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Transferring strong boundedness among Laguerre orthogonal systems

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Abstract. Given the family of Laguerre polynomials, it is known that several orthonormal systems of Laguerre functions can be considered. In this paper we prove that an exhaustive knowledge of the boundedness in weighted L^p of the heat and Poisson semigroups, Riesz transforms and g-functions associated to a particular Laguerre orthonormal system of functions, implies a complete knowledge of the boundedness of the corresponding operators on the other Laguerre orthonormal system of functions. As a byproduct, new weighted L^p boundedness are obtained. The method also allows us to get new weighted estimates for operators related with Laguerre polynomials.

Keyword. Laguerre functions.

1. Introduction

In this paper we will deal with the Laguerre second order differential operator defined by

$$L_\alpha = -y\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{\mathrm{d}}{\mathrm{d}y} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y > 0, \tag{1.1}$$

where $\alpha > -1$. Thus L_α is a nonnegative and self-adjoint operator with respect to the Lebesgue measure on $(0, \infty)$. The Laguerre functions, $\mathcal{L}_k^\alpha$, are defined as

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\right)^{1/2} \mathrm{e}^{-y/2} y^{\alpha/2} L_k^\alpha(y), \tag{1.2}$$

where $\{L_k^\alpha\}_{k=0}^\infty$ are the Laguerre polynomials of type α, see p. 100 of [10] and p. 7 of [16]. These functions $\mathcal{L}_k^\alpha$ are eigenfunctions of L_α. In fact

$$L_\alpha(\mathcal{L}_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\mathcal{L}_k^\alpha. \tag{1.3}$$

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Since the Laguerre polynomials are orthogonal with respect to the measure $e^{-y}y^{\alpha}$, it follows that the family $\{\mathcal{L}_k^{\alpha}\}_k$ is orthonormal in $L^2((0,\infty), dy)$.

Besides the orthonormal system $\{\mathcal{L}_k^{\alpha}\}$, some other types of Laguerre functions orthonormal systems like $\{\varphi_k^{\alpha}\}$, $\{\ell_k^{\alpha}\}$ and $\{\psi_k^{\alpha}\}$ (section 2), have been considered previously (see for instance [13], [16], [9], [6], [7]).

The orthogonality of these systems, with respect to the corresponding measure, is an immediate consequence of the orthogonality of the Laguerre polynomials. Differential operators similar to the operator in (1.1) can be defined, in such a way that the functions in the orthonormal systems become the eigenfunctions of these operators, with the same eigenvalues as in (1.3). This allows to define, in a natural way, semigroups associated to these differential operators that are related through isometries of the L^2 Hilbert spaces corresponding to the different measures that intervene. Therefore these isometries establish relations among operators defined canonically from the semigroups, as for instance the maximal operator and the infinitesimal generator.

Given a factorization of the infinitesimal generator by first order differential operators, following Stein [8], a notion of derivative can be given. Through the isometries defined above, we can obtain factorizations of the infinitesimal generators of the semigroups we are dealing with. Thus operators arising from these notions of derivatives and the semigroup, as for instance Riesz transforms and g-functions, can be transferred from one system to another (see Proposition 3.25). Likewise, the L^2 results obtained for one orthonormal system can be transferred to another of the orthonormal systems under consideration.

The isometries relating the different orthonormal systems are 'also isometries' on L^p with respect to power weighted measures (see Lemma 3.22), and therefore the results in L^p transfer from system to system. Thus, we conclude that an exhaustive knowledge of the boundedness of the operators, in L^p associated to a particular Laguerre orthonormal system, implies a complete knowledge of the boundedness of the corresponding operators on the other Laguerre orthonormal system (see Theorem 3.27).

Among other results in [7], sufficient conditions on weights are given for the boundedness of the Riesz transforms associated to the Laguerre orthonormal system $\{\varphi_k^{\alpha}\}$, in the case $\alpha > -\frac{1}{2}$. Theorem 3.27 contains sufficient conditions that, in the case of Riesz transforms and maximal operators, are also necessary for power weights in the range $\alpha > -1$, for all the Laguerre orthonormal systems mentioned above.

There is also a natural isometry among Laguerre functions and Laguerre polynomials. However, this isometry does not preserve infinitesimal generators (see Lemma 4.30). In the case of the maximal operator we overcome the difficulties by means of a result stated in Lemma 4.38. When dealing with the Riesz transforms we use results that are obtained by transferring similar results for Hermite functions (see Lemma 4.36). With these ingredients, our method works well, allowing new estimates with weights, for the maximal operator and the Riesz transforms associated to Laguerre polynomials, for $\alpha > -1$ (see Theorem 4.32). We point out that the Riesz transforms for polynomials were studied in the case $\alpha \geq -1/2$ in [2] and [6].

2. Preliminaries

Following Stein, [8], given a second order, non negative and self-adjoint differential operator L, taking $T_t = e^{-tL}$, its heat semigroup, we can introduce

(i) Maximal operator: $T^* f(x) = \sup_{t>0} |T_t f(x)|$,
(ii) Maximal operator of the subordinated Poisson semigroup: $P^* f(x) = \sup_{t>0} |P_t f(x)|$, where P_t is defined by the subordination formula

$$P_t f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-t^2/4s} T_s f(x) s^{-3/2} ds,$$

(iii) Riesz potentials: $L^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} T_t f(x) dt$, for $0 < \sigma$, derived from the identity, $s^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-ts} dt$,
(iv) Littlewood–Paley g-function: $g(f)(x) = \left(\int_0^\infty \left|t \frac{\partial}{\partial t} T_t f(x)\right|^2 \frac{dt}{t}\right)^{1/2}$, and
(v) Riesz transforms: $R_j = \partial_j (L)^{-1/2}$, where by ∂_j we mean a kind of 'derivation' which appears in a factorization of L.

In the context of Laguerre functions $\{\mathcal{L}_k^\alpha\}_k$, $\alpha > -1$, in order to define the Riesz transforms, appropriate first order derivatives were introduced in [12], that is

$$D_\alpha = \sqrt{y} \frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\alpha}{\sqrt{y}}\right). \tag{2.4}$$

The actions on the corresponding Laguerre functions are given by

$$D_\alpha(\mathcal{L}_k^\alpha) = -\sqrt{k}\mathcal{L}_{k-1}^{\alpha+1} \quad \text{and} \quad (D_\alpha)^*(\mathcal{L}_k^{\alpha+1}) = -\sqrt{k+1}\mathcal{L}_{k+1}^\alpha, \tag{2.5}$$

where $(D_\alpha)^* = -\sqrt{y}\frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\alpha+1}{\sqrt{y}}\right)$, is the formal adjoint of D_α with respect to the Lebesgue measure. From these definitions it follows that

$$L_\alpha - \left(\frac{\alpha+1}{2}\right) = (D_\alpha)^* D_\alpha.$$

Accordingly, we can define the Riesz transforms for the Laguerre function expansions as

$$R_+^\alpha = D_\alpha (L_\alpha)^{-1/2}, \alpha > -1 \quad \text{and} \quad R_-^\beta = (D_{\beta-1})^* (L^\beta)^{-1/2}, \beta > 0. \tag{2.6}$$

Hence,

$$R_+^\alpha(\mathcal{L}_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k + \frac{\alpha+1}{2}}}\mathcal{L}_{k-1}^{\alpha+1} \quad \text{and} \quad R_-^\beta(\mathcal{L}_k^\beta) = -\frac{\sqrt{k+1}}{\sqrt{k + \frac{\beta+1}{2}}}\mathcal{L}_{k+1}^{\beta-1}.$$

Remark 2.7. The definition of R_-^β only for $\beta > 0$, can be argued firstly by observing that when applying the operator $(D\alpha)^*$ to the function $(L_\alpha)^{-1/2} L_0^\alpha = \left(\frac{\alpha+1}{2}\right)^{-1/2} \Gamma(\alpha+1)^{-1/2} e^{-y/2} y^{\alpha/2}$, we have

$$(D_\alpha)^*(L_\alpha)^{-1/2} L_0^\alpha = \left(\frac{\alpha+1}{2}\right)^{-1/2} \Gamma(\alpha+1)^{-1/2} \left(\sqrt{y} - \frac{\alpha+1}{\sqrt{y}}\right) e^{-y/2} y^{\alpha/2},$$

where the last function belongs to $L^2(dx)$ if and only if $\alpha > 0$.

