T_{i,1}(\xi,\eta) &= \tau\frac{(1+\eta)}{2}. \end{aligned} \tag{3.3}$$

Let S^p denote the space of polynomials of degree p. Then $u^p_{i,j}$ belongs to the tensor product space $(S^p \times S^p)^k$ for $1 \le i \le N$, $1 \le j \le M$.

Let us assume that we have solved for $u^p_{i,j-1}$ for $1 \le i \le N$. We now choose $\{u^p_{1,j}, \ldots, u^p_{N,j}\}$, where $u^p_{i,j} \epsilon (S^p \times S^p)^k$ for $1 \le i \le N$, to minimize the functional

$$\begin{aligned} & r_j(v^p_{1,j}, \ldots, v^p_{N,j}) \\ &\quad = \sum_{i=1}^{N}\int_{W_{i,j}} |\mathcal{L}v^p_{i,j} - \mathcal{F}|^2 \mathrm{d}x\,\mathrm{d}t \\ &\qquad + 2N\left(\int_{\{0\}\times(t_{j-1},t_j)} |\mathcal{M}v^p_{1,j} - g|^2 \mathrm{d}t + \int_{\{a\}\times(t_{j-1},t_j)} |\mathcal{N}v^p_{N,j} - h|^2 \mathrm{d}t\right. \\ &\qquad \left. + \sum_{i=1}^{N-1}\int_{l_i\cap\{t_{j-1}<t<t_j\}} |[v^p]|^2 \mathrm{d}s\right) + \sum_{i=1}^{N}\int_{\bar{W}_{i,j}\cap\{t=t_{j-1}\}} |(v^p_{i,j} - u^p_{i,j-1})|^2 \mathrm{d}x \end{aligned} \tag{3.4}$$

over all $\{v^p_{i,j}\}_{1\le i\le N}$. Let $u^p_{i,0}(x, t=0) = f(x)$, which is an abuse of notation.

Consider the quadratic form

$$\begin{aligned} & H_j(\{v^p_{i,j}\}_{1\le i\le N}) \\ &\quad = \sum_{i=1}^{N}\int_{W_{i,j}} |\mathcal{L}v^p_{i,j}|^2 \mathrm{d}x\,\mathrm{d}t + 2N\left(\int_{\{0\}\times(t_{j-1},t_j)} |\mathcal{M}v^p_{1,j}|^2 \mathrm{d}t\right. \\ &\qquad \left. + \int_{\{a\}\times(t_{j-1},t_j)} |\mathcal{N}v^p_{N,j}|^2 \mathrm{d}t + \sum_{i=1}^{N-1}\int_{l_i\cap\{t_{j-1}<t<t_j\}} |[v^p]|^2 \mathrm{d}s\right) \\ &\qquad + \sum_{i=1}^{N}\int_{\bar{W}_i\cap\{t=t_{j-1}\}} |v^p_{i,j}|^2 \mathrm{d}x. \end{aligned} \tag{3.5}$$

For the error estimates that follow we shall need to write $H_1(\{v^p_{i,1}\}_i)$ in (ξ, η) variables, viz.

$$
\begin{aligned}
&H_1(\{v^p_{i,1}\}_{1\le i\le N})\\
&= \sum_{i=1}^{N}\int_S |\mathcal{L}_{i,1}v^p_{i,1}|^2 \mathrm{d}\xi\, \mathrm{d}\eta + 2N\left(\frac{\tau}{2}\int_{\{-1\}\times(-1,1)}|\mathcal{M}_1 v^p_{1,1}|^2\mathrm{d}\eta\right.\\
&\quad + \frac{\tau}{2}\int_{\{1\}\times(-1,1)}|\mathcal{N}_1 v^p_{N,1}|^2\mathrm{d}\eta\\
&\quad \left.+\frac{\sqrt{1+4c^2}}{2}\tau\sum_{i=1}^{N-1}\int_{(-1,1)}|v^p_{i-1,1}(1,\eta)-v^p_{i,1}(-1,\eta)|^2\mathrm{d}\eta\right)\\
&\quad + \frac{h}{2}\sum_{i=1}^{N}\int_{(-1,1)\times\{-1\}}|v^p_{i,1}|^2\mathrm{d}\xi. \qquad (3.6)
\end{aligned}
$$

