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A note on a paraholomorphic Cheeger–Gromoll metric

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Abstract. The aim of this note is to study a paraholomorphic Cheeger–Gromoll metric on the tangent bundle of Riemannian manifolds.

Keywords. Cheeger–Gromoll metric; pure metric; paracomplex structure; paraholomorphic tensor field.

1. Introduction

In [1], Cheeger and Gromoll study complete manifolds of nonnegative curvature and suggest a construction of Riemannian metrics useful in that context. Inspired by a paper of Cheeger and Gromoll, Musso and Tricerri [7] defined a new Riemannian metric ${}^{CG}g$ on tangent bundle of a Riemannian manifold which they called the Cheeger–Gromoll metric. The Levi-Civita connection of ${}^{CG}g$ and its Riemannian curvature tensor are calculated by Sekizawa in [11] (for more details, see [4], [5]). The main purpose of this paper is to investigate a paraholomorphic Cheeger–Gromoll metric with respect to the natural paracomplex structure on the tangent bundle.

1.1

Let M_n be a Riemannian manifold with metric g. We denote by $\Im_q^p(M_n)$ the set of all tensor fields of type (p, q) on M_n. Manifolds, tensor field and connections are always assumed to be differentiable and of class C^∞.

Let $T(M_n)$ be a tangent bundle of M_n, and π the projection $\pi\colon T(M_n) \to M_n$. Let the manifold M_n be covered by a system of coordinate neighbourhoods (U, x^i), where (x^i), $i = 1, \ldots, n$ is a local coordinate system defined in the neighbourhood U. Let (y^i) be the cartesian coordinates in each tangent spaces $T_p(M_n)$ at $p \in M_n$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}\right\}$, p being an arbitrary point in U whose coordinates are x^i. Then we can introduce local coordinates (x^i, y^i) in the open set $\pi^{-1}(U)$ of $T(M_n)$. We call them coordinates induced in $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, y^i) \to (x^i)$. We use the notations $x^I = (x^i, x^{\bar{i}})$ and $x^{\bar{i}} = y^i$. The indices $I, J, \ldots$ run from 1 to $2n$, the indices $i, j, \ldots$ from 1 to n and the indices $\bar{i}, \bar{j}, \ldots$ from $n+1$ to $2n$.

Let $X \in \Im_0^1(M_n)$, which are locally represented by

$$X = X^i \partial_i \quad \left(\partial_i = \frac{\partial}{\partial x^i}\right).$$

Then the vertical and horizontal lifts ${}^{\mathrm{V}}X$ and ${}^{\mathrm{H}}X$ of X (see [15]) are given by

$$ {}^{\mathrm{V}}X = X^i \partial_{\bar{i}} \quad \left(\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}\right) $$

and

$$ {}^{\mathrm{H}}X = X^i \partial_i - \Gamma^i_{jk} x^{\bar{j}} X^k \partial_{\bar{i}}, $$

where Γ^i_{jk} are the coefficients of the Levi-Civita connection on M_n.

Suppose that we are given in $U \subset M_n$ a vector field $X \in \Im^1_o(M_n)$ and a covector field $g_X = (g_{ij} X^i \mathrm{d}x^j)$. Then we define a function γg_X in $\pi^{-1}(U) \subset T(M_n)$ by $\gamma g_X = x^{\bar{j}} g_{ij} X^i$ with respect to the induced coordinates $(x^i, x^{\bar{i}})$. The function γg_X defined in each $\pi^{-1}(U)$ determine global function on $T(M_n)$, which is also denoted by γg_X. Now, let r be the norm vector $y = (y^i) = (x^{\bar{i}})$, i.e. $r^2 = g_{ij} x^{\bar{i}} x^{\bar{j}}$. The Cheeger–Gromoll metric ${}^{\mathrm{CG}}g$ on the tangent bundle $T(M_n)$ is given by

(i) ${}^{\mathrm{CG}}g({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y) = {}^{\mathrm{V}}(g(X, Y))$,
(ii) ${}^{\mathrm{CG}}g({}^{\mathrm{H}}X, {}^{\mathrm{V}}Y) = 0$,
(iii) ${}^{\mathrm{CG}}g({}^{\mathrm{V}}X, {}^{\mathrm{V}}Y) = \frac{1}{1+r^2}[{}^{\mathrm{V}}(g(X, Y)) + (\gamma g_X)(\gamma g_Y)]$

for all vector field $X, Y \in \Im^1_0(M_n)$, where ${}^{\mathrm{V}}(g(X, Y)) = (g(X, Y)) \circ \pi$.