Secondly, we recall that one of the main interests in studying Riesz transforms lies into their intimate connection with Sobolev spaces. It is easy to check that $R_-^{\alpha+1} \circ R_+^\alpha = T_m$, the multiplier operator associated with the sequence

$$m_k = \frac{k}{\sqrt{\left(k + \frac{\alpha+1}{2}\right)\left(k + \frac{\alpha}{2}\right)}}.$$

The boundedness of T_m and $T_{m^{-1}}$ in $L^2((0,\infty), \mathrm{d}x)$ is obvious and we may write

$$\begin{aligned} \|f\|_2 &= \|T_{m^{-1}} \circ R_-^{\alpha+1} \circ R_+^\alpha f\|_2 \le C\|R_+^\alpha f\|_2 \\ &= C\|D_\alpha (L_\alpha)^{-1/2} f\|_2 \le C\|f\|_2. \end{aligned}$$

Therefore

$$\|D_\alpha f\|_2 \sim \|(L_\alpha)^{1/2} f\|_2.$$

This equivalence, jointly with its analogous for $1 < p < \infty$, are the keys to define the Sobolev spaces associated to this Laplacian in terms of the derivative $D_\alpha$and only R_-^β, $\beta > 0$, intervene.

Consequently, it is natural to introduce the Riesz transform vector, $\mathcal{R}^\alpha$, associated to L_α as in [4]

$$\mathcal{R}^\alpha = (R_+^\alpha, R_-^{\alpha+1}) = (D_\alpha (L_\alpha)^{-1/2}, (D_\alpha)^* (L_{\alpha+1})^{-1/2}).$$

Moreover, in [4] it is proved.

Theorem 2.8 (Riesz transforms theorem). *Let $\alpha > -1$, $1 < p < \infty$ and δ be real numbers. Assume that $-\frac{\alpha}{2}p - 1 < \delta < p - 1 + \frac{\alpha}{2}p$. Then the operator $\|\mathcal{R}^\alpha\|$ is defined as*

$$\|\mathcal{R}^\alpha\|(f) = (|R_+^\alpha f|^2 + |R_-^{\alpha+1} f|^2)^{1/2},$$

which maps $L^p(y^\delta \mathrm{d}y)$ boundedly into $L^p(y^\delta \mathrm{d}y)$.

We shall also need the following results that can be found in [5].

Theorem 2.9 (Multiplier theorem). *Let $-1 < \alpha$, $1 < p < \infty$ and $m \in C^\infty[0,\infty)$, such that*

$$|D^\ell m(\xi)| \le C_\ell (1+\xi)^{-\ell}, \quad \xi \ge 0, \ell = 0, 1, 2 \dots. \tag{2.10}$$

Consider the operator $T_m f = \sum_{k\ge 0} m(k) < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\alpha$, defined at least for $f \in L^2((0,\infty), \mathrm{d}y)$. Then T_m admits a bounded extension to $L^p((0,\infty), y^\delta \mathrm{d}y)$ whenever $-\frac{\alpha}{2}p - 1 < \delta < p - 1 + \frac{\alpha}{2}p$.

The *transplantation operators*, for $\alpha, \beta > -1$ and $f \in L^2(\mathrm{d}y)$ are defined by

$$T_\beta^\alpha f = \sum_{k=0}^{\infty} < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\beta.$$

The following theorem is an important result regarding these operators.

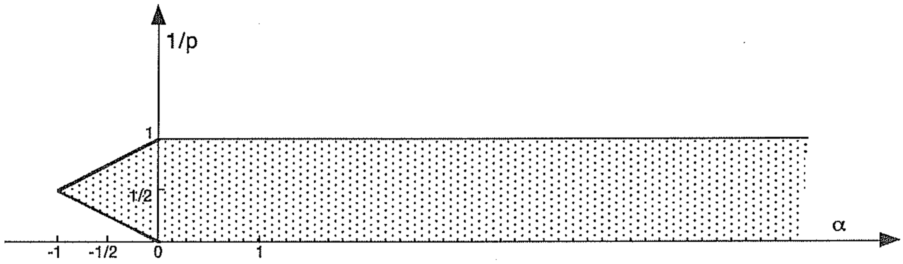
Theorem 2.11 (Transplantation theorem). *Let $-1 < \alpha < \beta$ and $1 < p < \infty$. Then the operators T^{α}_{β} and T^{β}_{α} admit a bounded extension to $L^p((0,\infty), y^{\delta}\mathrm{d}y)$ if and only if $-\frac{\alpha}{2}p - 1 < \delta < p - 1 + \frac{\alpha}{2}p$.*

For a proof of Theorem 2.9, see [5]. A weaker version was given by Thangavelu in [16]. Theorem 2.11 was proved by Kanjin, when $\delta = 0$, in [3]. For multiple Laguerre expansions, a result is due to Thangavelu (see [17]). In the weighted case, a more restricted version of Theorem 2.11 was proved by Stempak and Trebels (see [15]). Theorem 2.11 as stated was proved in [5].

Remark 2.12. In the unweighted case, $\delta = 0$, the restriction in p that appears in Theorems 2.8, 2.9 and 2.11 can be rewritten as

$$-\frac{\alpha}{2} - \frac{1}{p} < 0 < 1 - \frac{1}{p} + \frac{\alpha}{2}.$$

Then the region $\left(\alpha, \frac{1}{p}\right)$, for α and p satisfying the above conditions, can be visualized as



For obvious reasons we call this situation '*pencil phenomenon*' for the system of the Laguerre functions $\{\mathcal{L}^{\alpha}_k\}$.

As it was announced in the Introduction, in addition to the $\mathcal{L}^{\alpha}_k$ system we shall deal with other orthonormal systems closely related with it.

The Laguerre functions $\{\varphi^{\alpha}_k\}_{k=0}^{\infty}$, $\alpha > -1$.

We consider the orthonormal system in $L^2((0,\infty), \mathrm{d}y)$ given by

$$\varphi^{\alpha}_k(y) = \mathcal{L}^{\alpha}_k(y^2)(2y)^{1/2}, \tag{2.13}$$

where $\mathcal{L}^{\alpha}_k$ are the functions defined in (1.2). The functions φ^{α}_k are eigenfunctions of the operator

$$\mathbf{L}_{\alpha} = \frac{1}{4}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}.$$

In fact,

$$\mathbf{L}_{\alpha}(\varphi^{\alpha}_k) = \left(k + \frac{\alpha+1}{2}\right)\varphi^{\alpha}_k. \tag{2.14}$$

The operator $\mathbf{L}_{\alpha}$ can be 'factorized' as

$$\mathbf{L}_{\alpha} - \left(\frac{\alpha+1}{2}\right) = (\mathbf{D}_{\alpha})^* \mathbf{D}_{\alpha},$$

being $\mathbf{D}_\alpha = \frac{1}{2}\big\{\frac{\mathrm{d}}{\mathrm{d}y} + y - \frac{1}{y}\big(\alpha + \frac{1}{2}\big)\big\}$ and $\mathbf{D}_\alpha{}^* = \frac{1}{2}\big\{-\frac{\mathrm{d}}{\mathrm{d}y} + y - \frac{1}{y}\big(\alpha + \frac{1}{2}\big)\big\}$, where $\mathbf{D}_\alpha{}^*$ is the formal adjoint of $\mathbf{D}_\alpha$ with respect to the Lebesgue measure. Then

$$\mathbf{D}_\alpha(\varphi_k^\alpha) = -\sqrt{k}\varphi_{k-1}^{\alpha+1} \quad \text{and} \quad (\mathbf{D}_{\beta-\mathbf{1}})^*(\varphi_k^\beta) = -\sqrt{k+1}\varphi_{k+1}^{\beta-1}. \tag{2.15}$$

According to [12] the Riesz transforms can be defined as

$$\mathbf{R}_+^\alpha = \mathbf{D}_\alpha(\mathbf{L}_\alpha)^{-1/2}, \alpha > -1 \quad \text{and} \quad \mathbf{R}_-^\beta = (\mathbf{D}_{\beta-\mathbf{1}})^*(\mathbf{L}_\beta)^{-1/2}, \beta > 0. \tag{2.16}$$

Hence, $\mathbf{R}_+^\alpha(\varphi_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\varphi_{k-1}^{\alpha+1}$ and $\mathbf{R}_-^\beta(\varphi_k^\beta) = -\frac{\sqrt{k+1}}{\sqrt{k+\frac{\beta+1}{2}}}\varphi_{k+1}^{\beta-1}$. Then, the Riesz transform vector is

$$(\mathbf{R}_+^\alpha, \mathbf{R}_-^{\alpha+1}) = (\mathbf{D}_\alpha(\mathbf{L}_\alpha)^{-1/2}, (\mathbf{D}_\alpha)^*(\mathbf{L}_{\alpha+\mathbf{1}})^{-1/2}).$$

The Laguerre functions ℓ_k^α, $\alpha > -1$.