Here $\mathcal{M}_j(\eta)=\mathcal{M}(T_{1,j}(-1,\eta))$ and $\mathcal{N}_j(\eta)=\mathcal{N}(T_{N,j}(1,\eta))$. It is easy to show that

$$
\begin{aligned}
\mathcal{L}_{1,1}v^p_{1,1} &= \frac{1}{\sqrt{J_{1,1}}}\Big(\frac{(h+c\tau(1+\eta))}{2}\frac{\partial v^p_{1,1}}{\partial\eta}\\
&\quad -\left(A_{1,1}\frac{\tau}{2}+\frac{c\tau(1+\xi)}{2}\right)\frac{\partial v^p_{1,1}}{\partial\xi} - B_{1,1}J_{1,1}v^p_{1,1}\Big), \qquad (3.7a)
\end{aligned}
$$

where $J_{1,1}(\xi,\eta)$ is defined in (3.2) and $A_{i,j}(\xi,\eta)=A(X_{i,j}(\xi,\eta),T_{i,j}(\xi,\eta))$, etc. Moreover for $i\ne 1$ or N

$$
\mathcal{L}_{i,1}v^p_{i,1} = \frac{\sqrt{h\tau}}{2}\left(\frac{2}{\tau}\frac{\partial v^p_{i,1}}{\partial\eta}+\left(-\frac{4c}{h}\mathrm{I}-\frac{2}{h}A_{i,1}\right)\frac{\partial v^p_{i,1}}{\partial\xi}-B_{i,1}v^p_{i,1}\right). \qquad (3.7b)
$$

Recall that $\tau/h=\lambda$, the CFL constant. Let $u_{i,j}(\xi,\eta)=u(X_{i,j}(\xi,\eta),T_{i,j}(\xi,\eta))$. We now prove an approximation result which is similar to Theorem 3.3 in [8].

Lemma 3.6. *There exist polynomials* $\bar{u}^p_{i,j}(\xi,\eta)\epsilon(S^p\times S^p)^k$ *such that*:

$$
\bar{u}^p_{i,j}(1,\eta)=\bar{u}^p_{i+1,j}(-1,\eta) \text{ for } 1\le i\le N-1,\ 1\le j\le Q, \quad \text{and} \qquad (3.8a)
$$

$$
\bar{u}^p_{i,j}(\xi,1)=\bar{u}^p_{i,j+1}(\xi,-1) \text{ for } 1\le i\le N,\ 1\le j\le Q-1. \qquad (3.8b)
$$

Moreover the error estimate

$$
\sum_{j=1}^{Q}\sum_{i=1}^{N}\|u_{i,j}-\bar{u}^p_{i,j}\|^2_{1,S}\le K\,(ch)^{2s}\,\frac{(p-s)!}{(p+s)!}\,\|u\|^2_{s+2,\Omega\times(0,T)} \qquad (3.9)
$$

holds for $0\le s\le p$. *In the above* K *and* c *are constants and* $h=a/N$ *denotes the mesh width.*

Let $\hat{\mathrm{I}}=(-1,1)$ and π^p_ξ be the operator mapping $H^1(\hat{\mathrm{I}})\to S^p(\hat{\mathrm{I}})$ as in Theorem 3.3 of [8]. Define $\Pi^p=\pi^p_\xi\pi^p_\eta$ to be the tensor product operator. Let $\omega\in H^{k+2}(S)$ and $\phi^p=\Pi^p\omega$. Then there holds

$$
\phi^p(\pm1,\eta)=\pi^p_\eta\omega(\pm1,\eta), \text{ and}
$$

$$
\phi^p(\xi,\pm1)=\pi^p_\xi\omega(\xi,\pm1).
$$

Moreover the error estimate

$$\|\omega - \phi^p\|_{1,S}^2 \le 3\frac{(p-s)!}{(p+s)!}(\|\partial_\xi^{s+1}\omega\|_{0,S}^2 + \|\partial_\eta^{s+1}\omega\|_{0,S}^2) + \frac{10}{p(p+1)}\frac{(p-s)!}{(p+s)!}(\|\partial_\xi^{s+1}\partial_\eta\omega\|_{0,S}^2 + \|\partial_\xi\partial_\eta^{s+1}\omega\|_{0,S}^2) \tag{3.10}$$

holds for any $0 \le s \le \min(p, k)$.

