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A generalization of the finiteness problem in local cohomology modules

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R with non-zero identity and let N be a weakly Laskerian R-module and M be a finitely generated R-module. Let t be a non-negative integer. It is shown that if $H^i_{\mathfrak{a}}(N)$ is a weakly Laskerian R-module for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ is weakly Laskerian R-module. Also, we prove that $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(N))$ is weakly Laskerian R-module for all $i = 0, 1$. In particular, if $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(N))$ is a finite set for all $i < t$, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(N))$ is weakly Laskerian R-module for all $i = 0, 1$.

Keywords. Local cohomology modules; cofiniteness; weakly Laskerian; spectral sequences.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R and t a non-negative integer. For any unexplained notation or terminology, we refer the reader to [3].

It is well-known that for a Noetherian ring R, an ideal $\mathfrak{a}$ of R, and M a finitely generated R-module, the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are not always finitely generated. On the other hand, if R is local and $\mathfrak{m}$ is a maximal ideal of R, then $H^i_{\mathfrak{m}}(M)$ are Artinian modules. It is an easy consequence of Matlis duality [19] and the basic work of Grothendieck [10] that this property is equivalent to say that $\mathrm{Supp}_R(H^i_{\mathfrak{m}}(M)) \subseteq \{\mathfrak{m}\}$ and $\mathrm{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{m}}(M))$ is finitely generated. In view of these facts, the following conjecture was made by Grothendieck (see Expose XIII, Conjecture 1.2 of [11]):

Conjecture 1.1. Let M be a finitely generated R-module and let $\mathfrak{a}$ be an ideal of R. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \geq 0$.

This conjecture is false in general, as was shown by Hartshorne [12] with a counterexample, which states that the conjecture is not true even if the ring is regular. Hartshorne defined a module N to be $\mathfrak{a}$-cofinite if the support of N is contained in $V(\mathfrak{a})$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, N)$ is finitely generated for all j and he asked the following question:

Question 1.2. If $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module, when are $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ finitely generated for all i and all j?

By working in the derived category, Hartshorne showed that if M is a finitely generated R-module, where R is a complete regular local ring, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite in two cases:

(i) $\mathfrak{a}$ is a non-zero principal ideal (Corollary 6.3 of [12]);
(ii) $\mathfrak{a}$ is a prime ideal with dimension 1 (Corollary 7.7 of [12]).

There are several papers devoted to the extension of Hartshorne's results to more general situations. We refer the reader to [15], [5], [6], [23], [16], [4] and [20]. Also, there are some generalizations of the theory of local cohomology modules. The following generalization of local cohomology theory is given by Herzog [13] (see also [22]). The generalized local cohomology functor $H^i_{\mathfrak{a}}(\cdot, \cdot)$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N),$$

for all R-modules M and N. Recently, there is some new interest in generalized local cohomology (see, for e.g., [1], [14] and [18]). The purpose of this note is to make a suitable generalization of Conjecture 1.1 and Question 1.2 in the context of generalized local cohomology and usual local cohomology modules and then obtain some results as above, especially for Hartshorne's first result.

2. Preliminaries

In the present section, we recall briefly definitions and basic properties of weakly Laskerian modules that we shall use.

An R-module N is said to be weakly Laskerian if the set of associated primes of any quotient module of N is finite. Obviously, any finitely generated module, any module with finite support, and any minimax module are weakly Laskerian modules. Also, an R-module N is $\mathfrak{a}$-weakly cofinite if $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, N)$ is weakly Laskerian for all $j \geq 0$. It is clear that every $\mathfrak{a}$-cofinite module is $\mathfrak{a}$-weakly cofinite. These definitions were introduced in [8] and [9] and have led to some interesting results. Throughout the paper, we shall appeal to the following lemma without further comments.

