

On a class of semicommutative modules

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Abstract. Let R be a ring with identity, M a right R -module and $S = \text{End}_R(M)$. In this note, we introduce S -semicommutative, S -Baer, S -q.-Baer and S -p.q.-Baer modules. We study the relations between these classes of modules. Also we prove if M is an S -semicommutative module, then M is an S -p.q.-Baer module if and only if $M[x]$ is an $S[x]$ -p.q.-Baer module, M is an S -Baer module if and only if $M[x]$ is an $S[x]$ -Baer module, M is an S -q.-Baer module if and only if $M[x]$ is an $S[x]$ -q.-Baer module.

Keywords. Baer modules; principally quasi-Baer modules; quasi-Baer modules; semicommutative modules.

1. Introduction

Throughout this paper R will denote an associative ring with identity, $\text{Mod-}R$ will be the category of unitary right R -modules. For a module M , $S = \text{End}_R(M)$ will denote the ring of right R -module endomorphisms of M . Then M is a left S -module, right R -module and S - R -bimodule. In this work, for any rings S and R and any S - R -bimodule M , $r_R(\cdot)$ and $l_M(\cdot)$ will denote the right annihilator of a subset of M with elements from R and the left annihilator of a subset of R with elements from M , respectively. Similarly, $l_S(\cdot)$ and $r_M(\cdot)$ will be the left annihilator of a subset of M with elements from S and the right annihilator of a subset of S with elements from M , respectively. In [10], Rizvi and Roman called M a *Baer module* if the right annihilator in M of any left ideal of S is generated by an idempotent of S , i.e., for any left ideal I of S , $r_M(I) = eM$ for some $e^2 = e \in S$ (or equivalently, for all R -submodules N of M , $l_S(N) = Se$ with $e^2 = e \in S$). M is said to be a *quasi-Baer module* if the right annihilator in M of any ideal of S is generated by an idempotent of S (or equivalently, for all fully invariant R -submodules N of M , $l_S(N) = Se$ with $e^2 = e \in S$). To avoid confusion with definitions in [6], we will call Baer modules S -Baer modules and quasi-Baer modules S -quasi-Baer modules. Among other results they have proved that any direct summand of an S -Baer (resp. S -quasi-Baer) module M is again an S -Baer (resp. S -quasi-Baer) module, and the endomorphism ring $S = \text{End}_R(M)$ of an S -Baer (resp. S -quasi-Baer) module M is an S -Baer (resp. S -quasi-Baer) ring (see Theorem 4.1 in [10]). They gave several results for a direct sum of S -Baer (resp. S -quasi-Baer) modules to be an S -Baer (resp. S -quasi-Baer) module.

Let M be an R -module. Recall that M is called a *semicommutative module* if for any $a \in R$ and $m \in M$, $ma = 0$ implies $mRa = 0$ and R is called a semicommutative ring if R_R is a semicommutative module. In this work we will call M *S-semicommutative* if for any $f \in S$ and $m \in M$, $f(m) = 0$ implies $fg(m) = 0$ for every $g \in S$. Then a ring R is a semicommutative ring if and only if R_R is an S -semicommutative module where $S = \text{End}_R(R_R) \cong R$. Note that any submodule N of an S -semicommutative module M is S -semicommutative. M is *S-principally quasi-Baer* (or *S-p.q.-Baer* for short) if for any $m \in M$, $l_S(m) = Se$ (which is equal to $l_S(mR)$) for some $e^2 = e \in S$. A ring is called an *abelian ring* if its idempotents are central. And also note that if M is an S -semicommutative module, then for all $\alpha \in S$, $\text{Ker}(\alpha)$ is a fully invariant submodule of M . In particular every direct summand of M is a fully invariant submodule of M and so M satisfies summand intersection property, that is, intersection of two direct summand of M is again direct summand.

2. Preliminaries

In this section we study some elementary properties of S -semicommutative modules. We start with

Lemma 2.1. Let M be an S -semicommutative module. Then S is a semicommutative, hence an abelian ring.