Note that the estimate (3.10), though slightly different from (3.8) and (3.9) of [8], is proved in the same way.

Let $\bar{u}_{i,j}^p = \Pi^p u_{i,j}$. Then clearly (3.8a), (3.8b) hold. Hence we conclude

$$\sum_{i=1}^N \sum_{j=1}^Q \|u_{i,j} - \bar{u}_{i,j}^p\|_{1,S}^2 \le 5\frac{(p-s)!}{(p+s)!}\left(\sum_{l=0}^1 \|\partial_\xi^{s+1}\partial_\eta^l u_{i,j}\|_{0,S}^2 + \sum_{l=0}^1 \|\partial_\xi^l \partial_\eta^{s+1} u_{i,j}\|_{0,S}^2\right) \tag{3.11}$$

for

$$u_{i,j}(\xi,\eta) = u(X_{i,j}(\xi,\eta), T_{i,j}(\xi,\eta)).$$

From the form of the mappings in (3.1) and (3.3) we have

$$(X_{i,j})_\xi = \Theta\left(\frac{1}{N}\right), \ (X_{i,j})_\eta = \Theta\left(\frac{1}{N}\right), \ (X_{i,j})_{\xi\eta} = O\left(\frac{1}{N}\right)$$

and all other derivatives $D_{\xi,\eta}^\alpha X_{i,j} = 0$. Moreover

$$(T_{i,j})_\eta = \Theta\left(\frac{1}{N}\right)$$

and all other derivatives $D_{\xi,\eta}^\alpha T_{i,j} = 0$. Hence

$$\partial_\eta^r u_{i,j} = \sum_{k=0}^r \binom{r}{k}((X_{i,j})_\eta)^k((T_{i,j})_\eta)^{r-k}\partial_x^k\partial_t^{r-k}u,$$

and

$$\partial_\xi\partial_\eta^r u_{i,j} = \sum_{k=0}^r \binom{r}{k}(k((X_{i,j})_\eta)^{k-1}(X_{i,j})_{\xi\eta}((T_{i,j})_\eta)^{r-k})\partial_x^k\partial_t^{r-k}u + \sum_{k=0}^r \binom{r}{k}((X_{i,j})_\eta)^k((T_{i,j})_\eta)^{r-k}\{(X_{i,j})_\xi\partial_x^{k+1}\partial_t^{r-k}u + (T_{i,j})_\xi\partial_x^k\partial_t^{r-k+1}u\}.$$

Now using the bounds for the derivative of $X_{i,j}$ and $T_{i,j}$ we obtain that there are constants C and β such that

$$\sum_{i=1}^N \sum_{j=1}^Q \left(\sum_{l=0}^1 \|\partial_\xi^{s+1}\partial_\eta^l u_{i,j}\|_{0,S}^2 + \sum_{l=0}^1 \|\partial_\xi^l\partial_\eta^{s+1}u_{i,j}\|_{0,S}^2\right) \le C\left(\frac{\beta}{N}\right)^{2s}\|u\|_{s+2,\Omega\times(0,T)}^2. \tag{3.12}$$

Combining (3.11) and (3.12) we obtain the result. ■

In what follows we shall assume that the coefficients of the IBVP (2.1) are smooth and the data is smooth and satisfies the compatibility conditions at the space-time corners for the solution to be smooth.

Theorem 3.7. *Let $u^p = \{u^p_{i,j}\}_{i,j}$ be the approximate solution of the IBVP* (2.1). *Then the following error estimate holds*

$$\|u(x,t) - u^p(x,t)\|^2_{\Omega\times(0,T)} + \|u(x,T) - u^p(x,T)\|^2_{\Omega\times\{T\}} \leq K(ch)^{2s}\frac{(p-s)!}{(p+s)!}\|u\|^2_{s+2,\Omega\times(0,T)} \tag{3.13}$$

for all $0 \leq s \leq p$.