1.2

An almost paracomplex manifold is an almost product manifold (M_n, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M_n and T^-M_n associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the following set of affinors on M_{2k}: $\{id, \varphi\}$, $\varphi^2 = id$, which form a bases of a representation of the algebra of order 2 over the field of real numbers R, which is called the algebra of paracomplex (or double) numbers and is denoted by $R(j) = \{a_0 + a_1 j / j^2 = 1; a_0, a_1 \in R\}$. Obviously, it is associative, commutative and unital, i.e., it admits principal unit 1. The canonical bases of this algebra has the form $\{1, j\}$. Structural constants of an algebra are defined by the multiplication law of the base units of this algebra: $e_i e_j = C^k_{ij} e_k$. The components of C^k_{ij} are given by $C^1_{11} = C^2_{12} = C^2_{21} = C^1_{22} = 1$, all the others being zero, with respect to the canonical bases of $R(j)$.

Consider $R(j)$ endowed with the usual topology of R^2 and a domain U of $R(j)$. Let

$$ X = x^1 + jx^2 $$

be a variable in $R(j)$, where x^i are real coordinates of a point of a certain domain U for $i = 1, 2$. Using two real-valued functions $f^i(x^1, x^2)$, $i = 1, 2$, we introduce a paracomplex function

$$ F = f^1 + jf^2 $$

of variable X. It is said to be paraholomorphic if we have

$$ \mathrm{d}F = F'(X)\mathrm{d}X $$

for the differentials $\mathrm{d}X = \mathrm{d}x^1 + j\mathrm{d}x^2$, $\mathrm{d}F = \mathrm{d}f^1 + j\mathrm{d}f^2$ and the derivative $F'(X)$. The paraholomorphy of the function $F = f^1 + jf^2$ in the variable $X = x^1 + jx^2$ is equivalent to the fact that the Jacobian matrix $D = (\partial_k f^i)$ commutes with the matrix

$$C_2 = (C^k_{2j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see p. 87 of [12]). It follows that F is paraholomorphic if and only if f^1 and f^2 satisfy the para-Cauchy–Riemann equations:

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}, \quad \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}.$$

For almost paracomplex structure the integrability is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y].$$

On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion-free linear connection such that $\nabla\varphi = 0$. A paracomplex manifold is an almost paracomplex manifold (M_{2k}, φ) such that the G-structure defined by the affinor field φ is integrable. We can give another equivalent definition of paracomplex manifold in terms of local homeomorphisms in the space $R^k(j) = \{(X^1, \ldots, X^k)/X^i \in R(j), i = 1, \ldots k\}$ and paraholomorphic changes of charts in a way similar to [2] (see also [3]), i.e. a manifold M_{2k} with an integrable paracomplex structure φ is a real realization of the paraholomorphic manifold $\mathbf{X}_k(R(j))$ over the algebra $R(j)$. Let t^* be a paracomplex tensor field on $\mathbf{X}_k(R(j))$. The real model of such a tensor field is a tensor field on M_{2k} of the same order that is independent of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [6], [8]–[10], [12]–[14]). In particular, being applied to a $(0, q)$-tensor field ω, purity means that for any $X_1, \ldots, X_q \in \Im^1_0(M_{2k})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \cdots = \omega(X_1, X_2, \ldots, \varphi X_q).$$

We define an operator

$$\Phi_\varphi \colon \Im^0_q(M_{2k}) \to \Im^0_{q+1}(M_{2k})$$

associated with φ and applied to the pure tensor field ω by [14]

$$\begin{aligned}(\Phi_\varphi\omega)&(X, Y_1, Y_2, \ldots, Y_q)\\ &= (\varphi X)(\omega(Y_1, Y_2, \ldots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \ldots, Y_q))\\ &\quad + \omega((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q) + \cdots + \omega(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X),\end{aligned}$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a paracomplex structure on M_{2k} and the tensor field $\Phi_\varphi\omega$ vanishes, the paracomplex tensor field ω^* on $\mathbf{X}_k(R(j))$ is said to be paraholomorphic [6]. Thus a paraholomorphic tensor field ω^* on $\mathbf{X}_k(R(j))$ is realized on M_{2k} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi\omega)(X, Y_1, Y_2, \ldots, Y_q) = 0$$

for any $X, Y_1, \ldots, Y_q \in \Im^1_0(M_{2k})$. Therefore such a tensor field ω on M_{2k} is also called paraholomorphic tensor field.