The orthonormal system, $\{\ell_k^\alpha\}_{k=0}^\infty$ in $L^2((0,\infty), \mathrm{d}\mu_\alpha(y))$, $\mathrm{d}\mu_\alpha(y) = y^\alpha \mathrm{d}y$ is given by

$$\ell_k^\alpha(y) = \mathcal{L}_k^\alpha(y) y^{-\alpha/2},$$

where $\mathcal{L}_k^\alpha$ are the functions defined in (1.2). The functions ℓ_k^α are eigenfunctions of the differential operator

$$\mathbb{L}_\alpha = -y\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha+1)\frac{\mathrm{d}}{\mathrm{d}y} + \frac{y}{4}.$$

More explicitly

$$\mathbb{L}_\alpha \ell_k^\alpha = \left(k + \frac{\alpha+1}{2}\right)\ell_k^\alpha. \tag{2.17}$$

The operator $\mathbb{L}_\alpha$ can be 'factorized' as

$$\mathbb{L}_\alpha - \left(\frac{\alpha+1}{2}\right) = (\mathbb{D}_\alpha)^*\mathbb{D}_\alpha,$$

where $\mathbb{D}_\alpha = \sqrt{y}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}\sqrt{y}$ and $(\mathbb{D}_\alpha)^* = -\sqrt{y}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}\sqrt{y} - \frac{\alpha}{\sqrt{y}} - \frac{1}{2\sqrt{y}}$ is the formal adjoint of $\mathbb{D}_\alpha$ with respect to the measure $\mathrm{d}\mu_\alpha$. Furthermore,

$$\mathbb{D}_\alpha \ell_k^\alpha(y) = -\sqrt{k}\sqrt{y}\ell_{k-1}^{\alpha+1}(y). \tag{2.18}$$

It is easy to check that $(\mathbb{D}_\alpha)^*(\sqrt{(\cdot)}\ell_{k-1}^{\alpha+1}(\cdot))$ is a function in $L^2(\mathrm{d}\mu_\alpha)$, namely, $(\mathbb{D}_\alpha)^*(\sqrt{(\cdot)}\ell_{k-1}^{\alpha+1}(\cdot))(y) = -\sqrt{k}\ell_k^\alpha(y)$. On the other hand, the family $\{\sqrt{y}\ell_k^{\alpha+1}(y)\}_k$ is orthonormal with respect to the measure $\mathrm{d}\mu_\alpha$. Clearly, $\mathbb{D}_\alpha(\mathbb{D}_\alpha)^*(\sqrt{(\cdot)}\ell_k^{\alpha+1}(\cdot))(y) = (k+1)\sqrt{y}\ell_k^{\alpha+1}(y)$. In this situation the Riesz transform vector becomes

$$(\mathbb{R}_+^\alpha, \mathbb{R}_-^{\alpha+1}) = \left(\mathbb{D}_\alpha(\mathbb{L}_\alpha)^{-1/2}, (\mathbb{D}_\alpha)^*\left[\mathbb{D}_\alpha(\mathbb{D}_\alpha)^* + \frac{\alpha}{2}\right]^{-1/2}\right).$$

Hence, $\mathbb{R}_+^\alpha(\ell_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\sqrt{y}\ell_{k-1}^{\alpha+1}(y)$ and $\mathbb{R}_-^{\alpha+1}(\sqrt{(\cdot)}\ell_k^{\alpha+1}(\cdot))(y) = -\frac{\sqrt{k+1}}{(\sqrt{k+1+\frac{\alpha}{2}})}$ $\ell_{k+1}^\alpha(y)$.

The Laguerre functions $\{\psi_k^\alpha\}_k$, $\alpha > -1$.

Let $\{\psi_k^\alpha\}_{k=0}^\infty$ be the orthonormal system, in $L^2((0,\infty), y^{2\alpha+1}\mathrm{d}y)$, given by $\psi_k^\alpha(y) = \sqrt{2}y^{-\alpha}\mathcal{L}_k^\alpha(y^2)$, where $\mathcal{L}_k^\alpha(y)$ are the functions defined in (1.2). The functions ψ_k^α are eigenfunctions for the operator $\mathfrak{L}_\alpha = -\frac{1}{4}\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{2\alpha+1}{y}\right)\frac{\mathrm{d}}{\mathrm{d}y} - y^2\right\}$, in effect

$$\mathfrak{L}_\alpha(\psi_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\psi_k^\alpha. \tag{2.19}$$

Furthermore, the operator $\mathfrak{L}_\alpha$ can be ‘factorized’ as

$$\mathfrak{L}_\alpha - \left(\frac{\alpha+1}{2}\right) = (\mathfrak{D}_\alpha)^*\mathfrak{D}_\alpha,$$

with $\mathfrak{D}_\alpha = \frac{1}{2}\left\{\frac{\mathrm{d}}{\mathrm{d}y} + y\right\}$ and $(\mathfrak{D}_\alpha)^* = -\frac{1}{2}\left\{\frac{\mathrm{d}}{\mathrm{d}y} + \frac{(2\alpha+1)}{y} - y\right\}$, $(\mathfrak{D}_\alpha)^*$ turns out to be the adjoint of $\mathfrak{D}_\alpha$ with respect to the measure $\mathrm{d}\omega_\alpha(y) = y^{2\alpha+1}\mathrm{d}y$. Because of this

$$\mathfrak{D}_\alpha(\psi_k^\alpha)(y) = -\sqrt{k}y\psi_{k-1}^{\alpha+1}(y) \quad \text{and} \quad (\mathfrak{D}_\alpha)^*((\cdot)\psi_{k-1}^{\alpha+1}(\cdot))(y) = -\sqrt{k}\psi_k^\alpha(y). \tag{2.20}$$

As a consequence, the Riesz transforms become

$$\mathfrak{R}_+^\alpha = \mathfrak{D}_\alpha(\mathfrak{L}_\alpha)^{-1/2} \quad \text{and} \quad \mathfrak{R}_-^\alpha = (\mathfrak{D}_\alpha)^*\left[\mathfrak{D}_\alpha(\mathfrak{D}_\alpha)^* + \frac{\alpha}{2}\right]^{-1/2}.$$

Hence,

$$\mathfrak{R}_+^\alpha(\psi_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\sqrt{y}\psi_{k-1}^{\alpha+1}(y) \quad \text{and}$$

$$\mathfrak{R}_-^{\alpha+1}(\sqrt{(\cdot)}\psi_k^{\alpha+1}(\cdot))(y) = -\frac{\sqrt{k+1}}{\sqrt{k+1+\frac{\alpha}{2}}}\psi_{k+1}^\alpha(y).$$

Notation. The following typographical convention will be used. The font-type of a letter, T, $\mathbf{T}$, $\mathbb{T}$ and $\mathfrak{T}$, will identify the orthonormal system under consideration, namely $\{\mathcal{L}_k^\alpha\}$, $\{\varphi_k^\alpha\}$, $\{\ell_k^\alpha\}$ and $\{\psi_k^\alpha\}$, respectively. For instance, L_α, $\mathbf{L}_\alpha$, $\mathbb{L}_\alpha$, $\mathfrak{L}_\alpha$, are the corresponding differential operators. Lemma 2.21 belongs to the folklore of the subject and it is not difficult to obtain a proof by means of the arguments in Theorem 5.7.1 of [10].

Lemma 2.21. *Let* $\alpha > -1$, $1 < p < \infty$, δ, γ *and* ρ *be real numbers*, $\mathrm{d}\mu_\alpha(y) = y^\alpha\mathrm{d}y$ *and* $\mathrm{d}\omega_\alpha(y) = y^{2\alpha+1}\mathrm{d}y$.

(i) *The functions* $\{\mathcal{L}_k^\alpha\}_k$ *are in* $L^p((0,\infty), y^\delta\mathrm{d}y) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\delta}\mathrm{d}y)$, *if and only if*, $-1-\alpha\frac{p}{2} < \delta < \alpha\frac{p}{2} + (p-1)$. *Moreover the set* S_α *of finite linear combinations of Laguerre functions,* $\{\mathcal{L}_k^\alpha\}_k$, *is dense in* $L^p((0,\infty), y^\delta\mathrm{d}y)$ *and* $L^{p'}((0,\infty), y^{-\frac{p'}{p}\delta}\mathrm{d}y)$.