Since $\{u^p_{i,1}(\xi,\eta)\}_i$ minimizes $r_1(\{v^p_{i,1}\}_i)$ we have

$$r_1(\{\bar{u}^p_{i,1}\}_i) = r_1(\{u^p_{i,1}\}_i) + H_1(\{u^p_{i,1} - \bar{u}^p_{i,1}\}_i). \tag{3.14}$$

Hence

$$H_1(\{u^p_{i,1} - \bar{u}^p_{i,1}\}_i) \leq r_1(\{\bar{u}^p_{i,1}\}_i). \tag{3.15}$$

Let $\lambda = 2\max(1,\eta_1)$. Then by Theorem 2.5 there exists a positive constant α such that

$$\begin{aligned}\sum_{i=1}^N(\alpha\|u^p_{i,1}(x,t) - \bar{u}^p_{i,1}(x,t)\|^2_{W_{i,1}} + \|u^p_{i,1}(x,t) - \bar{u}^p_{i,1}(x,t)\|^2_{\bar{W}_i\cap\{t=t_1\}})\\ \leq e^{\lambda\tau}H_1(\{u^p_{i,1} - \bar{u}^p_{i,1}\}_i)\\ \leq e^{\lambda\tau}r_1(\{\bar{u}^p_{i,1}\}_i).\end{aligned} \tag{3.16}$$

Let

$$\tilde{r}_j(\{v^p_{i,j}\}_i) = r_j(\{v^p_{i,j}\}_i) - \sum_{i=1}^N\|v^p_{i,j} - u^p_{i,j-1}\|^2_{\bar{W}_i\cap\{t=t_{j-1}\}} \text{ for } j \geq 2.$$

Now just as in (3.14)

$$H_2(\{u^p_{i,2} - \bar{u}^p_{i,2}\}_i) \leq r_2(\{\bar{u}^p_{i,2}\}).$$

Hence as in (3.16) we get

$$\begin{aligned}\sum_{i=1}^N\alpha\|u^p_{i,2}(x,t) - \bar{u}^p_{i,2}(x,t)\|^2_{W_{i,2}} + \|u^p_{i,2}(x,t) - \bar{u}^p_{i,2}(x,t)\|^2_{\bar{W}_i\cap\{t=t_2\}}\\ \leq e^{\lambda\tau}\left(\sum_{i=1}^N\|u^p_{i,1}(x,t) - \bar{u}^p_{i,2}(x,t)\|^2_{\bar{W}_i\cap\{t=t_1\}} + \tilde{r}_2(\{\bar{u}^p_{i,2}\}_i)\right).\end{aligned} \tag{3.17}$$

Now $\bar{u}^p_{i,2}(x,t_1) = \bar{u}^p_{i,1}(x,t_1)$. Multiplying (3.16) by $e^{\lambda\tau}$ and adding $e^{\lambda\tau}\tilde{r}_2(\{\bar{u}^p_{i,2}\}_i)$ to it gives

$$\begin{aligned}\sum_{i=1}^N\alpha(\|u^p_{i,2}(x,t) - \bar{u}^p_{i,2}(x,t)\|^2_{W_{i,2}} + e^{\lambda\tau}\|u^p_{i,1}(x,t) - \bar{u}^p_{i,1}(x,t)\|^2_{W_{i,1}})\\ + \|u^p_{i,2}(x,t) - \bar{u}^p_{i,2}(x,t)\|^2_{\bar{W}_i\cap\{t=t_2\}}\\ \leq e^{2\lambda\tau}r_1(\{\bar{u}^p_{i,1}\}_i) + e^{\lambda\tau}\tilde{r}_2(\{\bar{u}^p_{i,2}\}_i).\end{aligned}$$

We use (3.17) to obtain the above. Continuing in this way yields

$$\alpha \|u^p(x,t) - \bar{u}^p(x,t)\|^2_{\Omega\times(0,T)} + \|u^p(x,t) - \bar{u}^p(x,t)\|^2_{\Omega\times\{T\}} \tag{3.18}$$
$$\leq \mathrm{e}^{\lambda T}\left(r_1(\{\bar{u}^p_{i,1}\}_i) + \sum_{j=2}^{Q} \tilde{r}_j(\{\bar{u}^p_{i,j}\}_i)\right).$$

Now

$$r_1(\{\bar{u}^p_{i,1}\}_i) \leq C\sum_{i=1}^{N} \|u_{i,1} - \bar{u}^p_{i,1}(\xi,\eta)\|^2_{1,S} \tag{3.19a}$$

and

$$\tilde{r}_j(\{\bar{u}^p_{i,j}\}_i) \leq C\sum_{i=1}^{N} \|(u_{i,j} - \bar{u}^p_{i,j})(\xi,\eta)\|^2_{1,S} \text{ for } j \geq 2. \tag{3.19b}$$

