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Sign (di)lemma for dimension shifting

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Abstract. There is a surprising occurrence of some minus signs in the isomorphisms produced in the well-known technique of dimension shifting in calculating derived functors in homological algebra. We explicitly determine these signs. Getting these signs right is important in order to avoid basic contradictions. We illustrate the result – which we call as the *sign lemma for dimension shifting* – by some de Rham cohomology and Chern class considerations for compact Riemann surfaces.

Keywords. Abelian categories; derived functors; dimension shifting.

1. Statement of the main result

Let $\mathcal{A}$ be an abelian category with enough injectives, and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive, left-exact functor from $\mathcal{A}$ to another abelian category $\mathcal{B}$. For each object M of $\mathcal{A}$, we choose an injective resolution $0 \to M \to I^\bullet$, and define the value of the derived functor $R^i F$ on M to be $H^i(FI^\bullet)$. (To make such a choice for each object, we need to assume some foundational framework, which is fairly standard so we will omit all reference to it.) If $0 \to M \to J^\bullet$ is any other resolution of M, we have a homomorphism of complexes $f^\bullet\colon J^\bullet \to I^\bullet$ which is unique up to homotopy, and is identity on M. If $J^\bullet$ is F-acyclic, that is, if $R^i F$ is zero on J^k for each $i \geq 1$ and $k \geq 0$, then for each $n \geq 1$, $f^\bullet$ induces an isomorphism

$$c^n = H^n(Ff^\bullet)\colon H^n(FJ^\bullet) \to H^n(FI^\bullet) = R^n FM.$$

We will call $c^n\colon H^n(FJ^\bullet) \to R^n FM$ as the *canonical isomorphism* for the acyclic resolution J of M.

There is another very useful isomorphism $d^n\colon H^n(FJ^\bullet) \to R^n FM$ for $n \geq 1$, known as the *dimension shifting isomorphism*. To define it, we begin by breaking-up the resolution J into a sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of short exact sequences

$$\mathcal{E}_1 = (0 \to Z^0 \to J^0 \to Z^1 \to 0),\ \text{where } Z^0 = M.$$

$$\mathcal{E}_2 = (0 \to Z^1 \to J^1 \to Z^2 \to 0),$$

$$\ldots$$

$$\mathcal{E}_n = (0 \to Z^{n-1} \to J^n \to Z^n \to 0).$$

As the J^i are F-acyclic, the corresponding connecting homomorphisms are isomorphisms for $p \geq 1$ and $q \geq 1$, which we denote by

$$\partial^p_{\mathcal{E}_q} : R^p F Z^q \to R^{p+1} F Z^{q-1}.$$

For $p = 0$ and $q \geq 1$, the connecting homomorphism $\partial^0_{\mathcal{E}_q} : FZ^q \to R^1 F Z^{q-1}$ is epic, and induces an isomorphism

$$\overline{\partial}^0_{\mathcal{E}_q} : \frac{FZ^q}{\operatorname{im} FJ^{q-1}} \to R^1 F Z^{q-1}.$$

For $q = n$, we thus have a sequence of isomorphisms

$$H^n(FJ^\bullet) = \frac{FZ^n}{\operatorname{im} FJ^n} \xrightarrow{\overline{\partial}} R^1 F Z^{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} R^n F M.$$

The composite of these is an isomorphism

$$d^n : H^n(FJ^\bullet) \to R^n F M$$

which is by definition the dimension shifting isomorphism for $n \geq 1$.

In this note, we compare the two isomorphisms $c^n : H^n(FJ^\bullet) \to R^n F M$ and $d^n : H^n(FJ^\bullet) \to R^n F M$ for all $n \geq 1$, and we find the following, which is our main result.

Sign lemma for dimension shifting. *With notation as above, the canonical isomorphism c^n and the dimension shifting isomorphism d^n are related by*

$$d^n = (-1)^{(n^2+n)/2} c^n.$$

To prove the above lemma, we need some preliminaries.