2. Results

2.1

Let M_{2k} be an almost paracomplex manifold with structure φ. If M_{2k} admits a pure Riemannian metric g such that

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1}$$

for $X, Y \in \Im_0^1(M_n)$, then M_{2k} is called an almost paracomplex Riemannian manifold (p. 423 of [15]).

We define now an operator $\Phi_\varphi\colon \Im_2^0(M_n) \to \Im_3^0(M_n)$ applied to the pure metric g by

$$\begin{aligned}(\Phi_\varphi g)&(X, Y_1, Y_2)\\ &= (\varphi X)(g(Y_1, Y_2)) - X(g(\varphi Y_1, Y_2)) + g((L_{Y_1}\varphi)X, Y_2) + g(Y_1, (L_{Y_2}\varphi)X).\end{aligned} \tag{2}$$

If (M_{2k}, φ, g) is an almost paracomplex Riemannian manifold with condition $\Phi_\varphi g = 0$, we say that (M_{2k}, φ, g) is an almost paraholomorphic Riemannian manifold. If φ is integrable, we say that (M_{2k}, φ, g) is a paraholomorphic Riemannian manifold. In some aspects, paraholomorphic Riemannian manifolds are similar to Kahler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kahler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 1. *An almost paracomplex Riemannian manifold M_{2k} is paraholomorphic if and only if the almost paracomplex structure is parallel with respect to the Levi-Civita connection of g.*

Proof. By virtue of (1) and $\nabla g = 0$ we have

$$g(Z, (\nabla_Y\varphi)X) = g((\nabla_Y\varphi)Z, X). \tag{3}$$

Using (3) and $[X, Y] = \nabla_X Y - \nabla_Y X$, we have transform (2) as follows:

$$\begin{aligned}(\Phi_\varphi g)&(X, Z_1, Z_2)\\ &= -g((\nabla_X\varphi)Z_1, Z_2) + g((\nabla_{Z_1}\varphi)X, Z_2) + g(Z_1, (\nabla_{Z_2}\varphi)X).\end{aligned} \tag{4}$$

From this we have

$$\begin{aligned}(\Phi_\varphi g)&(Z_2, Z_1, X)\\ &= -g((\nabla_{Z_2}\varphi)Z_1, X) + g((\nabla_{Z_1}\varphi)Z_2, X) + g(Z_1, (\nabla_X\varphi)Z_2).\end{aligned} \tag{5}$$

If we add (4) and (5), we get

$$(\Phi_\varphi g)(X, Z_1, Z_2) + (\Phi_\varphi g)(Z_2, Z_1, X) = 2g(X, (\nabla_{Z_1}\varphi)Z_2). \tag{6}$$

Putting $\Phi_\varphi g = 0$ in (6), $\nabla\varphi = 0$. Conversely, if $\nabla\varphi = 0$, then the condition $\Phi_\varphi g = 0$ follows from (4) (or (5)).

COROLLARY 1

The structure φ on almost paracomplex Riemannian manifold is integrable if $\Phi_\varphi g = 0$.

2.2

The diagonal lift ${}^{\mathrm{D}}\varphi$ on $T(M_n)$ is defined by

$$\begin{cases} {}^{\mathrm{D}}\varphi {}^{\mathrm{H}}X = {}^{\mathrm{H}}(\varphi X), \\ {}^{\mathrm{D}}\varphi {}^{\mathrm{V}}X = -{}^{\mathrm{V}}(\varphi X), \end{cases}$$

for any $X \in \Im^1_0(M_n)$ and $\varphi \in \Im^1_1(M_n)$. The diagonal lift ${}^{\mathrm{D}}I$ of the identity tensor field $I \in \Im^1_1(M_n)$ has the components

$${}^{\mathrm{D}}I = \begin{pmatrix} \delta^j_i & 0 \\ -2y^t\Gamma^j_{ti} & -\delta^j_i \end{pmatrix}$$

with respect to the induced coordinates and satisfies $({}^{\mathrm{D}}I)^2 = I_{T(M_n)}$. Thus ${}^{\mathrm{D}}I$ is an almost paracomplex structure determining the horizontal distribution and the distribution consisting of the tangent planes to fibres.