Lemma 2.1 (*see Lemma* 2.3 *of* [8])

(i) *Let* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, *be an exact sequence of R-modules. Then M is weakly Laskerian if and only if L and N are both weakly Laskerian. Thus any subquotient of a weakly Laskerian module is weakly Laskerian.*
(ii) *Let M and N be two R-modules. If N is weakly Laskerian and M is finitely generated, then* $\operatorname{Ext}^j_R(M, N)$ *is weakly Laskerian for all* $j \geq 0$.

3. The results

We start this section with the following lemmas which are useful in our arguments.

Lemma 3.1. *Let M be a finitely generated R-module and N be an R-module. Let $H^i_{\mathfrak{a}}(N)$ be a weakly Laskerian module for all $i < t$. Then $H^i_{\mathfrak{a}}(M, N)$ is weakly Laskerian module for all $i < t$.*

Proof. By Theorem 11.38 of [21], there is a Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}_R^p(M, H_{\mathfrak{a}}^q(N)) \underset{p}{\Longrightarrow} H_{\mathfrak{a}}^{p+q}(M, N).$$

Since $E_i^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $i \geq 2$, our hypotheses give us that $E_i^{p,q}$ is weakly Laskerian for all $i \geq 2$, $p \geq 0$ and $q < t$. For all $i \geq 2$, $p \geq 0$ and $r < t$, we consider the exact sequence

$$0 \longrightarrow \ker d_i^{p,r} \longrightarrow E_i^{p,r} \overset{d_i^{p,r}}{\longrightarrow} E_i^{p+i,r-i+1}. \tag{1}$$

Since $E_i^{p,r} = \ker d_i^{p,r}/\operatorname{im} d_i^{p-i,r+i-1}$ and $E_i^{p,j} = 0$ for all $j < 0$, we may use (1) to obtain $\ker d_{r+2}^{i,r-i} \cong E_{r+2}^{i,r-i} \cong \cdots \cong E_{\infty}^{i,r-i}$ for all $0 \leq i \leq r$. There is a finite filtration of the module $H^r = H_{\mathfrak{a}}^r(M, N)$,

$$0 = \phi^{r+1}H^r \subseteq \phi^r H^r \subseteq \cdots \subseteq \phi^1 H^r \subseteq \phi^0 H^r = H^r,$$

such that $E_{\infty}^{i,r-i} = \phi^i H^r/\phi^{i+1}H^r$ for all $0 \leq i \leq r$. Now, the exact sequence

$$0 \longrightarrow \phi^{i+1}H^r \longrightarrow \phi^i H^r \longrightarrow E_{\infty}^{i,r-i} \longrightarrow 0, \qquad (0 \leq i \leq r)$$

yields that $H_{\mathfrak{a}}^i(M, N)$ is weakly Laskerian for all $i < t$. ■

Lemma 3.2. *Let M be a finitely generated R-module and N be an $\mathfrak{a}$-torsion R-module. Then $H_{\mathfrak{a}}^i(M, N) \cong \operatorname{Ext}_R^i(M, N)$ for all i.*

Proof. By Corollary 2.1.6 of [3], we have an injective resolution $I^\bullet$ of N in which all terms are injective $\mathfrak{a}$-torsion module. Hence, for all $i \in \mathbb{N}_0$,

$$H_{\mathfrak{a}}^i(M, N) \cong H^i(\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M, I^\bullet))) \cong H^i(\operatorname{Hom}_R(M, I^\bullet)) \cong \operatorname{Ext}_R^i(M, N).$$

■

The following theorem is a generalization of Theorem 1.2 of [1].