2. Preliminary lemmas

Any object X in a category $\mathcal{C}$ defines a contravariant functor $h_X = \operatorname{Hom}_{\mathcal{C}}(-, X)$, which we call its *functor of elements*, and for any other object T, an element $x \in h_X(T)$ will be called a *T-valued element* of X. When we do not want to explicitly mention T, then such an x will be just called a *valued element* of X, and by abuse of notation we write it as $x \in X$. The object T is called as the *level of definition* of x. Given any $\phi : T' \to T$, we denote by $\phi^*(x)$ (or simply by $x|_{T'}$ when ϕ is understood) the composite $x \circ \phi : T' \to X$. Note that if ϕ is epic then x is uniquely determined by $x|_{T'}$. By the Yoneda lemma, any morphism $f : X \to Y$ in $\mathcal{C}$ is determined by its effect on all valued elements of X where the level of definition T varies over all objects of $\mathcal{C}$.

If $g : X' \to X$ is epic, then even though not all T-valued elements of X lift to T-valued elements of X' with the same T, any morphism $f : X \to Y$ is determined by the effect of $g \circ f$ on all valued elements of X' as the level of definition T varies over all objects of $\mathcal{C}$. If $\mathcal{C}$ admits fibered products, then valued elements can themselves be 'lifted' under an epimorphism, provided we change their levels of definition, as follows. For any epic $g : X' \to X$ and any T-valued point x of X, taking $T' = T \times_{x,X,g} X'$ and the projections $\phi : T' \to T$ and $x' : T' \to X'$, we see that $g(x') = x|_{T'}$. Note that in this case $\phi : T' \to T$

is the pull-back of g and hence is epic, so x is uniquely determined by x', and thus x' can be said to be a 'lift' x under the epic $g: X' \to X$.

If $\mathbf{E} = (0 \to U^\bullet \xrightarrow{u} V^\bullet \xrightarrow{v} W^\bullet \to 0)$ is a short-exact sequence of co-chain complexes of abelian groups, then recall that the *connecting homomorphism* $\delta^i_{\mathbf{E}}: H^i(W) \to H^{i+1}(U)$ is defined as follows. Represent a cohomology class in $H^i(W)$ by a cycle $z \in Z^i(W)$, and lift it to an element $y \in V^i$ which is possible as v^i is epic. Then $y = u^i(x)$ for a unique x in $Z^{i+1}(U)$, which defines the class $\delta^i_{\mathbf{E}}(z)$ in $H^{i+1}(U)$. The same prescription, with elements suitably replaced by valued elements defined at appropriate levels, defines the connecting homomorphism $\delta^i_{\mathbf{E}}: H^i(W) \to H^{i+1}(U)$ if $\mathbf{E}$ is a short-exact sequence of co-chain complexes of objects in any abelian category. In the above we have to use the fact that abelian categories admit fibered products and so valued elements can be 'lifted' under an epimorphism, as described earlier. (The above elementary account of the connecting homomorphism is designed to avoid the use of the Freyd–Mitchell embedding theorem.)

In what follows, $\mathcal{A}$ will be an abelian category with enough injectives, and $F: \mathcal{A} \to \mathcal{B}$ will be an additive, left-exact functor from $\mathcal{A}$ to another abelian category $\mathcal{B}$.

For any short-exact sequence $\mathcal{E} = (0 \to A \to C \to B \to 0)$ in $\mathcal{A}$, recall that the connecting homomorphism $\partial^i_{\mathcal{E}}: R^iFB \to R^{i+1}FA$ is defined as follows. Let $0 \to A \to I^\bullet_A$ and $0 \to B \to I^\bullet_B$ be the chosen injective resolutions of A and B, so that $R^iFA = H^i(FI^\bullet_A)$ and $R^iFB = H^i(FI^\bullet_B)$. Then by the so-called 'horse-shoe lemma', there exists a resolution $0 \to C \to M^\bullet$ which fits in a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & A & \to & C & \to & B & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & I^\bullet_A & \to & M^\bullet & \to & I^\bullet_B & \to 0 \end{array},$$

where the rows are short-exact. Let $\mathbf{e} = (0 \to I^\bullet_A \to M^\bullet \to I^\bullet_B \to 0)$ denote the second row of the above diagram. Then $F\mathbf{e} = (0 \to FI^\bullet_A \to FM^\bullet \to FI^\bullet_B \to 0)$ is again short-exact in the category $\mathcal{B}$ as the I^n_B are injective objects. By definition, $\partial^i_{\mathcal{E}}: R^iFB \to R^{i+1}FA$ is the connecting homomorphism $\delta^i_{F\mathbf{e}}: H^i(FI^\bullet_B) \to H^{i+1}(FI^\bullet_A)$. This can be shown to be well-defined, independent of the choices made.