(ii) *The functions* $\{\varphi_k^\alpha\}_k$ *are in* $L^p((0,\infty), y^\gamma \mathrm{d}y) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\gamma}\mathrm{d}y)$, *if and only if,* $-1-\alpha p-\frac{p}{2} < \gamma < \alpha p+\frac{p}{2}+(p-1)$. *Besides the set* $\mathbf{S}_\alpha$ *of finite linear combinations of Laguerre functions* $\{\varphi_k^\alpha\}_k$ *is dense in* $L^p((0,\infty), y^\gamma \mathrm{d}y)$ *and* $L^{p'}((0,\infty), y^{-\frac{p'}{p}\gamma}\mathrm{d}y)$.

(iii) *The functions* $\{\ell_k^\alpha\}_k$ *are in* $L^p((0,\infty), y^\rho \mathrm{d}\mu_\alpha) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\rho}\mathrm{d}\mu_\alpha)$, *if and only if,* $-1-\alpha < \rho < (\alpha+1)(p-1)$. *Furthermore the set* $\mathbb{S}_\alpha$ *of finite linear combinations of Laguerre functions* $\{\ell_k^\alpha\}_k$ *is dense in* $L^p((0,\infty), y^\rho \mathrm{d}y)$ *and* $L^{p'}((0,\infty), y^{-\frac{p'}{p}\rho}\mathrm{d}y)$.

(iv) *The functions* $\{\psi_k^\alpha\}_k$ *are in* $L^p((0,\infty), y^\eta \mathrm{d}\omega_\alpha(y)) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\eta}\mathrm{d}\omega_\alpha(y))$, *if and only if,* $-2(1+\alpha) < \eta < 2(p-1)(\alpha+1)$. *In addition, the set* $\mathfrak{S}_\alpha$ *of finite linear combinations of Laguerre functions* $\{\psi_k^\alpha\}_k$ *is dense in* $L^p((0,\infty), y^\eta \mathrm{d}\omega_\alpha)$ *and* $L^{p'}((0,\infty), y^{-\frac{p'}{p}\eta}\mathrm{d}\omega_\alpha(y))$.

3. The isometries V, W^α and Z^α connecting the different Laguerre function systems

Let V, W^α and Z^α be the operators defined by

$$Vf(y) = (2y)^{1/2} f(y^2), \quad W^\alpha f(y) = y^{-\frac{\alpha}{2}} f(y) \quad \text{and} \quad Z^\alpha f(y) = \sqrt{2} y^{-\alpha} f(y^2),$$

for f a measurable function with domain on $(0,\infty)$. For further reference we state the following lemma, whose simple proof is left to the reader.

Lemma 3.22. *Let* $\alpha > -1$.

(i) *Let* $2\delta = \gamma + \frac{p}{2} - 1$, *then* $\|Vf\|_{L^p(y^\gamma \mathrm{d}y)} = 2^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(y^\delta \mathrm{d}y)}$.

(ii) *Let* $\delta = \rho - \alpha\big(\frac{p}{2}-1\big)$, *then* $\|W^\alpha f\|_{L^p(y^\rho \mathrm{d}\mu_\alpha)} = \|f\|_{L^p(y^\delta \mathrm{d}y)}$, *where* $\mathrm{d}\mu_\alpha(y) = y^\alpha \mathrm{d}y$.

(iii) *Let* $\delta = \frac{\eta}{2} - \alpha\big(\frac{p}{2}-1\big)$, *then* $\|Z^\alpha f\|_{L^p(y^\eta \mathrm{d}\omega_\alpha)} = 2^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(y^\delta \mathrm{d}y)}$, *where* $\mathrm{d}\omega_\alpha(y) = y^{2\alpha+1}\mathrm{d}y$.

Remark 3.23. Given a Banach space B, and a strongly measurable B-valued function f, we can define the operators

$$V_B f(y) = (2y)^{1/2} f(y^2), \quad W_B^\alpha f(y) = y^{-\alpha/2} f(y) \quad \text{and} \quad Z_B^\alpha f(y) = 2y^{-\alpha} f(y^2).$$

Hence

$$\|V_B f(y)\|_B = V(\|f\|_B)(y), \|W_B^\alpha f(y)\|_B = W^\alpha(\|f\|_B)(y) \quad \text{and}$$
$$\|Z_B^\alpha f(y)\|_B = Z^\alpha(\|f\|_B)(y).$$

Therefore, under the conditions of Lemma 3.22, for any Banach space B the identities

$$\|V_B f\|_{L_B^p(y^\gamma \mathrm{d}y)} = 2^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L_B^p(y^\delta \mathrm{d}y)}, \quad \|W_B^\alpha f\|_{L_B^p(y^\rho \mathrm{d}\mu_\alpha)} = \|f\|_{L_B^p(y^\delta \mathrm{d}y)} \quad \text{and}$$
$$\|Z_B^\alpha f\|_{L_B^p(y^\eta \mathrm{d}\omega_\alpha)} = 2^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L_B^p(y^\delta \mathrm{d}y)}$$

hold. Moreover if $B = \ell^\infty$ and $f = (f_j)_j$ is a strongly measurable ℓ^∞-valued function then

$$V_{\ell^\infty} f = (V f_j)_j, \quad W^\alpha_{\ell^\infty} f = (W^\alpha f_j)_j \quad \text{and} \quad Z^\alpha_{\ell^\infty} f = (Z^\alpha f_j)_j.$$

Analogously, if $B = L^2(\Omega, \mathrm{d}\mu)$ and $f(\cdot) = g(\cdot, z), z \in \Omega$, is a $L^2(\Omega, \mathrm{d}\mu)$-valued function it follows that

$$V_{L^2(\Omega,\mathrm{d}\mu)} f(y) = V g(y, z) = g(y^2, z)(2y)^{1/2},$$

$$W^\alpha_{L^2(\Omega,\mathrm{d}\mu)} f(y) = W^\alpha g(y, z) = g(y, z) y^{-\alpha/2} \quad \text{and}$$

$$Z^\alpha_{L^2(\Omega,\mathrm{d}\omega)} f(y) = Z^\alpha g(y, z) = \sqrt{2} g(y^2, z) y^{-\alpha}.$$

PROPOSITION 3.24

Let $1 < p < \infty, \delta, \gamma$ and ρ, η be real numbers. Let B_1, B_2 be Banach spaces and T be an operator defined over the set of finite linear combination of Laguerre functions $\{\mathcal{L}^\alpha_k\}_k$.

(i) *The operator T has a bounded extension from $L^p_{B_1}((0, \infty), y^\delta \mathrm{d}y)$ into $L^p_{B_2}((0, \infty), y^\delta \mathrm{d}y)$ if and only if the operator $\mathbf{T} = V_{B_2} T V^{-1}_{B_1}$ has a bounded extension from $L^p_{B_1}((0, \infty), y^\gamma \mathrm{d}y)$ into $L^p_{B_2}((0, \infty), y^\gamma \mathrm{d}y)$, where $2\delta = \gamma + \frac{p}{2} - 1$.*

(ii) *The operator T has a bounded extension from $L^p_{B_1}(y^\delta \mathrm{d}y)$ into $L^p_{B_2}(y^\delta \mathrm{d}y)$ if and only if the operator $\mathbb{T} = W^\alpha_{B_2} T (W^\alpha_{B_1})^{-1}$ has a bounded extension from $L^p_{B_1}(y^\rho \mathrm{d}\mu_\alpha(y)$ into $L^p_{B_2}(y^\rho \mathrm{d}\mu_\alpha(y))$, with $\delta = \rho - \alpha\left(\frac{p}{2} - 1\right)$ and $\mathrm{d}\mu_\alpha(y) = y^\alpha \mathrm{d}y$.*

(iii) *The operator T has a bounded extension from $L^p_{B_1}(y^\delta \mathrm{d}y)$ into $L^p_{B_2}(y^\delta \mathrm{d}y)$ if and only if the operator $\mathfrak{T} = Z^\alpha_{B_2} T (Z^\alpha_{B_1})^{-1}$ has a bounded extension from $L^p_{B_1}(y^\eta \mathrm{d}\omega_\alpha(y)$ into $L^p_{B_2}(y^\eta \mathrm{d}\omega_\alpha(y))$, if $\delta = \frac{\eta}{2} - \alpha\left(\frac{p}{2} - 1\right)$ and $\mathrm{d}\omega_\alpha(y) = y^{2\alpha+1} \mathrm{d}y$.*

Furthermore, the norms of the operators T, $\mathbf{T}$, $\mathbb{T}$ and $\mathfrak{T}$ coincide.