Proof. Let $f, g \in S$ and assume $fg = 0$. Then $fg(m) = 0$ for all $m \in M$. By hypothesis $fhg(m) = 0$ for all $m \in M$ and $h \in S$. Hence $fhg = 0$ for all $h \in S$ and so $fSg = 0$. Let $e, f \in S$ with $e^2 = e$. Then $(e(1 - e))M = 0$. By hypothesis $(ef(1 - e))M = 0$. Hence $ef(1 - e) = 0$ for all $f \in S$. Similarly $(1 - e)fe = 0$ for all $f \in S$. Thus $ef = fe$ for all $f \in S$. \square

We do not know whether or not the converse of Lemma 2.1 is true in general. Now we investigate at least when the converse of Lemma 2.1 is possible.

Lemma 2.2. Let R be a ring and eRe be a semicommutative subring where $e^2 = e \in R$. If $ere = 0$ implies $er = 0$, then eR is an S -semicommutative module where $r \in R$, $S = \text{End}_R(eR)$.

Proof. Let $f(er) = 0$ where $f \in S$. Then for all $g \in S$, $fg(er) = er_1er_2er$ where $f(e) = er_1$ and $g(e) = er_2$. Since $S = \text{End}_R(eR) \cong eRe$, eRe is a semicommutative ring. Also since $f(er) = 0$, $er_1er = 0$. Thus $er_1ere = 0$ and $er_1er_2ere = 0$ for all $er_2e \in eRe$. By the hypothesis of lemma, $er_1er_2er = 0$. Therefore $fg(er) = er_1er_2er = 0$. \square

Let R be a ring without identity. If $r_1Rr_2 = 0$ whenever $r_1r_2 = 0$, then it will be called that R has the semicommutative property.

Lemma 2.3. Let $e^2 = e \in R$ and $S = \text{End}_R(eR)$. Then

- (1) *If eR is a semicommutative module (and so eRe is a semicommutative ring), then eR is an S -semicommutative module.*
- (2) *Let R be a ring and Re be a semicommutative module where $e^2 = e \in R$ but eR has not the semicommutative property. Then eR is not an S -semicommutative module where $S = \text{End}_R(eR) \cong eRe$ but S is a semicommutative ring.*

Proof.

- (1) Let $f \in S$ and $f(er_1) = 0$ where $r \in R$. Then $f(e)er = 0$. Let $g \in S$. By $S = \text{End}_R(eR) \cong eRe$, $g(e) = ese$ and $f(e) = ete$ and so $f(g(er)) = er_3eser$ and $f(er) = eter = 0$. Since eR is semicommutative, $eteser = 0$ and therefore $f(g(er)) = 0$. Then eR is an S -semicommutative module.
- (2) Assume that eR is an S -semicommutative module where $S = \text{End}_R(eR)$. Take any elements $r_1, r_2 \in R$ such that $er_1er_2 = 0$. Since $S = \text{End}_R(eR) \cong eRe$, then for fixed $r \in R$, note that $f: eR \rightarrow eR$, $f(e) = ere$, $f(es) = eres$, $s \in S$ is an R -homomorphism. So, if we take $f(e) = er_1e$, then $f(er_2) = er_1er_2 = 0$. Since eR is an S -semicommutative module, for all $g(e) = er_3e \in S$ we obtain that $fg(er_2) = 0$ and so $fg(er_2) = er_1er_3er_2 = 0$. Thus we obtain that if $er_1r_2 = 0$, then for all $er_3 \in eR$, $er_1er_3er_2 = 0$. So eR has the semicommutative property. This is a contradiction. Therefore eR is not an S -semicommutative module. But since Re is a semicommutative module and $S = \text{End}_R(eR) \cong eRe$, eRe is a semicommutative subring of R , S is a semicommutative ring. \square

We investigate in Lemma 2.4 the conditions under which the semicommutativity of S implies S -semicommutativity of M .