Let there be given objects A and B in $\mathcal{A}$, together with F-acyclic resolutions $0 \to A \to J^\bullet$ and $0 \to B \to K^\bullet$. Let $c^i_B: H^i(FJ^\bullet) \to R^iFA$ and $c^i_B: H^i(FK^\bullet) \to R^iFB$ be the corresponding canonical isomorphisms.

Let there be chosen an F-acyclic resolution $0 \to C \to L^\bullet$, together with a short-exact sequence of complexes

$$\mathbf{E} = (0 \to J^\bullet \to L^\bullet \to K^\bullet \to 0)$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 \to & A & \to & C & \to & B & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & J^\bullet & \to & L^\bullet & \to & K^\bullet & \to 0 \end{array}.$$

Note that by the horse-shoe lemma, such a resolution $L^\bullet$ always exists. As $J^\bullet$ is F-acyclic, applying F gives a short-exact sequence of complexes

$$F\mathbf{E} = (0 \to FJ^\bullet \to FL^\bullet \to FK^\bullet \to 0).$$

Let $\delta^i_{F\mathbf{E}}: H^i(FK^\bullet) \to H^{i+1}(FJ^\bullet)$ be the corresponding connecting homomorphism.

Lemma A. *With notation as above, the following diagram commutes.*

$$\begin{array}{ccc} H^i(FK^\bullet) & \overset{\delta^i_{F\mathbf{E}}}{\to} & H^{i+1}(FJ^\bullet) \\ c^i \downarrow & & \downarrow c^{i+1} \\ R^iFA & \overset{\partial^i_{\mathcal{E}}}{\to} & R^{i+1}FB \end{array} .$$

Proof. We leave the proof of Lemma A as an exercise to the reader.

Remark. In particular, the homomorphism $\delta^i_{F\mathbf{E}}: H^i(FK^\bullet) \to H^{i+1}(FJ^\bullet)$ does not depend on the choice of the resolution $0 \to C \to L^\bullet$, or on the short exact sequence of complexes $\mathbf{E}$, but only depends on the given short exact sequence $\mathcal{E} = (0 \to A \to C \to B \to 0)$. Hence in what follows we will denote it simply by $\delta^i_{\mathcal{E}}: H^i(FK^\bullet) \to H^{i+1}(FJ^\bullet)$.

We now return to the situation of our main result, the sign lemma. Recall that we began with an F-acyclic resolution $0 \to M \to J^\bullet$ of an object M of $\mathcal{A}$. Let $\delta^i_J: J^i \to J^{i+1}$ denote the differentials. We break up the resolution into short-exact sequences

$$\mathcal{E}_q = (0 \to Z^{q-1} \overset{u_{q-1}}{\to} J^{q-1} \overset{v_{q-1}}{\to} Z^q \to 0)$$

for $1 \le q \le n$, with $u_{i+1}v_i = \delta^i_J$. For each $r \ge 0$, we get an F-acyclic resolution

$$0 \to Z^r \to K^\bullet_r$$

of Z^r defined by $K^p_r = J^{p+r}$, $\delta^r_K = \delta^{p+r}_J$, and $Z^r \to K^0_r$ the 'inclusion' homomorphism $u_r: Z^r \to J^r$.

Lemma B. *For all* $n \ge 1$ *and* $p \ge 1$*, the following diagrams are commutative.*