Proof. Let f be a finite linear combination of the Laguerre functions φ^α_k. Then $V^{-1} f$ is a finite linear combination of Laguerre functions $\mathcal{L}^\alpha_k$. By using the relation between the operators $\mathbf{T}$ and T and appropriate changes of variables, it follows

$$\begin{aligned}
\int_0^\infty \|\mathbf{T} f(y)\|^p_{B_2} y^\gamma \mathrm{d}y &= \int_0^\infty \|T V^{-1}_{B_1} f(y^2)\|^p_{B_2} (2y)^{p/2} y^\gamma \mathrm{d}y \\
&= \int_0^\infty \|T V^{-1}_{B_1} f(u)\|^p_{B_2} 2^{p/2} u^{p/4} u^{\gamma/2} \frac{1}{2} u^{-1/2} \mathrm{d}u \\
&= 2^{p/2-1} \int_0^\infty \|T V^{-1}_{B_1} f(u)\|^p_{B_2} u^\delta \mathrm{d}u \\
&\le 2^{p/2-1} \|T\|^p \int_0^\infty \|V^{-1}_{B_1} f(u)\|^p_{B_1} u^\delta \mathrm{d}u
\end{aligned}$$

$$= 2^{p/2-1}\|T\|^p \int_0^\infty \|f(u^{1/2})\|_{B_1}^p (2u^{1/2})^{-p/2} u^\delta \mathrm{d}u$$
$$= \|T\|^p \int_0^\infty \|f(y)\|_{B_1}^p y^{-p/2} y^{2\delta} y\mathrm{d}y = \|T\|^p \int_0^\infty \|f(y)\|_{B_1}^p y^\gamma \mathrm{d}y.$$

Then Lemma 2.21 gives (i). The proof of (ii) and (iii) are analogous. ■

Observe that as a byproduct of this proof we get

$$\begin{aligned}\|T\|_{\mathcal{L}(L^p(y^\delta,\mathrm{d}y),L^p(y^\delta,\mathrm{d}y))} &= \|\mathbf{T}\|_{\mathcal{L}(L^p(y^\gamma,\mathrm{d}y),L^p(y^\gamma,\mathrm{d}y))}\\ &= \|\mathbb{T}\|_{\mathcal{L}(L^p(y^\rho\mathrm{d}\mu_\alpha),L^p(y^\rho\mathrm{d}\mu_\alpha))}\\ &= \|\mathfrak{T}\|_{\mathcal{L}(L^p(y^\eta\mathrm{d}\omega_\alpha),L^p(y^\eta\mathrm{d}\omega_\alpha))}.\end{aligned}$$

The following Proposition shows how the operators, defined at the beginning of §2, for different Laguerre function systems are related by means of the isometries V, W^α, Z^α.

PROPOSITION 3.25

Let $\alpha > -1$ and let f be a finite linear combination of Laguerre functions $\{\mathcal{L}_k^\alpha\}$. Therefore

(i) $\mathrm{e}^{-tL_\alpha} f = V^{-1}\mathrm{e}^{-t\mathbf{L}_\alpha} Vf = (W^\alpha)^{-1}\mathrm{e}^{-t\mathbb{L}_\alpha} W^\alpha f = (Z^\alpha)^{-1}\mathrm{e}^{-t\mathfrak{L}_\alpha} Z^\alpha f$,
(ii) $\sup_t \mathrm{e}^{-tL_\alpha} f(y) = V^{-1}\sup_t \mathrm{e}^{-t\mathbf{L}_\alpha} Vf(y) = (W^\alpha)^{-1}\sup_t \mathrm{e}^{-t\mathbb{L}_\alpha} W^\alpha f(y) = (Z^\alpha)^{-1}\sup_t \mathrm{e}^{-t\mathfrak{L}_\alpha} Z^\alpha f(y)$,
(iii) *Given* $s > 0$, $(L_\alpha)^{-s} f = V^{-1}(\mathbf{L}_\alpha)^{-s} Vf = (W^\alpha)^{-1}(\mathbb{L}_\alpha)^{-s} W^\alpha f = (Z^\alpha)^{-1}(\mathfrak{L}_\alpha)^{-s} Z^\alpha f$,
(iv) $D_\alpha f = V^{-1}\mathbf{D}_\alpha Vf = (W^\alpha)^{-1}\mathbb{D}_\alpha W^\alpha f = (Z^\alpha)^{-1}\mathfrak{D}_\alpha Z^\alpha f$.
(v) $R_+^\alpha f = V^{-1}\mathbf{R}_+^\alpha Vf = (W^\alpha)^{-1}\mathbb{R}_+^\alpha W^\alpha f = (Z^\alpha)^{-1}\mathfrak{R}_+^\alpha Z^\alpha f$ *and*
(vi) $\frac{\partial}{\partial t}\mathrm{e}^{-t\sqrt{L_\alpha}} f = V^{-1}\frac{\partial}{\partial t}\mathrm{e}^{-t\sqrt{\mathbf{L}_\alpha}} Vf = (W^\alpha)^{-1}\frac{\partial}{\partial t}\mathrm{e}^{-t\sqrt{\mathbb{L}_\alpha}} W^\alpha f, = (Z^\alpha)^{-1}\frac{\partial}{\partial t}\mathrm{e}^{-t\sqrt{\mathfrak{L}_\alpha}} Z^\alpha f.$

Besides if g is a finite combination of Laguerre functions $\{\mathcal{L}_k^{\alpha+1}\}_k$, then

$$R_-^{\alpha+1} g = V^{-1}\mathbf{R}_-^{\alpha+1} Vf = (W^\alpha)^{-1}\mathbb{R}_-^{\alpha+1} W^\alpha f = (Z^\alpha)^{-1}\mathfrak{R}_-^{\alpha+1} Z^\alpha f.$$

Proof. Observe that $\mathcal{L}_k^\alpha = V^{-1}\varphi_k^\alpha = (W^\alpha)^{-1}\ell_k^\alpha = (Z^\alpha)^{-1}\psi_k^\alpha$. Hence, in order to prove (i), (ii) and (iii) we just use (1.3), (2.14), (2.17) and (2.19). On the other hand, $\mathcal{L}_{k-1}^{\alpha+1} = (W^\alpha)^{-1}(\sqrt{(\cdot)}\ell_{k-1}^{\alpha+1}(\cdot)) = (Z^\alpha)^{-1}((\cdot)\psi_{k-1}^{\alpha+1}(\cdot))$. Thus (2.5), (2.15), (2.18),(2.20) give (iv), (v) and (vi). ■

Theorem 3.26. *Let $\alpha > -1$, $1 < p < \infty$, δ, ρ, η and γ be real numbers, such that $2\delta = \gamma + \frac{p}{2} - 1$, $\delta = \rho + \alpha - \frac{\alpha p}{2}$, and $\delta = \frac{\eta}{2} - \frac{\alpha p}{2} + \alpha$. Let S stand for any of the operators e^{-tL_α}, T^*, P^*, g_α, R_+^α, $R_-^{\alpha+1}$ defined in §2. Then the following conditions are equivalent*:

(i) *The operator S has a bounded extension from $L^p((0,\infty), y^\delta\mathrm{d}y)$ into itself.*
(ii) *The operator $\mathbf{S}$ has a bounded extension from $L^p((0,\infty), y^\gamma\mathrm{d}y)$ into itself.*
(iii) *The operator $\mathbb{S}$ has a bounded extension from $L^p((0,\infty), y^\rho\mathrm{d}\mu_\alpha)$ into itself.*
(iv) *The operator $\mathfrak{S}$ has a bounded extension from $L^p((0,\infty), y^\eta\mathrm{d}\omega_\alpha)$ into itself.*

In addition, the norms of the operators S, $\mathbf{S}$, $\mathbb{S}$ and $\mathfrak{S}$ coincide.

Proof. If S is any of the operators e^{-tL_α}, T^*, P^*, $L^{-1/2}$, R_+^α, R_-^α, the proof follows directly from Propositions 3.25 and 3.24. When $S = g_\alpha$, we observe that S is bounded from

$L^p((0,\infty), y^\delta \mathrm{d}y)$ into $L^p((0,\infty), y^\delta \mathrm{d}y)$, if and only if the operator $f \to \frac{\partial}{\partial t}\mathrm{e}^{-t\sqrt{L_\alpha}} f$ is bounded from $L^p((0,\infty), y^\delta \mathrm{d}y)$ into $L^p((0,\infty), y^\delta \mathrm{d}y)_{L^2\left((0,\infty),\frac{\mathrm{d}t}{t}\right)}$. Consequently, the result follows by using (vi) in Propositions 3.25 and 3.24. ■

The above theorem shows the equivalence of the boundedness of the corresponding operators for the different systems of Laguerre functions. In the next theorem, we prove that they are actually bounded.