Lemma 2.4. Let M be a module with endomorphism ring S . Then the following are satisfied:

- (1) Assume that S is a semicommutative ring, and for every $m \in M$, there exists $g \in S$ such that $g(M) = mR$, then M is an S -semicommutative module.
- (2) If M is an S -p.p. module and S is a semicommutative ring, then M is an S -semicommutative module.
- (3) If M is an indecomposable S -Baer module, then M is an S -semicommutative module and so S is semicommutative.
- (4) Let M be an S -semicommutative module. Assume that for every submodule N of M there exist $e^2 = e \in S$, and $\alpha \in S$ such that $N \subseteq eM$ and $\alpha(N) = eM$. Then M is a Baer module.
- (5) If M is an S -semicommutative module and every fully invariant submodule is a direct summand of M , then M is an S -Baer module.

Proof.

- (1) Let $f(m) = 0$ where $S = \text{End}_R(M)$. Then by theorem there exists $g \in S$ such that $g(M) = mR$ and so $f(g(mR)) = f(g(M)) = 0$, that is $fg = 0$. Since S is a semicommutative ring for all $h \in S$, $fhg = 0$ and therefore $fh(m) = 0$. Thus M is an S -semicommutative module.
- (2) Let $\varphi(m) = 0$ where $\varphi \in S$ and $m \in M$. Since M is an S -p.p. module, there exists $e^2 = e \in S$ such that $l_S(mR) = Se$. Since $\varphi(m) = 0$, $\varphi \in l_S(mR) = Se$ and then $\varphi\beta \in Se\beta$ for all $\beta \in S$. Since S is semicommutative, $e\beta = \beta e$ for all $\beta \in S$ and so $\varphi\beta \in S\beta e \subseteq Se = l_S(mR)$. This implies that $\varphi\beta(m) = 0$.
- (3) Let $\varphi(m) = 0$ where $\varphi \in S$ and $m \in M$. Then $\varphi \in l_S(m) = Se$ for some $e^2 = e$. Hence $M = eM \oplus (1 - e)M$ and so $e = 0$ or $e = 1$. It follows that $\varphi = 0$ or $m = 0$.

- (4) Let N be a submodule of M . Then there exists an idempotent homomorphism $e \in S$ and $\alpha \in S$ such that $N \subseteq eM$ and $\alpha(N) = eM$. We prove that $l_S(N) = S(1 - e)$. It is trivial that $S(1 - e) \leq l_S(N)$ since $N \subseteq eM$. Let $\beta \in l_S(N)$. By hypothesis $\beta(N) = 0$ implies $\beta\alpha(N) = 0$. Then $\beta\alpha(N) = \beta eM = 0$, and so $\beta e = 0$. Hence $\beta = \beta(1 - e) \in S(1 - e)$. So $l_S(N) \leq S(1 - e)$. This completes the proof.
- (5) Since M is an S -semicommutative module, if $f(n) = 0$ where $f \in S$, then for all $g \in S$, $f(g(n)) = 0$. This implies that for all $f \in S$, $\text{Ker}(f)$ is a fully invariant submodule of M . Let I be an ideal of S . Since $r_M(I) = \bigcap_{\alpha \in I} \text{Ker}(\alpha)$ and all the $\text{Ker}(\alpha)$ are fully invariant submodules of M , $r_M(I)$ is a fully invariant submodule of M . So it is a direct summand of M and therefore M is an S -Baer module. \square

Lemma 2.5. Let M_R be a cyclic module. Assume that either M is a semicommutative R -module or R is a commutative ring. Then M is an S -semicommutative module if and only if S is a semicommutative ring.