We put

$$S(\tilde{X}, \tilde{Y}) = {}^{\mathrm{CG}}g({}^{\mathrm{D}}I\tilde{X}, \tilde{Y}) - {}^{\mathrm{CG}}g(\tilde{X}, {}^{\mathrm{D}}I\tilde{Y}).$$

If $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ${}^{\mathrm{V}}X$, ${}^{\mathrm{V}}Y$ or ${}^{\mathrm{H}}X$, ${}^{\mathrm{H}}Y$, then $S = 0$. By virtue of ${}^{\mathrm{D}}I{}^{\mathrm{V}}X = -{}^{\mathrm{V}}X, {}^{\mathrm{D}}I{}^{\mathrm{H}}X = {}^{\mathrm{H}}X$ and (i)–(iii) we have

$$S({}^{\mathrm{V}}X, {}^{\mathrm{V}}Y) = {}^{\mathrm{CG}}g(-{}^{\mathrm{V}}X, {}^{\mathrm{V}}Y) - {}^{\mathrm{CG}}g({}^{\mathrm{V}}X, -{}^{\mathrm{V}}Y) = 0,$$

$$S({}^{\mathrm{V}}X, {}^{\mathrm{H}}Y) = {}^{\mathrm{CG}}g(-{}^{\mathrm{V}}X, {}^{\mathrm{H}}Y) - {}^{\mathrm{CG}}g({}^{\mathrm{V}}X, {}^{\mathrm{H}}Y) = 0,$$

$$S({}^{\mathrm{H}}X, {}^{\mathrm{V}}Y) = {}^{\mathrm{CG}}g({}^{\mathrm{H}}X, {}^{\mathrm{V}}Y) - {}^{\mathrm{CG}}g({}^{\mathrm{H}}X, -{}^{\mathrm{V}}Y) = 0,$$

$$S({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y) = {}^{\mathrm{CG}}g({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y) - {}^{\mathrm{CG}}g({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y) = 0,$$

i.e. ${}^{\mathrm{CG}}g$ is pure metric with respect to ${}^{\mathrm{D}}I$.

We hence have:

Theorem 2. $(T(M_n), {}^{\mathrm{D}}I, {}^{\mathrm{CG}}g)$ *is an almost paracomplex Riemannian manifold.*

Let ${}^{\mathrm{CG}}\nabla$ be the Levi-Civita connection of ${}^{\mathrm{CG}}g$. Using properties of ${}^{\mathrm{V}}X, {}^{\mathrm{H}}X$ and $\gamma R(X, Y) = y^s R^k_{ijs} X^i Y^j \frac{\partial}{\partial x^k}$ (see [15]), i.e.

$${}^{\mathrm{V}}X{}^{\mathrm{V}}(g(Y, Z)) = 0,$$

$$[{}^{\mathrm{V}}X, {}^{\mathrm{V}}Y] = 0,$$

$$[{}^{\mathrm{V}}X, {}^{\mathrm{H}}Y] = {}^{\mathrm{V}}[X, Y] - {}^{\mathrm{V}}(\nabla_X Y) = -{}^{\mathrm{V}}(\nabla_Y X),$$

$${}^{\mathrm{H}}X{}^{\mathrm{V}}(g(Y, Z)) = {}^{\mathrm{V}}(Xg(Y, Z)),$$

$^{CG}\nabla_{^{H}X}{}^{V}Y = \frac{1}{2(1+r^2)}{}^{H}(R(x^{\bar{i}}, Y)X) + {}^{V}(\nabla_X Y)$ (R-curvature tensor field of ∇)

$$[{}^{H}X, {}^{H}Y] = {}^{H}[X, Y] - \gamma R(X, Y),$$
$$^{D}I\gamma R(X, Y) = -\gamma R(X, Y),$$
$$^{CG}g(\gamma R(X, Y), {}^{H}Z) = 0.$$