Theorem 3.27. *Let $\alpha > -1$, $1 < p < \infty$, δ, ρ, η and γ be real numbers. Let S be any one of the operators e^{-tL_α}, T^*, P^*, g_α, R_+^α, $R_-^{\alpha+1}$ and L^{-s}, $s > 0$. Therefore*

(i) *The operator S has a bounded extension from $L^p((0,\infty), y^\delta \mathrm{d}y)$ into itself, for δ such that*

$$-1 - \frac{\alpha p}{2} < \delta < \frac{\alpha p}{2} + p - 1.$$

(ii) *The operator $\mathbf{S}$ has a bounded extension from $L^p((0,\infty), y^\gamma \mathrm{d}y)$ into itself, for γ satisfying*

$$-1 - \alpha p - \frac{p}{2} < \gamma < \alpha p + \frac{3p}{2} - 1.$$

(iii) *The operator $\mathbb{S}$ has a bounded extension from $L^p((0,\infty), y^\rho \mathrm{d}\mu_\alpha)$ into itself, for ρ in the range*

$$-1 - \alpha < \rho < (\alpha + 1)(p - 1).$$

(iv) *The operator $\mathfrak{S}$ has a bounded extension from $L^p((0,\infty), y^\eta \mathrm{d}\omega_\alpha)$ into itself, for η*

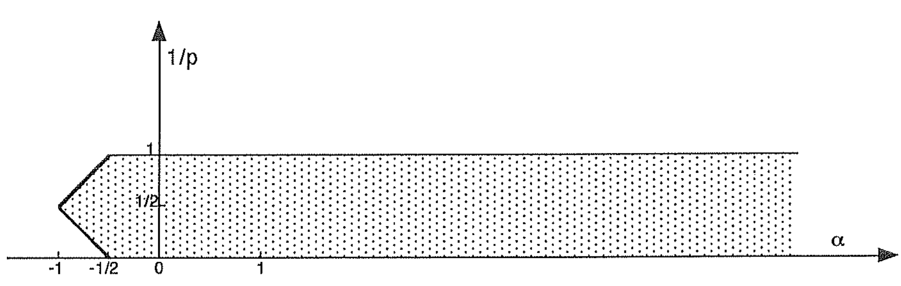
$$-2(1 + \alpha) < \eta < 2(p - 1)(\alpha + 1).$$

Proof. The boundedness of the maximal operator, the Riesz transform and the g-function in case (i) was proved in [1], [4], [5], respectively. The boundedness of L^{-s} is due to B Bongioanni (oral communication). The theorem follows by applying Theorem 3.26.

Remark 3.28. We observe that in the case $\mathbf{S}$, that is for the system $\{\varphi_k^\alpha\}$, and $\gamma = 0$, we get the condition

$$-\alpha - \frac{1}{2} < \frac{1}{p} < \alpha + \frac{3}{2}.$$

This restriction on α and $\frac{1}{p}$ can be visualized as the shaded region in the following picture



However, in the cases $\mathbb{S}$, $\mathfrak{S}$ and $\rho = 0$, $\eta = 0$ we get

$$-(1+\alpha) < 0 < (p-1)(1+\alpha) \quad \text{and} \quad -2(1+\alpha) < 0 < 2(p-1)(\alpha+1). \tag{3.29}$$

In other words, for the system $\{\varphi_k^\alpha\}_k$ there is a 'pencil phenomenon' for $-1 < \alpha < -\frac{1}{2}$.

Since the conditions (3.29) are fulfilled for any $\alpha > -1$ and every $1 < p < \infty$, there is not 'pencil phenomenon' for the systems $\{\ell_k^\alpha\}_k$ and $\{\psi_k^\alpha\}_k$.

4. Application to operators related with Laguerre polynomials

As is well-known the one-dimensional Laguerre polynomials of type $\alpha > -1$ can be defined by $L_k^\alpha(y) = \frac{1}{k!}\mathrm{e}^y y^{-\alpha}\frac{\mathrm{d}^k}{\mathrm{d}y^k}(\mathrm{e}^{-y}y^{k+\alpha})$. They form a complete orthogonal system in $L^2((0,\infty), \mathrm{d}\gamma_\alpha(y))$ where $\mathrm{d}\gamma_\alpha(y) = y^\alpha \mathrm{e}^{-y}\mathrm{d}y$. For a given $\alpha > -1$, the Laguerre differential operator is

$$\Pi_\alpha = -y\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha + 1 - y)\frac{\mathrm{d}}{\mathrm{d}y}.$$

The polynomials L_k^α satisfy $\Pi_\alpha L_k^\alpha(y) = kL_k^\alpha(y)$.

If $\mathrm{grad}_\alpha f(y) = \sqrt{y}\frac{\mathrm{d}}{\mathrm{d}y}f(y)$ and $\mathrm{div}_\alpha f(y) = -\sqrt{y}\big(\frac{\mathrm{d}}{\mathrm{d}y}f(y) + \big(\frac{\alpha+1/2}{y} - 1\big)f(y)\big)$, then $\Pi_\alpha = \mathrm{div}_\alpha \mathrm{grad}_\alpha$. The following lemma establishes a connection between Laguerre polynomials and Laguerre functions. The proof is simple and we leave it to the reader.

Lemma 4.30. *Let* $\Lambda_\alpha f(y) = f(y)y^{-\alpha/2}\mathrm{e}^{y/2}$. *For any weight* ω, *the operator* Λ_α *is an isometry from* $L^p((0,\infty), \omega(y)\mathrm{d}y)$ *into* $L^p\big((0,\infty), \omega(y)y^{-\alpha(1-\frac{p}{2})}\mathrm{e}^{y(1-\frac{p}{2})}\mathrm{d}\gamma_a(y)\big)$. *Moreover, the following identities are satisfied for any polynomial*:

(i) $\Pi_\alpha f = \Lambda_\alpha \circ \big(L_\alpha - \frac{\alpha+1}{2}\big) \circ (\Lambda_\alpha)^{-1} f$,
(ii) $\sup_t \mathrm{e}^{-t\Pi_\alpha} f = \Lambda_\alpha \circ \sup_t \mathrm{e}^{-t\left(L_\alpha - \frac{\alpha+1}{2}\right)} \circ (\Lambda_\alpha)^{-1} f$,
(iii) $\mathrm{grad}_\alpha f = \Lambda_\alpha \circ D_a \circ (\Lambda_\alpha)^{-1} f$ *and* $\mathrm{div}_\alpha f = \Lambda_\alpha \circ (D_a)^* \circ (\Lambda_\alpha)^{-1} f$,

where D_α, $(D_\alpha)^*$ *and* L_α *are defined in* (2.4) *and* (1.1).

In the setting of Laguerre polynomials the Riesz transforms can be defined by

$$\mathfrak{R}_\alpha = \mathrm{grad}_\alpha (\Pi_a)^{-1/2}, \tag{4.31}$$

(see [2]). In that paper it was proved that these operators are bounded in $L^p(\mathrm{d}\gamma_a)$ for $1 < p < \infty$ and $\alpha = \frac{n}{2} - 1$, $n = 1, 2, \ldots$. Later on in [6] the boundedness in $L^p(\mathrm{d}\gamma_\alpha)$ was proved for any $\alpha \geq -\frac{1}{2}$.

Theorem 4.32. *Let* $\alpha > -1$. *Let* $\mathcal{T}_\alpha$ *be either the heat maximal semigroup*, $\sup_t \mathrm{e}^{-t\Pi_\alpha}$, *or the Riesz transform*, $\mathfrak{R}_\alpha$, *associated to the Laguerre operator* Π_α. *Then*, $\mathcal{T}_\alpha$ *is a bounded operator from* $L^p(y^\sigma \mathrm{e}^{y(1-p/2)}\mathrm{d}\gamma_\alpha)$ *into itself for* $-(1+\alpha) < \sigma < (p-1)(1+\alpha)$.

In order to prove this theorem we need some known results on Hermite functions. The Hermite polynomials, H_k, are given by the formula $H_k(t) = (-1)^k \mathrm{e}^{t^2}\frac{\mathrm{d}^k \mathrm{e}^{-t^2}}{\mathrm{d}t^k}$, $t \in \mathbb{R}$. The Hermite normalized functions with respect to Lebesgue measure turn out to be $h_k(t) =$

$(2^k k!\sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2}$, $t \in \mathbb{R}$. Given the multiindex $\mu \in \mathbb{Z}_+^n$, the n-dimensional Hermite functions h_μ are defined by

$$h_\mu(x) = \Pi_{i=1}^n h_{\mu_i}(x_i), \quad \mu = (\mu_1, \ldots, \mu_n).$$

The second order Hermite differential operator on $\mathbb{R}^n$ is

$$\mathbf{H}_n = -\Delta + |x|^2 = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + |x|^2.$$

This operator is nonegative and self-adjoint with respect to the Lebesgue measure on $\mathbb{R}^n$. Its eigenfunctions are the n-dimensional Hermite functions, in fact

$$\mathbf{H}_n(h_\mu) = (2|\mu| + n) h_\mu. \tag{4.33}$$

It is known that if

$$A_j g(x) = \frac{\partial g}{\partial x_j}(x) + x_j \text{ and } A_j^* g(x) = -\frac{\partial g}{\partial x_j}(x) + x_j,\ j \in \{1, \ldots, n\}, \tag{4.34}$$

then $\mathbf{H}_n - n = \sum_{j=1}^n A_j^* A_j$.