Proof. Let $M = xR$. Assume that S is a semicommutative ring, and let $f \in S, m \in M$ with $f(m) = 0$. Then $m = xt$ for some $t \in R$. Define $g(xs) = ms$ where $xs \in M$. We prove M is S -semicommutative. For if $x \in M$ and $r \in R$ with $xr = 0$, then, by hypothesis, $xtr = 0$ so $g(xr) = mr = xtr = 0$. Hence g becomes a well-defined endomorphism of M in the cases where M is a semicommutative R -module or R is a commutative ring. But then $0 = f(m) = fg(x)$. Hence $fg = 0$. By assumption $fhg = 0$ for every $h \in S$. So $0 = fhg(x) = fh(m) = 0$. Thus M is an S -semicommutative module. The rest is clear from Lemma 2.1. \square

The following two examples shows that it is not necessary that if M is a semicommutative R -module, then M is an S -semicommutative R -module and if M is an S -semicommutative R -module, then M is a semicommutative R -module, respectively.

Example A. There exists a semicommutative R -module M such that it is not S -semicommutative.

Proof. Let F be a field and $R = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ where F is a field and $M = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ and $S = \text{End}_R(M)$. Then M is a right R -module by usual matrix operations. Let $f, g \in S$ be defined by

$$f \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, g \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in M.$$

Then $f \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $fg \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. That is, M is not S -semicommutative. Since R is commutative, M is a semicommutative R -module.

Example B. There exists a module M with $S = \text{End}_R(M)$ such that M is S -semicommutative but not semicommutative.

Proof. Let \mathbb{Z} denote the ring of integers, $M = \mathbb{Z} \times \mathbb{Z}$, $R = \text{End}_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z})$ and $S = \text{End}_R(\mathbb{Z} \times \mathbb{Z})$. Then M is an S -semicommutative module. But M_R is not a semicommutative R -module. For if, let f and $g \in R$ be defined by $(a, b)f = (a, 0)$ and $(a, b)g = (b, 0)$ where $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Then $(0, 1)f = (0, 0)$ but $(0, 1)gf \neq (0, 0)$. Therefore M is not a semicommutative R -module. \square

Lemma 2.6. Let M be an S -semicommutative module. Then the following are satisfied:

- (1) If M is a quasi-injective module, then every submodule N of M is an S' -semicommutative module where $S' = \text{End}_R(N)$.
- (2) If $M = R$, then for all $a \in R$, aR is an S' -semicommutative module where $S' = \text{End}_R(aR)$.

Proof.

- (1) Let $f \in S'$ and $f(n) = 0, n \in N$. Since f is an endomorphism from N to M and M is a quasi-injective module, f is extended to the function $\bar{f} \in S$ such that $\bar{f}(n') = f(n')$ for all $n' \in N$ and also for all $g \in S'$, there exists a function $\bar{g} \in S$ such that $\bar{g}(n') = g(n')$ for all $n' \in N$. Since M is an S -semicommutative module, $\bar{f}\bar{g}(n) = 0$ for all $g \in S'$. This implies that $\bar{f}\bar{g}(n) = fg(n) = 0$ for all $g \in S'$.
- (2) Since R is S -semicommutative, S is semicommutative. Let $f(ar) = 0$ where $f \in \text{End}_R(aR)$ and $ar \in aR$. Then for all $g \in \text{End}_R(aR)$, $fg(ar) = ar_1r_2r$ where $f(a) = ar_1$ and $g(a) = ar_2$. Since $f(ar) = 0, ar_1r = 0$ and $S \cong R$ is semicommutative, we get $ar_1r_2r = 0$. This completes the proof. \square

COROLLARY 2.7

Every direct summand M'_R of M_R is S' -semicommutative, where $S' = \text{End}_R(M')$.

Proof. From the proof of Lemma 2.6(1) we conclude that direct summand M' of M is also S' -semicommutative with respect to its endomorphism ring $S' = \text{End}(M')$. \square

The following example shows that if M is an S -semicommutative module, then any submodule N of M may not be a T -semicommutative module where $S = \text{End}_R(M)$ and $T = \text{End}_R(N)$.