From (2) we have

$$\begin{aligned}
\text{(i)}\ \ (\Phi_{^{D}I}{}^{CG}g)({}^{V}X, {}^{V}Y, {}^{V}Z) &= ({}^{D}I{}^{V}X)({}^{CG}g({}^{V}Y, {}^{V}Z)) - {}^{V}X({}^{CG}g({}^{D}I{}^{V}Y, {}^{V}Z)) \\
&\quad + {}^{CG}g((L_{^{V}Y}{}^{D}I){}^{V}X, {}^{V}Z) + {}^{CG}g({}^{V}Y, (L_{^{V}Z}{}^{D}I){}^{V}X) \\
&= -{}^{V}X({}^{CG}g({}^{V}Y, {}^{V}Z)) + {}^{V}X({}^{CG}g({}^{V}Y, {}^{V}Z)) \\
&\quad + {}^{CG}g(L_{^{V}Y}({}^{D}I{}^{V}X) - {}^{D}IL_{^{V}Y}{}^{V}X, {}^{V}Z) + {}^{CG}g({}^{V}Y, (L_{^{V}Z}({}^{D}I{}^{V}X) - {}^{D}IL_{^{V}Z}{}^{V}X) \\
&= {}^{CG}g([{}^{V}X, {}^{V}Y] - {}^{D}I[{}^{V}Y, {}^{V}X], {}^{V}Z) + {}^{CG}g({}^{V}Y, [{}^{V}X, {}^{V}Z] - {}^{D}I[{}^{V}Z, {}^{V}X]) \\
&= {}^{CG}g(0, {}^{V}Z) + {}^{CG}g({}^{V}Y, 0) = 0. \\
\text{(ii)}\ \ (\Phi_{^{D}I}{}^{CG}g)({}^{V}X, {}^{V}Y, {}^{H}Z) &= ({}^{D}I{}^{V}X)({}^{CG}g({}^{V}Y, {}^{H}Z)) - {}^{V}X({}^{CG}g({}^{D}I{}^{V}Y, {}^{H}Z)) \\
&\quad + {}^{CG}g((L_{^{V}Y}{}^{D}I){}^{V}X, {}^{H}Z) + {}^{CG}g({}^{V}Y, (L_{^{H}Z}{}^{D}I){}^{V}X) \\
&= -{}^{V}X.0 + {}^{V}X.0 + {}^{CG}g(L_{^{V}Y}({}^{D}I{}^{V}X) - {}^{D}IL_{^{V}Y}{}^{V}X, {}^{H}Z) \\
&\quad + {}^{CG}g({}^{V}Y, (L_{^{H}Z}({}^{D}I{}^{V}X) - {}^{D}IL_{^{H}Z}{}^{V}X) \\
&= {}^{CG}g(-[{}^{V}Y, {}^{V}X]{}^{D}I.0, {}^{H}Z) + {}^{CG}g({}^{V}Y, -[{}^{H}Z, {}^{V}X] - {}^{D}I[{}^{H}Z, {}^{V}X]) \\
&= {}^{CG}g(0, {}^{H}Z) + {}^{CG}g({}^{V}Y, {}^{V}[X, Z] - {}^{V}(\nabla_X Z) + {}^{D}I({}^{V}[X, Z] - {}^{V}(\nabla_X Z)) \\
&= {}^{CG}g({}^{V}Y, 0) = 0 \\
\text{(iii)}\ \ (\Phi_{^{D}I}{}^{CG}g)({}^{V}X, {}^{H}Y, {}^{V}Z) &= ({}^{D}I{}^{V}X)({}^{CG}g({}^{H}Y, {}^{V}Z)) - {}^{V}X({}^{CG}g({}^{D}I{}^{H}Y, {}^{V}Z)) \\
&\quad + {}^{CG}g((L_{^{H}Y}{}^{D}I){}^{V}X, {}^{V}Z) + {}^{CG}g({}^{H}Y, (L_{^{V}Z}{}^{D}I){}^{V}X) \\
&= -{}^{V}X.0 + {}^{V}X.0 + {}^{CG}g(L_{^{H}Y}({}^{D}I{}^{V}X) - {}^{D}IL_{^{H}Y}{}^{V}X, {}^{V}Z) \\
&\quad + {}^{CG}g({}^{H}Y, (L_{^{V}Z}({}^{D}I{}^{V}X) - {}^{D}IL_{^{V}Z}{}^{V}X) \\
&= {}^{CG}g(-[{}^{H}Y, {}^{V}X] - {}^{D}I[{}^{H}Y, {}^{V}X], {}^{V}Z) + {}^{CG}g({}^{H}Y, -[{}^{V}Z, {}^{V}X] - {}^{D}I[{}^{V}Z, {}^{V}X]) \\
&= {}^{CG}g({}^{V}[X, Y] - {}^{V}(\nabla_X Y) + {}^{D}I({}^{V}[X, Y] - {}^{V}(\nabla_X Y)), {}^{V}Z) + {}^{CG}g({}^{H}Y, 0) \\
&= {}^{CG}g({}^{V}[X, Y] - {}^{V}(\nabla_X Y) - {}^{V}[X, Y] + {}^{V}(\nabla_X Y), {}^{V}Z) \\
&= {}^{CG}g(0, {}^{V}Z) = 0.
\end{aligned}$$