Combining (3.15), (3.18) and (3.19) gives

$$\alpha \|u^p(x,t) - \bar{u}^p(x,t)\|^2_{\Omega\times(0,T)} + \|u^p(x,t) - \bar{u}^p(x,t)\|^2_{\Omega\times\{T\}}$$
$$\leq C\sum_{j=1}^{Q}\sum_{i=1}^{N} \|u_{i,j} - \bar{u}^p_{i,j}\|^2_{1,S}$$

The result now follows from Lemma 3.6. ■

Remark. We now examine the error estimate (3.13).

We first examine the case when p is fixed and h tends to zero. Then the error in the L^2 norm is $O(h^p)$ if the solution is smooth enough. Hence the h version of the method is pth. order accurate.

Next we look at the case when h is fixed and p tends to infinity. It can be shown by using arguments as in [8] that if u is analytic then the error in the L^2 norm satisfies the estimate

$$\|u - u^p\|_{L^2(\Omega\times(0,T))} \leq C\mathrm{e}^{-\beta p} h^{\alpha p}$$

for constants $0 < \alpha \leq 1$ and $0 < \beta$.

If u is smooth then the error is bounded by the estimate

$$\|u - u^p\|_{L^2(\Omega\times(0,T))} \leq K\frac{(ch)^s}{p^s}\|u\|_{s+2,\Omega\times(0,T)} \text{ for } 0 \leq s \leq p.$$

4. Parallelization and pre-conditioning

We now need to solve the normal equations we obtain from minimizing the functional $r_j(v^p_{1,j}, v^p_{2,j}, \ldots, v^p_{N,j})$. In [4] it has been shown that the residual in the normal equations can be computed in $O(p^3)$ operations without having to compute mass and stiffness matrices.

We now define a quadratic form

$$
\begin{aligned}
&\tilde{Q}_j(v^p_{1,j}, v^p_{2,j}, \ldots, v^p_{N,j}) \\
&= \sum_{i=1}^{N} \int_S |\tilde{\mathcal{L}}_{i,j} v^p_{i,j}|^2 \mathrm{d}\xi\, \mathrm{d}\eta + \sum_{\substack{i=1\\ j=\text{odd}}}^{N-1} \int_{(-1,1)} |v^p_{i,j}(1,\eta)|^2 \mathrm{d}\eta \\
&\quad + \sum_{\substack{i=1\\ j=\text{even}}}^{N-1} \int_{(-1,1)} |v^p_{i+1,j}(-1,\eta)|^2 \mathrm{d}\eta + \int_{(-1,1)} |\mathcal{M}_j v^p_{1,j}(-1,\eta)|^2 \mathrm{d}\eta \\
&\quad + \int_{(-1,1)} |\mathcal{N}_j v^p_{N,j}(1,\eta)|^2 \mathrm{d}\eta + \sum_{i=1}^{N} \int_{(-1,1)} |v^p_{i,j}(\xi,-1)|^2 \mathrm{d}\xi. \qquad (4.1)
\end{aligned}
$$

Here for j odd

$$
\tilde{\mathcal{L}}_{1,j} v = \frac{1}{\sqrt{h\tau}} \left(\left(\frac{h + c\tau(1+\eta)}{2} \right) \frac{\partial v}{\partial \eta} - \left(A_{1,j} \frac{\tau}{2} + \frac{c\tau(1+\xi)}{2} \right) \frac{\partial v}{\partial \xi} \right), \tag{4.2a}
$$

$$
\tilde{\mathcal{L}}_{i,j} v = \sqrt{h\tau} \left(\frac{2}{\tau} \frac{\partial v}{\partial \eta} + \left(-\frac{4c}{h} I - \frac{2}{h} A_{i,j} \right) \frac{\partial v}{\partial \xi} \right) \text{ for } 1 < i < N, \tag{4.2b}
$$

and the other terms are similarly defined. Note that in (4.2a) we replace $J_{1,1}$ in (3.7a) by $h\tau$ and drop the lower order terms. It is easy to show that

$$
e \left(\sum_{i=1}^{N} \int_S |v^p_{i,j}|^2 \mathrm{d}\xi\, \mathrm{d}\eta + \int_{\partial S} |v^p_{i,j}|^2 \mathrm{d}s \right) \le \tilde{Q}_j(v^p_{1,j}, \ldots, v^p_{N,j}) \tag{4.3}
$$

for some constant e.