For a measurable function $f\colon (0,\infty) \to \mathbb{R}$ we define the function $\mathcal{W}_\alpha(f)\colon \mathbb{R}^n \to \mathbb{R}$ as $\mathcal{W}_\alpha(f)(x) = \frac{f(|x|^2)}{|x|^\alpha}$. It is easy to see that

$$\|\mathcal{W}_\alpha f\|_{L^p(\mathbb{R}^n, |x|^\tau dx)} = C_{p,\alpha} \|f\|_{L^p((0,\infty), y^\delta dy)}, \quad \text{with} \quad \delta = \frac{\tau - \alpha p}{2} + \frac{n}{2} - 1. \tag{4.35}$$

Lemma 4.36. *Let D_α and A_j as in* (2.4) *and* (4.34) *respectively. For a function $f\colon (0,\infty) \to \mathbb{R}$, good enough, we have*

(i) $A_j(\mathcal{W}_\alpha(f))(x) = 2\frac{x_j}{|x|}\mathcal{W}_\alpha(D_\alpha f)(x)$, $j = 1, \ldots, n$.

(ii) *If $\alpha = \frac{n}{2} - 1$ then, when f is a finite linear combination of Laguerre functions $\mathcal{L}_k^\alpha$*

$$\mathcal{W}_\alpha(L_\alpha f)(x) = \tfrac{1}{4}\mathbf{H}_n \mathcal{W}_\alpha(f)(x),$$
$$\mathcal{W}_\alpha(e^{-tL_\alpha} f)(x) = e^{-t\frac{1}{4}\mathbf{H}_n}\mathcal{W}_\alpha(f)(x) \text{ and}$$
$$\mathcal{W}_\alpha((L_\alpha)^{-1/2} f)(x) = (\mathbf{H}_n)^{-1/2}\mathcal{W}_\alpha f(x),$$

hold.

Proof. Observe that given a function $g\colon \mathbb{R}^n \to \mathbb{R}$ it follows

$$\begin{aligned}
\frac{\partial}{\partial x_j}\frac{f(|x|^2)}{|x|^\alpha} &= f'(|x|^2)\frac{2x_j}{|x|^\alpha} - \alpha\frac{f(|x|^2)}{|x|^{\alpha+1}}\frac{x_j}{|x|} \\
&= \frac{2x_j}{|x|}\left\{\frac{|x|}{|x|^\alpha} f'(|x|^2) - \frac{1}{2}\left(\frac{\alpha}{|x|}\right)\frac{f(|x|^2)}{|x|^\alpha}\right\} \\
&= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)} f'(\cdot))(x) - \frac{1}{2}\left(\frac{\alpha}{|x|}\right)\mathcal{W}_\alpha f(x)\right\}.
\end{aligned}$$

Accordingly,

$$\begin{aligned} A_j(\mathcal{W}_\alpha f)(x) &= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) - \frac{1}{2}\left(\frac{\alpha}{|x|}\right)\mathcal{W}_\alpha f(x)\right\} + x_j\mathcal{W}_\alpha f(x) \\ &= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) - \frac{\alpha}{2|x|}\mathcal{W}_\alpha f(x) + \frac{|x|}{2}\mathcal{W}_\alpha f(x)\right\} \\ &= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) + \frac{1}{2}\left(|x| - \frac{\alpha}{|x|}\right)\mathcal{W}_\alpha f(x)\right\} \\ &= \frac{2x_j}{|x|}\mathcal{W}_\alpha(D_\alpha f)(x). \end{aligned}$$

In order to prove (ii), we recall the relationship between the families of Laguerre and Hermite functions given by

$$\mathcal{L}_k^\alpha(|x|^2) = \mathcal{W}_\alpha(\mathcal{L}_k^\alpha)(x)|x|^\alpha = c_k^\alpha \sum_{|r|=k} \frac{a_r}{b_{2r}} h_{2r}(x)|x|^\alpha, \quad x \in \mathbb{R}^n, \quad \alpha = \frac{n}{2} - 1, \tag{4.37}$$

where $h_s(x)$ are the Hermite functions on $\mathbb{R}^n$, of order $|s|$; c_k^α and b_k are the orthonormalization coefficients given by $c_k^\alpha = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2}$, $b_k = (2^k k!\sqrt{\pi})^{-1/2}$ and $b_\mu = \Pi_{i=1}^n b_{\mu_i}$ (see [12]). The coefficients a_r are those given in the known formula

$$L_k^\alpha(|x|^2) = \sum_{|r|=k} a_r H_{2r}(x), \quad r = (r_1, \ldots, r_n), \quad \alpha = \frac{n}{2} - 1.$$

See formula 5.6.1 of [10] in the case $n = 1$ and Lemma 1.1 of [2] in the general case. The proof of (ii) is a consequence of formulas (4.33), (1.3) and (4.37). ■

In order to prove Theorem 4.32 we will need the following.

Lemma 4.38. *Let* $-1 < \alpha$, $1 < p < \infty$ *and* δ *be real numbers such that* $-1 - \frac{\alpha p}{2} < \delta < +\frac{\alpha p}{2} + (p-1)$. *If* L_α *is defined as in* (1.1) *then the maximal operator*

$$\sup_t \left|\mathrm{e}^{-t\left(L_\alpha - \frac{\alpha+1}{2}\right)} f(y)\right|$$

is bounded from $L^p((0,\infty), y^\delta \mathrm{d}y)$ *into itself.*

Proof. We have

$$\mathrm{e}^{-t\left(L_\alpha - \frac{\alpha+1}{2}\right)} f(y) = \mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y).$$

Thus

$$\begin{aligned} \sup_t |\mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y)| &\le \sup_{t\le 1} |\mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y)| + \sup_{t>1} |\mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y)| \\ &\le \mathrm{e}^{\frac{\alpha+1}{2}} \sup_t |\mathrm{e}^{-tL_\alpha} f(y)| + \sup_{t>1} |\mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y)| \\ &= A + B. \end{aligned}$$

In [1] and [14] it is shown that $\|A\|_{L^p((0,\infty),y^\delta \mathrm{d}y)} \le C\|f\|_{L^p((0,\infty),y^\delta \mathrm{d}y)}$. As for B, taken a function f good enough, it follows that

$$\begin{aligned}\sup_{t\ge 1} |\mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y)| &= \sup_{t\ge 1} \left| \int_{[0,\infty)} \sum_k \mathrm{e}^{-tk} \mathcal{L}_k^\alpha(y)\mathcal{L}_k^\alpha(z) f(z)\mathrm{d}z \right| \\ &\le \sum_k \mathrm{e}^{-k} |\mathcal{L}_k^\alpha(y)| \left| \int_{[0,\infty)} \mathcal{L}_k^\alpha(z) f(z)\mathrm{d}z \right|.\end{aligned}$$

Then by Hölder's inequality we get

$$\sup_{t\ge 1} |\mathrm{e}^{t\frac{\alpha+1}{2}} \mathrm{e}^{-tL_\alpha} f(y)| \le \sum_k \mathrm{e}^{-k} |\mathcal{L}_k^\alpha(y)| \|\mathcal{L}_k^\alpha\|_{L^{p'}([0,\infty),y^{-\delta p'/p}\mathrm{d}y)} \|f\|_{L^p([0,\infty),y^\delta \mathrm{d}y)}.$$

Hence by Minkowski's inequality

$$\begin{aligned}&\|B\|_{L^p([0,\infty),y^\delta \mathrm{d}y)} \\ &\quad\le \sum_k \mathrm{e}^{-k} \|\mathcal{L}_k^\alpha\|_{L^p([0,\infty),y^\delta \mathrm{d}y)} \|\mathcal{L}_k^\alpha\|_{L^{p'}([0,\infty),y^{-\delta p'/p}\mathrm{d}y)} \|f\|_{L^p([0,\infty),y^\delta \mathrm{d}y)}.\end{aligned}$$