(iv) $(\Phi_{{}^{\mathrm{D}}I}{}^{CG}g)({}^{\mathrm{H}}X, {}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) = ({}^{\mathrm{D}}I{}^{\mathrm{H}}X)({}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z)) - {}^{\mathrm{H}}X({}^{CG}g({}^{\mathrm{D}}I{}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z))$

$$+ {}^{CG}g((L_{{}^{\mathrm{V}}Y}{}^{\mathrm{D}}I){}^{\mathrm{H}}X, {}^{\mathrm{V}}Z) + {}^{CG}g({}^{\mathrm{V}}Y, (L_{{}^{\mathrm{V}}Z}{}^{\mathrm{D}}I){}^{\mathrm{H}}X)$$

$$= {}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) + {}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z)$$

$$+ {}^{CG}g(L_{{}^{\mathrm{V}}Y}({}^{\mathrm{D}}I{}^{\mathrm{H}}X) - {}^{\mathrm{D}}I(L_{{}^{\mathrm{V}}Y}{}^{\mathrm{H}}X), {}^{\mathrm{V}}Z) + {}^{CG}g({}^{\mathrm{V}}Y, L_{{}^{\mathrm{V}}Z}({}^{\mathrm{D}}I{}^{\mathrm{H}}X) - {}^{\mathrm{D}}IL_{{}^{\mathrm{V}}Z}{}^{\mathrm{H}}X)$$

$$= 2{}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) + {}^{CG}g(L_{{}^{\mathrm{V}}Y}{}^{\mathrm{H}}X - {}^{\mathrm{D}}I({}^{\mathrm{V}}[Y, X] - {}^{\mathrm{V}}(\nabla_Y X)), {}^{\mathrm{V}}Z)$$

$$+ {}^{CG}g({}^{\mathrm{V}}Y, L_{{}^{\mathrm{V}}Z}{}^{\mathrm{H}}X - {}^{\mathrm{D}}I({}^{\mathrm{V}}[Z, X] - {}^{\mathrm{V}}(\nabla_Z X)))$$

$$= 2{}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) + {}^{CG}g({}^{\mathrm{V}}[Y, X] - {}^{\mathrm{V}}(\nabla_Y X) + {}^{\mathrm{V}}[Y, X] - {}^{\mathrm{V}}(\nabla_Y X), {}^{\mathrm{V}}Z)$$

$$+ {}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}[Z, X] - {}^{\mathrm{V}}(\nabla_Z X) + {}^{\mathrm{V}}[Z, X] - {}^{\mathrm{V}}(\nabla_Z X))$$

$$= 2{}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) + 2{}^{CG}g({}^{\mathrm{V}}[Y, X] - {}^{\mathrm{V}}(\nabla_Y X), {}^{\mathrm{V}}Z)$$

$$+ {}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}[Z, X] - {}^{\mathrm{V}}(\nabla_Z X))$$

$$= 2{}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) + 2({}^{CG}g(-{}^{\mathrm{V}}(\nabla_Y X), {}^{\mathrm{V}}Z) + {}^{CG}g({}^{\mathrm{V}}Y, -{}^{\mathrm{V}}(\nabla_Z X)))$$

$$= 2{}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) + 2\left({}^{CG}g\left(\frac{1}{2(1+r^2)}{}^{\mathrm{H}}(R(x^{\bar{i}}, Y)X) - {}^{CG}\nabla_{{}^{\mathrm{H}}X}{}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z\right)\right.$$

$$\left.+{}^{CG}g\left({}^{\mathrm{V}}Y, \frac{1}{2(1+r^2)}{}^{\mathrm{H}}(R(x^{\bar{i}}, Z)X) - {}^{CG}\nabla_{{}^{\mathrm{H}}X}{}^{\mathrm{V}}Z\right)\right)$$

$$= 2({}^{\mathrm{H}}X{}^{CG}g({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) - {}^{CG}g({}^{CG}\nabla_{{}^{\mathrm{H}}X}{}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) - {}^{CG}g({}^{\mathrm{V}}Y, {}^{CG}\nabla_{{}^{\mathrm{H}}X}{}^{\mathrm{V}}Z)$$

$$= 2({}^{CG}\nabla_{{}^{\mathrm{H}}X}{}^{CG}g)({}^{\mathrm{V}}Y, {}^{\mathrm{V}}Z) = 0.$$