Theorem 3.3. *Let M be a finitely generated R-module and N be an R-module. Let $\operatorname{Ext}_R^j(R/\mathfrak{a}, N)$ be a weakly Laskerian module for all $j \leq t$ and let $H_{\mathfrak{a}}^i(N)$ be a weakly Laskerian module for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, N))$ is a weakly Laskerian module.*

Proof. We use induction on t. If $t = 0$, then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^0(M, N))$ is equal to the weakly Laskerian R-module $\operatorname{Hom}_R(M/\mathfrak{a}M, N)$. Now the assertion holds. Suppose that $t > 0$ and the case $t - 1$ is settled. Since $\Gamma_{\mathfrak{a}}(N)$ is weakly Laskerian, we have $\operatorname{Ext}_R^i(M, \Gamma_{\mathfrak{a}}(N))$ is weakly Laskerian for all i. Now, by using the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0,$$

we get $\operatorname{Ext}_R^i(R/\mathfrak{a}, N/\Gamma_{\mathfrak{a}}(N))$ is weakly Laskerian for all $i \leq t$ and that $H_{\mathfrak{a}}^0(N/\Gamma_{\mathfrak{a}}(N)) = 0$ and $H_{\mathfrak{a}}^i(N/\Gamma_{\mathfrak{a}}(N)) \cong H_{\mathfrak{a}}^i(N)$ for all $i > 0$. On the other hand, we have the exact sequence

$$\operatorname{Ext}_R^{i-1}(M, \Gamma_{\mathfrak{a}}(N)) \longrightarrow H_{\mathfrak{a}}^{i-1}(M, N) \longrightarrow H_{\mathfrak{a}}^{i-1}(M, N/\Gamma_{\mathfrak{a}}(N)) \longrightarrow \operatorname{Ext}_R^i(M, \Gamma_{\mathfrak{a}}(N)),$$

for all i. Hence, we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$. Let E be an injective hull of N and put $L = E/N$. Then $\Gamma_{\mathfrak{a}}(E) = 0$, $H^0_{\mathfrak{a}}(M, E) = 0$ and $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$. Consequently, $\operatorname{Ext}^i_R(R/\mathfrak{a}, L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, N)$, $H^i_{\mathfrak{a}}(L) \cong H^{i+1}_{\mathfrak{a}}(N)$ and $H^i_{\mathfrak{a}}(M, L) \cong H^{i+1}_{\mathfrak{a}}(M, N)$ for all $i \geq 0$. Now the induction hypotheses yields that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M, L))$ is weakly Laskerian and hence $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ is weakly Laskerian too. ■

The following corollary immediately follows from Theorem 3.3.

COROLLARY 3.4

Let M be a finitely generated R-module and N be a weakly Laskerian R-module. Let $H^i_{\mathfrak{a}}(N)$ be a weakly Laskerian module for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ *is a weakly Laskerian module.*

The following corollary, which is an extension of Theorem 2.2 of [2], immediately follows from Corollary 3.4.

COROLLARY 3.5

Let N be a weakly Laskerian R-module. Let $H^i_{\mathfrak{a}}(N)$ be a weakly Laskerian module for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(N))$ *is a weakly Laskerian module. In particular,* $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(N))$ *is a finite set.*

All the statements and proofs of the paper would go through if one replaces 'the weakly Laskerian' condition by condition '($\star$)', where ($\star$) is any condition on R-modules such that the following holds:

(a) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of R-modules, then M has condition ($\star$) if and only if L and N both have condition ($\star$).
(b) If M and N are two R-modules, such that M is finitely generated and N satisfies condition ($\star$) then $\operatorname{Ext}^j_R(M, N)$ satisfies ($\star$) condition, for all $j \geq 0$.

The following result is an improvement of Theorem 1.2 and Theorem 1.3 of [1].

Remark 3.6. Let M and N be two R-modules such that M is finitely generated and N satisfies condition ($\star$). If $H^i_{\mathfrak{a}}(N)$ satisfies condition ($\star$) for all $i < t$, then the following are true:

(1) $H^i_{\mathfrak{a}}(M, N)$ satisfies condition ($\star$) for all $i < t$;
(2) $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ satisfies condition ($\star$).