$$\begin{array}{ccc} H^n(FJ^\bullet) & \overset{-1}{\to} & H^n(FJ^\bullet) \\ \| & & \| \\ \frac{H^0(FK^\bullet_n)}{\operatorname{im} FJ^{n-1}} & \overset{\overline{\delta}\mathcal{E}_n}{\to} & H^1(FK^\bullet_{n-1}) \\ \| & & \downarrow c^1 \\ \frac{FZ^n}{\operatorname{im} FJ^{n-1}} & \overset{\overline{\partial}\mathcal{E}_n}{\to} & R^1FZ^{n-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} H^n(FJ^\bullet) & \overset{(-1)^{p+1}}{\to} & H^n(FJ^\bullet) \\ \| & & \| \\ H^p(FK^\bullet_{n-p}) & \overset{\delta\mathcal{E}_{n-p}}{\to} & H^{p+1}(FK^\bullet_{n-p-1}) \\ c^p \downarrow & & \downarrow c^{p+1} \\ R^pFZ^{n-p} & \overset{\partial\mathcal{E}_{n-p}}{\to} & R^{p+1}FZ^{n-p-1} \end{array} .$$

Proof. By Lemma A, the homomorphisms $\delta_{\mathcal{E}_q}$ can be computed in terms of any resolution $L^\bullet_q$ of J^q which fits in a short-exact sequence $\mathbf{E}_q$ of commuting resolutions of $0 \to Z^q \to J^q \to Z^{q+1} \to 0$, and the lower squares in the above diagrams commute by Lemma A. To see that the upper squares commute, we construct a particular such resolution

$$0 \to J^q \to L^\bullet_q$$

as follows. For any p, we put

$$L^p_q = K^p_q \oplus K^p_{q+1} = J^{p+q} \oplus J^{p+q+1}.$$

We write valued elements of L^p_q as (2×1)-column vectors. With this notation, the inclusion of J^q into L^0_q is defined in matrix terms by

$$\begin{pmatrix} 1_{J^q} \\ \delta^q_J \end{pmatrix} : J^q \to J^q \oplus J^{q+1} = L^0_q.$$

The differential

$$\delta^p_{L_q}\colon L^p_q \to L^{p+1}_q$$

is defined in matrix terms (acting on column vectors) by

$$\delta^p_{L_q} = \begin{pmatrix} \delta^{p+q}_J & (-1)^{p+1} 1_{J^{p+q+1}} \\ 0 & \delta^{p+q+1}_J \end{pmatrix} \colon J^{p+q} \oplus J^{p+q+1} \to J^{p+q+1} \oplus J^{p+q+2}.$$

With these definitions, $0 \to J^q \to L^\bullet_q$ is indeed exact. Moreover, the following is a commutative diagram with exact rows, where the second row $\mathbf{E}_q$ is given by inclusions and projections for the level-wise direct sum $L^p_q = K^p_q \oplus K^p_{q+1} = J^{p+q} \oplus J^{p+q+1}$,

$$\begin{array}{ccccccccc} 0 \to & Z^q & \to & J^q & \to & Z^{q+1} & \to 0 & \\ & \downarrow & & \downarrow & & \downarrow & & . \\ 0 \to & K^\bullet_q & \to & L^\bullet_q & \to & K^\bullet_{q+1} & \to 0 & \end{array}$$

Now take $q = n - p - 1$ in the above. Given any valued element $x \in FZ^n$ which represents a valued element

$$\overline{x} \in FZ^n / \operatorname{im} FJ^{n-1} = H^p(FK^\bullet_{n-p}),$$

the valued element

$$y = \begin{pmatrix} 0 \\ x \end{pmatrix} \in FJ^{n-1} \oplus FJ^n = FK^p_{n-p-1} \oplus FK^p_{n-p} = FL^p_{n-p-1}$$

has the property that under the projection $FL^p_{n-p-1} \to FK^p_{n-p}$, we have $y \mapsto x$. Now note that as $\delta^n_{FJ} x = 0$, we have

$$\begin{pmatrix} \delta^{n-1}_{FJ} & (-1)^{p+1} 1_{FJ^n} \\ 0 & \delta^n_{FJ} \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} (-1)^{p+1} x \\ \delta^n_{FJ} x \end{pmatrix} = \begin{pmatrix} (-1)^{p+1} x \\ 0 \end{pmatrix} \in FL^{p+1}_{n-p-1}.$$

This is the image of $(-1)^{p+1} x \in FK^{p+1}_{n-p-1} = FJ^n$ under the inclusion $FK^{p+1}_{n-p-1} \to FL^{p+1}_{n-p-1}$. This shows that under the connecting morphism $\delta^p_{F\mathbf{E}_{n-p-1}}$, the image of $\overline{x}$ is $(-1)^{p+1}\overline{x}$. As $FZ^n \to FZ^n / \operatorname{im} FJ^{n-1} = H^p(FK^\bullet_{n-p})$ is epic, this calculation is enough to show that $\delta^p_{F\mathbf{E}_{n-p-1}}$ acts as $(-1)^{p+1}$ on all valued elements of $H^p(FK^\bullet_{n-p}) = FZ^n / \operatorname{im} FJ^{n-1}$. This completes the proof of Lemma B.