(v) $(\Phi_{{}^{\mathrm{D}}I}{}^{CG}g)({}^{\mathrm{H}}X, {}^{\mathrm{H}}Y, {}^{\mathrm{H}}Z) = ({}^{\mathrm{D}}I{}^{\mathrm{H}}X)({}^{CG}g({}^{\mathrm{H}}Y, {}^{\mathrm{H}}Z)) - {}^{\mathrm{H}}X({}^{CG}g({}^{\mathrm{D}}I{}^{\mathrm{H}}Y, {}^{\mathrm{H}}Z))$

$$+ {}^{CG}g((L_{{}^{\mathrm{H}}Y}{}^{\mathrm{D}}I){}^{\mathrm{H}}X, {}^{\mathrm{H}}Z) + {}^{CG}g({}^{\mathrm{H}}Y, (L_{{}^{\mathrm{H}}Z}{}^{\mathrm{D}}I){}^{\mathrm{H}}X)$$

$$= {}^{\mathrm{H}}X{}^{\mathrm{V}}(g(Y, Z)) - {}^{\mathrm{H}}X{}^{\mathrm{V}}(g(Y, Z))$$

$$+ {}^{CG}g(L_{{}^{\mathrm{H}}Y}({}^{\mathrm{D}}I{}^{\mathrm{H}}X) - {}^{\mathrm{D}}I(L_{{}^{\mathrm{H}}Y}{}^{\mathrm{H}}X), {}^{\mathrm{H}}Z) + {}^{CG}g({}^{\mathrm{H}}Y, L_{{}^{\mathrm{H}}Z}({}^{\mathrm{D}}I{}^{\mathrm{H}}X) - {}^{\mathrm{D}}IL_{{}^{\mathrm{H}}Z}{}^{\mathrm{H}}X)$$

$$= {}^{CG}g([{}^{\mathrm{H}}Y, {}^{\mathrm{H}}X] - {}^{\mathrm{D}}I[{}^{\mathrm{H}}Y, {}^{\mathrm{H}}X], {}^{\mathrm{H}}Z) + {}^{CG}g({}^{\mathrm{H}}Y, [{}^{\mathrm{H}}Z, {}^{\mathrm{H}}X] - {}^{\mathrm{D}}I[{}^{\mathrm{H}}Z, {}^{\mathrm{H}}X])$$

$$= {}^{CG}g({}^{\mathrm{H}}[Y, X] - \gamma R(Y, X) - {}^{\mathrm{D}}I({}^{\mathrm{H}}[Y, X] - \gamma R(Y, X)), {}^{\mathrm{H}}Z)$$

$$+ {}^{CG}g({}^{\mathrm{H}}Y, {}^{\mathrm{H}}[Z, X] - \gamma R(Z, X) - {}^{\mathrm{D}}I({}^{\mathrm{H}}[Z, X] - \gamma R(Z, X)))$$

$$= {}^{CG}g({}^{\mathrm{H}}[Y, X] - \gamma R(Y, X) - {}^{\mathrm{H}}[Y, X] + {}^{\mathrm{D}}I\gamma R(Y, X), {}^{\mathrm{H}}Z)$$

$$+ {}^{CG}g({}^{\mathrm{H}}Y, {}^{\mathrm{H}}[Z, X] - \gamma R(Z, X) - {}^{\mathrm{H}}[Z, X] + {}^{\mathrm{D}}I\gamma R(Z, X)))$$

$$= -2({}^{CG}g(\gamma R(Y, X), {}^{\mathrm{H}}Z) - {}^{CG}g({}^{\mathrm{H}}Y, \gamma R(Z, X)) = 0$$