It can be shown as in Theorem 4.2 of [4] that

$$
\frac{1}{CN^2} \tilde{Q}_j(v^p_{1,j}, \ldots, v^p_{N,j}) \le H_j(v^p_{1,j}, \ldots, v^p_{N,j}) \le C \tilde{Q}_j(v^p_{1,j}, \ldots, v^p_{N,j}) \tag{4.4}
$$

for some constant C.

Let

$$
A^{(m)}_{i,j} = \sum_{\alpha_1, \alpha_2 \le m} D^{\alpha}_{\xi,\eta} A_{i,j}(0,0) \xi^{\alpha} \eta^{\alpha}, \tag{4.5a}
$$

$$
\mathcal{M}^{(0)}_j = \mathcal{M}_j(0), \tag{4.5b}
$$

and

$$
\mathcal{N}^{(0)}_j = \mathcal{N}_j(0). \tag{4.5c}
$$

Now

$$
\| A_{i,j} - A^{(m)}_{i,j} \|_{0,\infty,S} \le \frac{1}{N^{m+1}} \text{ for } m = 0, 1 \tag{4.6a}
$$

$$
\| \mathcal{M}_j - \mathcal{M}^{(0)}_j \|_{0,\infty,S} \le \frac{1}{N}, \tag{4.6b}
$$

and

$$\|\mathcal{N}_j - \mathcal{N}_j^{(0)}\|_{0,\infty,S} \leq \frac{1}{N}. \tag{4.6c}$$

We shall choose as our preconditioner the quadratic form

$$\begin{aligned}\tilde{Q}_j^{(m)}(v_{1,j}^p, \ldots, v_{N,j}^p) = \sum_{i=1}^{N} \int_S |\tilde{\mathcal{L}}_{i,j}^{(m)} v_{i,j}^p|^2 \mathrm{d}\xi\, \mathrm{d}\eta \\ + \cdots + \int_{(-1,1)} |\mathcal{M}_j^{(0)} v_{1,j}^p(-1,\eta)|^2 \mathrm{d}\eta \\ + \int_{(-1,1)} |\mathcal{N}_j^{(0)} v_{N,j}^p(1,\eta)|^2 \mathrm{d}\eta + \cdots\end{aligned} \tag{4.7}$$

for $m = 0, 1$. Here $\tilde{\mathcal{L}}_{i,j}^{(m)}$ is the differential operator $\tilde{\mathcal{L}}_{i,j}$ in (4.2a), (4.2b) in which the matrix $A_{i,j}$ is replaced by $A_{i,j}^{(m)}$.

Now using (4.6) we obtain

$$\sum_{i=1}^{N} \int_S |(\tilde{\mathcal{L}}_{i,j} - \tilde{\mathcal{L}}_{i,j}^{(m)}) v_{i,j}^p|^2 \mathrm{d}\xi\, \mathrm{d}\eta \leq \frac{Cp^4}{N^{2m+2}} \sum_{i=1}^{N} \|v_{i,j}^p\|_{0,S}^2, \text{ and}$$

$$\int_{(-1,1)} |(\mathcal{M}_j - \mathcal{M}_j^{(0)}) v_{1,j}^p(-1,\eta)|^2 \mathrm{d}\eta \leq \frac{C}{N^2} \|v_{i,j}^p\|_{0,\partial S}^2.$$

Hence it follows that

$$\begin{aligned}\tilde{Q}_j^{(m)}(v_{1,j}^p, \ldots, v_{N,j}^P) \leq 2\tilde{Q}_j(v_{1,j}^p, \ldots, v_{N,j}^p) \\ + \frac{Cp^4}{N^{2m+2}} \left(\sum_{i=1}^{N} \|v_{i,j}^p\|_{0,S}^2 \right) + \frac{C}{N^2} \left(\sum_{i=1}^{N} \|v_{i,j}^p\|_{0,\partial S}^2 \right).\end{aligned} \tag{4.8}$$