From Lemma 1.5.4 of [16] we obtain that if $\delta > -1 - \frac{\alpha p}{2}$ then $\|\mathcal{L}_k^\alpha\|_{L^p([0,\infty),y^\delta \mathrm{d}y)} \le Ck^{\theta_1}$, for some $\theta_1 > 0$. Analogously when $\delta < p - 1 + \frac{\alpha p}{2}$, applying the same lemma we get $\|\mathcal{L}_k^\alpha\|_{L^{p'}([0,\infty),y^{-\delta p'/p}\mathrm{d}y)} \le Ck^{\theta_2}$, for some $\theta_2 > 0$. Therefore

$$\|B\|_{L^p([0,\infty),y^\delta \mathrm{d}y)} \le C\left(\sum_k \mathrm{e}^{-k} k^{\theta_1+\theta_2}\right) \|f\|_{L^p([0,\infty),y^\delta \mathrm{d}y)} \le C\|f\|_{L^p([0,\infty),y^\delta \mathrm{d}y)}.$$

■

Proof of Theorem 4.32. By Lemma 4.30 we have

$$\|\sup_t \mathrm{e}^{-t\mathcal{L}_\alpha} g\|_{L^p(y^\sigma \mathrm{e}^{y(1-p/2)}\mathrm{d}\gamma_\alpha)} = \|\sup_t \mathrm{e}^{-t\left(L-\frac{\alpha+1}{2}\right)} \circ (\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)}\mathrm{d}y)}.$$

Observe that

$$-1 - \alpha p/2 < \sigma + \alpha(1 - p/2) < p - 1 + \alpha p/2. \tag{4.38}$$

Hence by Lemma 4.38, for $\delta = \sigma + \alpha(1 - p/2)$, we have

$$\begin{aligned}&\|\sup_t \mathrm{e}^{-t\left(L-\frac{\alpha+1}{2}\right)} \circ (\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)}\mathrm{d}y)} \\ &\quad\le C_p \|(\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)}\mathrm{d}y)} \\ &\quad= C_p \|g\|_{L^p(y^\sigma \mathrm{e}^{y(1-p/2)}\mathrm{d}\gamma_\alpha)}.\end{aligned}$$

This finishes the proof for the maximal operator.

As for the Riesz transform, by Lemma 4.30 and the definition of R_+^α given in (2.6), we get

$$\begin{aligned}\mathfrak{R}_\alpha g &= \mathrm{grad}_\alpha (\mathcal{L}_\alpha)^{-1/2} g = \Lambda_\alpha \circ D_\alpha \circ \left(L_\alpha - \frac{\alpha+1}{2}\right)^{-1/2} \circ (\Lambda_\alpha)^{-1} g \\ &= \Lambda_\alpha \circ D_\alpha \circ \left(\left(L_\alpha - \frac{\alpha+1}{2}\right)^{-1/2} - (L_\alpha)^{-1/2}\right) \circ (\Lambda_\alpha)^{-1} g + \Lambda_\alpha \circ R_+^\alpha \circ (\Lambda_\alpha)^{-1} g \\ &= I + II.\end{aligned}$$

Again, by Lemma 4.30, we get $\|II\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} = \|R_+^\alpha \circ (\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)} dy)}$. By (4.38), applying Theorem 2.8 and Lemma 4.30 we have

$$\begin{aligned}\|II\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} &\le C\|(\Lambda_\alpha)^{-1} g\|_{L^p(y^{\sigma+\alpha(1-p/2)} dy)}\\ &= C\|g\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)}.\end{aligned}$$

To finish the proof for the Riesz transforms it is enough to show that

$$\|I\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} \le C\|g\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)}.$$

This is equivalent to showing the boundedness of $D_\alpha \circ \left(\left(L_\alpha - \frac{\alpha+1}{2}\right)^{-1/2} - (L_\alpha)^{-1/2}\right)$ from $L^p(y^\delta dy)$ into itself, for $-1 - \alpha p/2 < \delta < p - 1 + \alpha p/2$. Moreover

$$\begin{aligned}D_\alpha &\circ \left(\left(L_\alpha - \frac{\alpha+1}{2}\right)^{-1/2} - (L_\alpha)^{-1/2}\right)\\ &= T_\alpha^\beta \circ \tau_m \circ D_\beta \circ \left(\left(L_\beta - \frac{\beta+1}{2}\right)^{-1/2} - (L_\beta)^{-1/2}\right) \circ T_\beta^\alpha,\end{aligned}$$

where T_β^α is the transplantation operator, $T_\beta^\alpha(\sum c_\alpha \mathcal{L}_k^\alpha) = \sum c_\alpha \mathcal{L}_k^\beta$, and τ_m is the multiplier operator given by the $\mathcal{C}^\infty$ function

$$m(t) = \frac{\frac{\alpha+1}{2}\sqrt{t + \frac{\beta+1}{2}}\left(\sqrt{t + \frac{\beta+1}{2}} + \sqrt{t}\right)}{\frac{\alpha+1}{2}\sqrt{t + \frac{\alpha+1}{2}}\left(\sqrt{t + \frac{\alpha+1}{2}} + \sqrt{t}\right)}.$$

By the transplantation theorem 2.11 and the multiplier theorem 2.9 we just need to prove the boundedness of the operator $D_\beta \circ \left(\left(L_\beta - \frac{\beta+1}{2}\right)^{-1/2} - (L_\beta)^{-1/2}\right)$ from $L^p(y^\delta dy)$ into itself for $-1 - \beta p/2 < \delta < p - 1 + \beta p/2$, for a β bigger than α. Choosing $\beta = \frac{n}{2} - 1$, by Lemma 4.36 we obtain

$$\begin{aligned}&\left|\mathcal{W}_\beta \circ D_\beta \circ \left(\left(L_\beta - \frac{\beta+1}{2}\right)^{-1/2} - (L_\beta)^{-1/2}\right) f\right|\\ &\quad = \left\{\sum_{j=1}^n (A_j \circ ((\mathbf{H}_n - n)^{-1/2} - (\mathbf{H}_n)^{1/2}) \circ \mathcal{W}_\beta f)^2\right\}^{1/2}.\end{aligned}$$

Since $A_j \circ ((\mathbf{H}_n - n)^{-1/2} - (\mathbf{H}_n)^{-1/2})$, $j = 1, \ldots, n$; is bounded in $L^p(\mathbb{R}, |x|^\tau dx)$ for $-1-\beta p/2 < \frac{\tau - \beta p}{2} + \beta < p - 1 + \beta p/2$ (see Theorem 3.5 of [11]) and $\mathcal{W}_\beta$ is an isometry. Therefore

$$\left\|D_\beta \circ \left(\left(L_\beta - \frac{\beta+1}{2}\right)^{-1/2} - (L_\beta)^{-1/2}\right) f\right)\right\|_{L^p(y^\delta dy)} \le C_p \|f\|_{L^p(y^\delta dy)}.$$

■

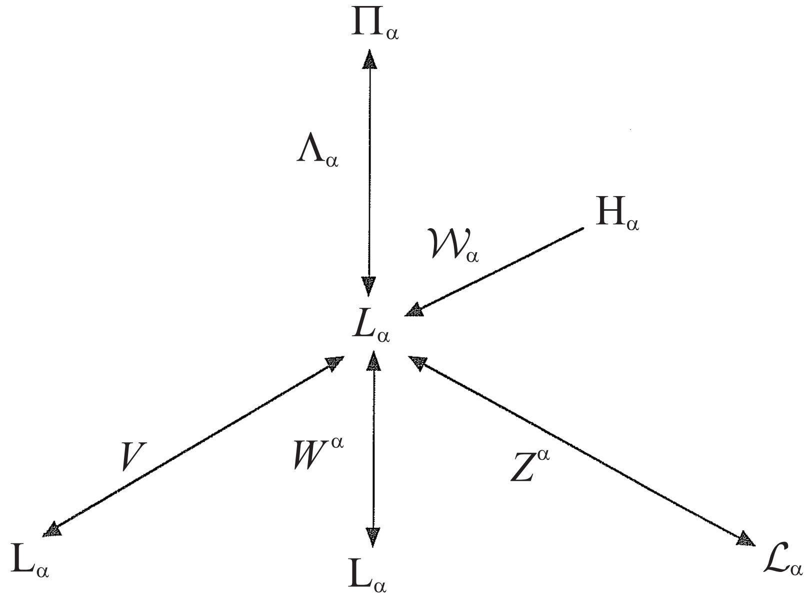


Figure 1. $\alpha > -1$ and for $\mathcal{W}_\alpha$, $\alpha = \frac{n}{2} - 1$.

Figure 1 illustrates the relations among the different infinitesimal generators and the isometries considered in this paper.

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