3. Proof of the sign lemma

By Lemma B, the following squares commute for all $n \geq 1$ and $p \geq 1$.

$$\begin{array}{ccc} H^n(FJ^\bullet) & \stackrel{-1}{\to} & H^n(FJ^\bullet) \\ \| & & \downarrow c^1 \\ \frac{FZ^n}{\operatorname{im} FJ^{n-1}} & \stackrel{\overline{\partial}}{\to} & R^1 FZ^{n-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} H^n(FJ^\bullet) & \stackrel{(-1)^{p+1}}{\to} & H^n(FJ^\bullet) \\ c^p \downarrow & & \downarrow c^{p+1} \\ R^p FZ^{n-p} & \stackrel{\partial}{\to} & R^{p+1} FZ^{n-p+1} \end{array} .$$

As $\sum_{p=0}^{n-1}(p+1) = (n^2+n)/2$, the horizontal composition of the above diagrams gives a commutative square

$$\begin{array}{ccc} H^n(FJ^\bullet) & \stackrel{(-1)^{(n^2+n)/2}}{\rightarrow} & H^n(FJ^\bullet) \\ \| & & \downarrow c^n \\ \frac{FZ^n}{\mathrm{im}\, FJ^{n-1}} & \stackrel{d^n}{\rightarrow} & R^n FZ^0 \end{array},$$

where d^n is the composite $H^n(FJ^\bullet) \stackrel{\overline{\partial}}{\rightarrow} R^1FZ^{n-1} \stackrel{\partial}{\rightarrow} \cdots \stackrel{\partial}{\rightarrow} R^nFM$, which is by definition the dimension-shifting isomorphism. Thus

$$d^n = (-1)^{(n^2+n)/2}\, c^n$$

which completes the proof of the sign lemma.

4. Illustration: Chern class and de Rham's theorem

Let X be a differential manifold (paracompact), and $\mathcal{A}$ the category of sheaves of real vector spaces (or complex vector spaces) on X. Let $\mathcal{B}$ be the category of real (resp. complex) vector spaces, and let $F\colon \mathcal{A} \to \mathcal{B}$ be the global section functor $\Gamma(X, -)$, with derived functors the sheaf cohomologies $H^i(X, -)$. Let $\mathcal{C}^i$ be the sheaf of real (resp. complex) differential i-forms, and let $\delta^i\colon \mathcal{C}^i \to \mathcal{C}^{i+1}$ be the exterior derivative. As this defines an F-acyclic resolution $0 \to \mathbb{R}_X \to \mathcal{C}^\bullet$ (resp. $0 \to \mathbb{C}_X \to \mathcal{C}^\bullet$), we get a canonical isomorphism

$$c^i\colon H^i_{dR}(X) = H^i(F\mathcal{C}^\bullet) \to R^iF(\mathbb{R}_X) = H^i(X, \mathbb{R}_X)$$

(a canonical isomorphism

$$c^i\colon H^i_{dR}(X) = H^i(F\mathcal{C}^\bullet) \to R^iF(\mathbb{C}_X) = H^i(X, \mathbb{C}_X)$$

in the complex case), where $H^i_{dR}(X)$ denotes the real (resp. complex) de Rham cohomology of X.

Some authors (for example, [G-H]) prove de Rham's theorem by identifying de Rham cohomology and sheaf cohomology by the dimension shifting isomorphism $d^i\colon H^i_{dR}(X) \to H^i(X, \mathbb{C}_X)$. By the sign lemma, this is $(-1)^{(i^2+i)/2}$-times the canonical isomorphism. Omission of this sign can lead to sign mistakes and confusion later on.