Example C. There exists a module M with a submodule $N, S = \text{End}_R(M)$ and $T = \text{End}_R(N)$ such that M is S -semicommutative, but N is not T -semicommutative.

Proof. Let F be any field, $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in F \right\}, M = R_R$ and $N = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$. Then $\text{End}_R(M) \cong R$ and M is R -semicommutative by [1]. Let $f \in T$ be defined by

$$f \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \in N.$$

Then $f \in T$. Let $N = N_1 \oplus N_2$ where $N_1 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ and $e \in T$ be the projection of N onto N_2 , i.e., $e(n_1 + n_2) = n_2$ where $n_1 \in N_1$ and $n_2 \in N_2$. Then $f \in T$ and $e \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$. But $ef \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. Hence N is not T -semicommutative. \square

Lemma 2.8. Let $M = M_1 \oplus M_2, M_1$ be S_1 -semicommutative and M_2 be S_2 -semicommutative, where $S_1 = \text{End}_R(M_1)$ and $S_2 = \text{End}_R(M_2)$. If $\text{Hom}(M_i, M_j) = 0$ for $1 \leq i \neq j \leq 2$, then M is S -semicommutative, where $S = \text{End}_R(M)$.

Proof. Let $fm = 0$ and $S = S_1 \oplus S_2$. Then $f = f_1 + f_2$ and $m = m_1 + m_2$ and $fm = f_1m_1 + f_2m_2$. So $f_1m_1 = 0, f_2m_2 = 0$. By hypothesis $f_1S_1m_1 = 0, f_2S_2m_2 = 0$. Hence $fSm = 0$. \square

COROLLARY 2.9

Let e be an idempotent in a ring R . Then R is a semicommutative ring if and only if e is a central idempotent, eR and $(1 - e)R$ are semicommutative rings.

Lemma 2.10. *Let $M = M_1 \oplus M_2$. If $\text{Hom}(M_2, M_1) = 0$ or $\text{Hom}(M_1, M_2) = 0$, then $S = \text{End}_R(M)$ is not semicommutative.*

Proof. Let $0 \neq f \in \text{Hom}(M_2, M_1)$ with $f(m_2) \neq 0$ where $m_2 \in M_2$. Then $(\pi_1 f \pi_2)(m_2) = \pi_1(f(m_2)) = f(m_2)$. Hence $\pi_1 f \pi_2 \neq 0$. This implies that S is not semicommutative since $\pi_1 \pi_2 = 0$. \square

COROLLARY 2.11

Let $M = M_1 \oplus M_2$. If $S = \text{End}_R(M)$ is a semicommutative ring, then $\text{Hom}(M_i, M_j) = 0$ where $1 \leq i \neq j \leq 2$.

Lemma 2.12. *Let M be a duo module. Then M is S -semicommutative.*

Proof. Let $f \in S$ and $m \in M$ with $f(m) = 0$. By hypothesis $g(m) \in mR$ for any $g \in S$. Then $fg(m) \in f(m)R = 0$. Hence $fg(m) = 0$ for all $g \in S$. So S is semicommutative. \square

Lemma 2.13. *Let $M = M_1 \oplus M_2$. If M is weak duo (see [8] in detail) and M_1 and M_2 are S_1 -semicommutative and S_2 -semicommutative submodules respectively where $\text{End}(M_1) = S_1$ and $\text{End}(M_2) = S_2$, then M is S -semicommutative.*

Proof. Since M is weak duo, M_1 and M_2 are fully invariant submodules. Let $f(m) = 0$. If $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$, then $f(m_1 + m_2) = 0$ and so $f(m_1) = f(m_2) = 0$. Since M_1 and M_2 are S_1 - and S_2 -semicommutative submodules respectively, for all $g \in S, fg(m) = 0$. \square

S -semicommutative modules are not closed under direct sums. Let $R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$, where F is a field. It is well known that R is not a semicommutative ring and thus $F \oplus F$ is not an S -semicommutative module, since $\text{End}_F(F \oplus F) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$. Also $F \oplus F$ is not an S -semicommutative module, but F is an S_1 -semicommutative module where $S = \text{End}_F(F \oplus F)$ and $S_1 = \text{End}_F(F)$. Also we understand from this example that this property is not extension closed.