Theorem 3.7. *Let N be a weakly Laskerian R-module. Let $H^i_{\mathfrak{a}}(N)$ be an $\mathfrak{a}$-weakly cofinite module for all $i < t$. Then* $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(N))$ *is weakly Laskerian module for all $i = 0, 1$.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(R/\mathfrak{a}, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, N).$$

For each $r \geq 2$ and $i = 0, 1$, we consider the exact sequence

$$0 \longrightarrow \ker d_r^{i,t} \longrightarrow E_r^{i,t} \xrightarrow{d_r^{i,t}} E_r^{i+r,t-r+1}. \qquad (*)$$

It follows from the hypotheses that the R-module $E_r^{i+r,t-r+1}$ is a weakly Laskerian module. Note that $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p, q \in \mathbb{N}_0$. There is an integer $l \geq 2$ such that $E_\infty^{i,t} = E_r^{i,t}$ for all $r \geq l$. Also, there is a bounded filtration

$$0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H^t,$$

for the module $H^t = \mathrm{Ext}_R^t(R/\mathfrak{a}, N)$ such that $E_\infty^{p,n-p} = \phi^p H^t/\phi^{p+1}H^t$ for all $p = 0, 1, \ldots, t$. Thus $E_\infty^{p,q}$ is weakly Laskerian for all p, q. Since $E_l^{i,t} = \ker d_{l-1}^{i,t}/\mathrm{im}\, d_{l-1}^{i-l+1,t+l-2}$ and $\mathrm{im}\, d_{l-1}^{i-l+1,t+l-2} = 0$ (for all $l > 2$ and $i = 0, 1$), it follows that $\ker d_{l-1}^{i,t}$ is a weakly Laskerian module. Hence by using the exact sequence $(*)$ for $r = l-1$, we deduce that $E_{l-1}^{i,t}$ is a weakly Laskerian module. By continuing this argument repeatedly for integer $l-1, l-2, \ldots, 3$ instead of l, we obtain that $E_2^{i,t}$ is a weakly Laskerian module for $i = 0, 1$. This completes the proof. ■

The following is an immediate consequence of Theorem 3.7.

COROLLARY 3.8 (see Theorem B of [17])

Let N be a weakly Laskerian module. Let $\mathrm{Supp}_R(H_\mathfrak{a}^i(N))$ *be a finite set for all* $i < t$. *Then* $\mathrm{Ext}_R^i(R/\mathfrak{a}, H_\mathfrak{a}^t(N))$ *is weakly Laskerian for all* $i = 0, 1$. *Moreover,* $\mathrm{Ass}_R(H_\mathfrak{a}^t(N))$ *is a finite set.*

By using finitely generated instead of weakly Laskerian in Theorem 3.7, we have the following corollary.

COROLLARY 3.9 (see Corollary 2.6 of [8] and Theorem 2.1 of [7])

Let N be a finitely generated module. Let $H_\mathfrak{a}^i(N))$ *be an* $\mathfrak{a}$*-cofinite module for all* $i < t$. *Then* $\mathrm{Ext}_R^i(R/\mathfrak{a}, H_\mathfrak{a}^t(N))$ *is a finitely generated module for all* $i = 0, 1$. *In particular,* $\mathrm{Ass}_R(H_\mathfrak{a}^t(N))$ *is a finite set.*

Example 3.10.

(i) Let $(R, \mathfrak{m})$ be a local ring of dimension 3 and let x, y, z be a system of parameters of R. Let $\mathfrak{a} = (xz, yz)$. Then by Corollary 3.4 and Example 3.5 of [9], $H_\mathfrak{a}^2(R)$ is not finitely generated. But $H_\mathfrak{a}^2(R)$ is weakly Laskerian.
(ii) Let k be a field, $R = (k[x, y, u, v]/(xu - yv))_{(x,y,u,v)}$ and $\mathfrak{a} = (x, y)R \cap (u, v)R$. Then by Example 3.5 of [9], $H_\mathfrak{a}^2(R)$ is not $\mathfrak{a}$-cofinite, while in each case $H_\mathfrak{a}^i(R)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$, by Corollary 3.4 of [9].

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