One such confusion, which we describe next to end this note, occurs in the basic calculation of Chern classes of line bundles on Riemann surfaces. Let $\mathcal{C}^*$ be the multiplicative sheaf non-vanishing complex-valued smooth functions on X. Let $\exp : \mathcal{C} \to \mathcal{C}^*$ denote the map defined at the level of local sections by $f \mapsto e^{2\pi i f}$. This map is surjective at the level of germs, and defines a short exact sequence of sheaves

$$0 \to \mathbb{Z}_X \to \mathcal{C} \stackrel{\exp}{\rightarrow} \mathcal{C}^* \to 0$$

called as the exponential sequence. Any complex line bundle on X defines an element $(L) \in H^1(X, \mathcal{C}^*)$, whose image

$$c_1(L) = \partial(L) \in H^2(X, \mathbb{Z}_X)$$

under the connecting homomorphism $\partial\colon H^1(X,\mathcal{C}^*) \to H^2(X,\mathbb{Z}_X)$ is the first Chern class of L.

For a compact Riemann surface X, let $\eta_X \in H^2(X,\mathbb{Z}_X)$ denote the positive generator. For a complex line bundle L on a compact Riemann surface it can be directly calculated that

$$\partial(L) = \deg(L)\,\eta_X \in H^2(X,\mathbb{Z}_X),$$

where $\deg(L)$ is the degree of L, which is positive for ample line bundles. Hence we get the relation $c_1(L) = \deg(L)\,\eta_X$ relating first Chern class and degree.

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}_X & \to \mathcal{C} \to & \mathcal{C}^* & \to 0 \\ & \downarrow & \| & \downarrow & \\ 0 \to & \mathbb{C}_X & \to \mathcal{C} \to & \mathcal{Z}^1 & \to 0 \end{array},$$

where $\mathcal{Z}^1$ is the sheaf of closed 1-forms, and $\mathcal{C}^*_X \to \mathcal{Z}^1$ is defined by $f \mapsto df/2\pi i f$. This gives a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{C}^*) & \stackrel{\partial}{\to} & H^2(X,\mathbb{Z}_X) \\ \downarrow & & \downarrow \\ H^1(X,\mathcal{Z}^1) & \stackrel{\partial}{\to} & H^2(X,\mathbb{C}_X) \end{array}$$

in which the bottom row is an isomorphism. Thus, any complex line bundle L defined by transition functions $(g_{a,b}) \in H^1(X,\mathcal{C}^*)$ defines a class $c(L) = (dg_{a,b}/2\pi i g_{a,b}) \in H^1(X,\mathcal{Z}^1)$ whose image $\partial(c(L)) \in H^2(X,\mathbb{C}_X)$ is $c_1(L)$. We have connecting isomorphisms

$$H^2_{dR}(X) = \frac{H^0(X,\mathcal{Z}^2)}{\operatorname{im} H^0(X,\mathcal{C}^1)} \stackrel{\partial}{\to} H^1(X,\mathcal{Z}^1) \stackrel{\partial}{\to} H^2(X,\mathbb{C}_X)$$

whose composite is the dimension shifting isomorphism $d^2\colon H^2_{dR}(X) \to H^2(X,\mathbb{C}_X)$.

A simple calculation (see for example [Na]) shows that if X is a compact Riemann surface and if $\alpha_X \in H^2_{dR}(X)$ is the positive integral generator (means $\int_X \alpha_X = 1$), then

$$c(L) = -\deg(L)\,\partial(\alpha_X).$$

This is consistent with the sign lemma, by which $d^2\colon H^2_{dR}(X) \to H^2(X,\mathbb{C}_X)$ is (-1)-times the canonical isomorphism, so that $\partial \circ \partial(\alpha_X) = -\eta_X$, and so $\partial(c(L)) = \deg(L)\,\eta_X$.

Not taking the sign (-1) into account will lead to a paradox at this point. Any attempt to resolve it by trying to define the first Chern class as $-\partial(L) \in H^2(X,\mathbb{Z}_X)$ will in turn be contradicted by a direct calculation of $\partial\colon H^1(X,\mathcal{C}^*) \to H^2(X,\mathbb{Z}_X)$. The author of this note actually got into this contradiction, which led to this work which in particular resolves it.

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