(vi) $(\Phi_{{}^{D}I}{}^{CG}g)({}^{V}X, {}^{H}Y, {}^{H}Z) = ({}^{D}I\,{}^{V}X)({}^{CG}g({}^{H}Y, {}^{H}Z)) - {}^{V}X({}^{CG}g({}^{D}I\,{}^{H}Y, {}^{H}Z))$

$$+ {}^{CG}g((L_{{}^{H}Y}{}^{D}I){}^{V}X, {}^{H}Z) + {}^{CG}g({}^{H}Y, (L_{{}^{H}Z}{}^{D}I){}^{V}X)$$

$$= -2{}^{V}X({}^{CG}g({}^{H}Y, {}^{H}Z)) + {}^{CG}g(L_{{}^{H}Y}({}^{D}I\,{}^{V}X) - {}^{D}I[{}^{H}Y, {}^{V}X], {}^{H}Z)$$

$$+ {}^{CG}g({}^{H}Y, L_{{}^{H}Z}({}^{D}I\,{}^{V}X) - {}^{D}I[{}^{H}Z, {}^{V}X])$$

$$= -2{}^{V}X({}^{CG}g({}^{H}Y, {}^{H}Z)) + {}^{CG}g(-[{}^{H}Y, {}^{V}X] - {}^{D}I({}^{V}[Y, X] - {}^{V}(\nabla_Y X)), {}^{H}Z)$$

$$+ {}^{CG}g({}^{H}Y, -[{}^{H}Z, {}^{V}X] - {}^{D}I({}^{V}[Z, X] - {}^{V}(\nabla_Z X)))$$

$$= -2{}^{V}X({}^{CG}g({}^{H}Y, {}^{H}Z)) + {}^{CG}g(-[{}^{H}Y, {}^{V}X] + {}^{V}[Y, X] - {}^{V}(\nabla_Y X)), {}^{H}Z)$$

$$+ {}^{CG}g({}^{H}Y, -[{}^{H}Z, {}^{V}X] + {}^{V}[Z, X] - {}^{V}(\nabla_Z X))$$

$$= -2{}^{V}X{}^{V}(g(Y, Z)) + {}^{CG}g(-{}^{V}[Y, X] + {}^{V}(\nabla_Y X) + {}^{V}[Y, X] - {}^{V}(\nabla_Y X)), {}^{H}Z)$$

$$+ {}^{CG}g({}^{H}Y, -{}^{V}[Z, X] + {}^{V}(\nabla_Z X) + {}^{V}[Z, X] - {}^{V}(\nabla_Z X))$$

$$= -2{}^{V}X{}^{V}(g(Y, Z)) + {}^{CG}g(0, {}^{H}Z) + + {}^{CG}g({}^{H}Y, 0) = 0.$$

(vii) $(\Phi_{{}^{D}I}{}^{CG}g)({}^{H}X, {}^{H}Y, {}^{V}Z) = ({}^{D}I\,{}^{H}X)({}^{CG}g({}^{H}Y, {}^{V}Z)) - {}^{H}X({}^{CG}g({}^{D}I\,{}^{H}Y, {}^{V}Z))$

$$+ {}^{CG}g((L_{{}^{H}Y}{}^{D}I){}^{H}X, {}^{V}Z) + {}^{CG}g({}^{H}Y, (L_{{}^{V}Z}{}^{D}I){}^{H}X)$$

$$= {}^{H}X({}^{CG}g({}^{H}Y, {}^{V}Z)) - {}^{H}X({}^{CG}g({}^{H}Y, {}^{V}Z))$$

$$+ {}^{CG}g(L_{{}^{H}Y}{}^{H}X - {}^{D}I[{}^{H}Y, {}^{H}X]), {}^{V}Z) + {}^{CG}g({}^{H}Y, L_{{}^{V}Z}{}^{H}X - {}^{D}I[{}^{V}Z, {}^{H}X])$$

$$= {}^{CG}g([{}^{H}Y, {}^{H}X] - {}^{D}I({}^{H}[Y, X] - \gamma R(Y, X)), {}^{V}Z)$$

$$+ {}^{CG}g({}^{H}Y, [{}^{V}Z, {}^{H}X] + {}^{D}I\,{}^{V}(\nabla_X Z))$$

$$= {}^{CG}g({}^{H}[Y, X] - \gamma R(Y, X) - {}^{H}[Y, X] - \gamma R(Y, X), {}^{V}Z)$$

$$+ {}^{CG}g({}^{H}Y, -2{}^{V}(\nabla_X Z)) = -2{}^{CG}g(\gamma R(Y, X), {}^{V}Z)$$

(viii) $(\Phi_{{}^{D}I}{}^{CG}g)({}^{H}X, {}^{V}Y, {}^{H}Z) = -2{}^{CG}g({}^{V}Y, \gamma R(Z, X))$is analogous to (vii).

Therefore we have

Theorem 3. *The almost paracomplex Riemannian manifold* $(T(M_n), {}^{D}I, {}^{CG}g)$ *is paraholomorphic if and only if* M_n *is flat.*

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