Now using (4.3) gives

$$\tilde{Q}_j^{(m)}(v_{1,j}^p, \ldots, v_{N,j}^p) \leq C \left(1 + \frac{p^4}{N^{2m+2}} + \frac{1}{N^2} \right) \tilde{Q}_j(v_{1,j}^p, \ldots, v_{N,j}^p). \tag{4.9}$$

In the same way it can be shown that

$$\tilde{Q}_j(v_{1,j}^p, \ldots, v_{N,j}^p) \leq C \left(1 + \frac{p^4}{N^{2m+2}} + \frac{1}{N^2} \right) \tilde{Q}_j^{(m)}(v_{1,j}^p, \ldots, v_{N,j}^p). \tag{4.10}$$

Hence we conclude that if $p = O(N^{1/2})$, then $\tilde{Q}_j(v_{1,j}^p, \ldots, v_{N,j}^p)$ and $\tilde{Q}_j^{(0)}(v_{1,j}^p, \ldots, v_{N,j}^p)$ are spectrally equivalent. If however, $p = O(N)$, then $\tilde{Q}_j(v_{1,j}^p, \ldots, v_{N,j}^p)$ and $\tilde{Q}_j^{(1)}(v_{1,j}^p, \ldots, v_{N,j}^p)$ are spectrally equivalent.

We now briefly indicate the advantage we obtain in replacing $\tilde{Q}_j(v^p_{1,j},\ldots,v^p_{N,j})$ by $\tilde{Q}^{(m)}_j(v^p_{1,j},\ldots,v^p_{N,j})$. As in §3.1.1. of [12] let us define the hierarchic shape functions to be

$$N_1(\alpha) = \frac{1-\alpha}{2},$$

$$N_2(\alpha) = \frac{1+\alpha}{2},$$

$$N_i(\alpha) = \frac{\sqrt{2i-3}}{2}\int_{-1}^{\alpha} L_{i-2}(t)\mathrm{d}t, \quad 3 \le i \le p.$$

Clearly

$$\int_{-1}^{1} \frac{\mathrm{d}N_i}{\mathrm{d}\alpha}\frac{\mathrm{d}N_j}{\mathrm{d}\alpha}\,\mathrm{d}\alpha = \delta_{i,j} \quad \text{for} \quad i,j \ge 3.$$

Moreover,

$$N_m(\alpha) = \frac{1}{\sqrt{2(2m-3)}}(L_{m-1}(\alpha) - L_{m-3}(\alpha)) \quad \text{for} \quad m \ge 3.$$

Finally, we have the recurrence relation

$$\left(\frac{2m+1}{m+1}\right)\alpha L_m(\alpha) = L_{m+1}(\alpha) + \left(\frac{m}{m+1}\right)L_{m-1}(\alpha).$$

We choose as our basis the tensor product of hierarchic shape functions arranged in lexicographic order. Thus, let $l = p(j-1)+i$. Define

$$\psi_l(\xi,\eta) = N_i(\xi)N_j(\eta).$$

Thus $\{\psi_1 e_r\}_r,\ldots,\{\psi_{p^2}e_r\}_r$ denotes the basis we have chosen. Here $e_l^T = (0,\ldots,1,\ldots,0)$ is the vector $\epsilon\ \mathbb{R}^k$ which has one in its l-th position and zero elsewhere. Let $\tilde{M}^{(m)}_j$ denote the matrix corresponding to the quadratic form $\tilde{Q}_j^{(m)}$. Now $\tilde{M}^{(m)}_j$ is a block diagonal matrix, where each block is a matrix of the order kp^2 and is inverted on a different processor. Let $\tilde{M}^{(m)}_{i,j}$ be the block which is inverted on the processor i. Then $\tilde{M}^{(m)}_{i,j}$ is essentially a banded block matrix consisting of p by p block matrices each of which is a kp by kp matrix. If we choose $m=0$ then the semi-bandwidth is 3 and if $m=1$ then the semi-bandwidth is 5. We should point out that there is some slight additional fill-in but the matrix is sparse and structured. Hence $\tilde{M}^{(m)}_{i,j}$ can be inverted on each processor in $O(p^4)$ operations and the action of its inverse on a vector can be computed in $O(p^3)$ operations and this is the number of operations required to compute the residual vector in the normal equations.

Finally, if we use $\tilde{Q}_j^{(m)}$ as a pre-conditioner then $O(N)$ iterations are required to reduce the error by an order of magnitude.

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