Now we investigate at least when this case can be possible?

Lemma 2.14. *Let R be a ring and I be a fully invariant reduced ideal of R . If R/I is an S -semicommutative ring where $S = \text{End}_R(R/I)$, then R is an S_1 -semicommutative where $S_1 = \text{End}_R(R)$.*

Proof. Let $f(a) = 0$ where $f \in S_1$ and $g \in S_1$. Let $f_1: R/I \rightarrow R/I$ and $g_1: R/I \rightarrow R/I$ such that $f_1(r + I) = f(r) + I$ and $g_1(r + I) = g(r) + I$. Then f_1 and g_1 are module homomorphisms over R . Since R/I is an S -semicommutative ring $f_1 g_1(a + I) =$

$fg(a) + I = I$ and so $fg(a) \in I$. Since $(aIf(1))^2 = 0$ and I is reduced, $aIf(1) = 0$. Then $(f(1)g(1)aI)^2 = f(1)g(1)(aIf(1))g(1)aI = 0$ and so $f(1)g(1)aI = 0$ and $f(g(a)) = 0$. Therefore R is an S_1 -semicommutative module. \square

Furthermore if R is a semicommutative ring and so an S -semicommutative module where $S = \text{End}_R(R)$, then R/I may not be an S_1 -semicommutative module where $S_1 = \text{End}_R(R/I)$ and I is a right ideal. Let

$$R = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e \in F \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a \in F \right\}$$

where F is a field. Let $f \left(\begin{pmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} + I \right) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & a & 0 & e \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} + I$ and $g \left(\begin{pmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} + I \right) = \begin{pmatrix} a & 0 & 0 & e \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} + I$. Then $f, g \in S_1$ and $f \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + I \right) = I$ and $f \left(g \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + I \right) \right) \neq I$ and so R/I is not an S_1 -semicommutative module.

Lemma 2.15. Let M be an S -semicommutative module. Consider the following:

- (1) M is an S -Baer module.
- (2) M is an S -quasi-Baer module.
- (3) M is an S -p.q.-Baer module.

Then (1) \Leftrightarrow (2) \Rightarrow (3).

Proof. (1) \Rightarrow (2) \Rightarrow (3) is clear. (2) \Rightarrow (1). Let N be any submodule of M and $n \in N$. By hypothesis $l_S(n) = l_S(SnR)$. Hence $l_S(N) = l_S(SN)$. Since SN is a fully invariant submodule of M , by (2) $l_S(SN) = Se$ for some $e^2 = e \in S$. This completes the proof. \square

PROPOSITION 2.16

Following are equivalent for an S -semicommutative module M .

- (1) M is an S -p.q.-Baer module.
- (2) The left annihilator in S of any finitely generated R -submodule of M is generated (as a left ideal) by an idempotent of S .

Proof. (2) \Rightarrow (1) is clear. (1) \Rightarrow (2): Assume that M is an S -p.q.-Baer module and let N be a finitely generated R -submodule of M . We will prove only for $n = 2$. Same proof will work for any n . Let $N = n_1R + n_2R$. By (1), $l_S(n_1R) = Se_1$ and $l_S(n_2R) = Se_2$. By Lemma 2.1, $e_1e_2 = e_2e_1$ and so e_1e_2 becomes an idempotent. Hence $l_S(N) = Se_1e_2$. This completes the proof. \square

A module M_R is called a *principally projective* (or simply p.p.-module) if, for any $m \in M, r_R(m) = eR$ where $e^2 = e \in R$ (see [6]). In [6] Lee-Zhou introduced the following notation. For a module M_R , we consider $M[x] = \left\{ \sum_{i=0}^s m_i x^i : s \geq 0, m_i \in M \right\}$, $M[x]$ is an Abelian group under an obvious addition operation. Moreover $M[x]$ becomes a right $R[x]$ -module under the following scalar product operation:

For

$$m(x) = \sum_{i=0}^s m_i x^i \in M[x] \quad \text{and} \quad f(x) = \sum_{i=0}^t a_i x^i \in R[x],$$

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i a_j \right) x^k.$$

By these operations $M[x]$ becomes a right module over $R[x]$. Similarly, $M[x]$ is a left $S[x]$ -module by the scalar product:

For

$$m(x) = \sum_{i=0}^s m_i x^i \in M[x] \quad \text{and} \quad \alpha(x) = \sum_{i=0}^t f_i x^i \in S[x],$$

$$\alpha(x)m(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} f_i m_j \right) x^k.$$

PROPOSITION 2.17

Let M be an S -p.q.-Baer module. Then M is an S -semicommutative module if and only if $fem = efm$, for any $m \in M$, $f \in S$, and $e^2 = e \in S$.

Proof. The necessity is clear from Lemma 2.1. Conversely, assume that $fem = efm$, for any $m \in M$, $f \in S$ and $e^2 = e \in S$. Let $fm = 0$ for some $f \in S$ and $m \in M$. There exists $e^2 = e \in S$ such that $f \in l_S(m) = Se$. Then $f = fe$ and $em = 0$. For any $g \in S$, by assumption $fgm = fegm = fgem = 0$. Hence M is S -semicommutative. \square

Lemma 2.18. *Let M be a module and $S = \text{End}(M)$. If M is an S -p.q.-Baer module, then M is S -semicommutative if and only if $M[x]$ is $S[x]$ -semicommutative.*

Proof. Assume that M is S -semicommutative module. Let $m(x) = \sum m_i x^i \in M[x]$, $f(x) = \sum f_j x^j \in S[x]$ satisfy $f(x)m(x) = 0$. Then

$$f_0 m_0 = 0, \tag{1}$$

$$f_0 m_1 + f_1 m_0 = 0, \tag{2}$$

$$f_0 m_2 + f_1 m_1 + f_2 m_0 = 0, \tag{3}$$

$$\dots \tag{4}$$

Let $l_S(m_0) = Se_0$, $l_S(m_1) = Se_1$, $l_S(m_2) = Se_2$, \dots where $e_i^2 = e_i \in S$. By hypothesis S is abelian and by (1), $f_0 e_0 = e_0 f_0 = f_0$. Left multiply (2) by e_0 to obtain $f_0 m_1 = 0$. Hence $f_1 m_0 = 0$. So $f_0 e_1 = e_1 f_0 = f_0$ and $f_1 e_0 = e_0 f_1 = f_1$. Left multiply (3) by e_0 to obtain $f_2 m_0 = 0$ so (3) becomes $f_0 m_2 + f_1 m_1 = 0$. Multiply this equality by e_1 from left to have $f_2 m_0 = 0$. Hence $f_1 m_1 = 0$. Continuing in this way we may obtain $f_i m_j = 0$ for all i and j . The rest is clear. \square

To get rid of confusions we recall that $M[x]$ is an $S[x]$ -p.q.-Baer module if for any $m(x) \in M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(m(x)) = eS[x]$, and $M[x]$ is an $S[x]$ -Baer-module if for any $R[x]$ -submodule A of $M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(A) = eS[x]$, and $M[x]$ is an $S[x]$ -q.-Baer module if for any fully invariant $R[x]$ -submodule A of $M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}m(x) = eS[x]$.

Lemma 2.19. Let M be a module such that $S = \text{End}(M)$ is a semicommutative ring. Then

- (1) Every idempotent of $S[x]$ is in S and $S[x]$ is abelian.
- (2) Every idempotent of $S[[x]]$ is in the S and $S[[x]]$ is abelian.

Proof. Clear from Lemma 8 of [5].

Theorem 2.20. Let M be an S -semicommutative module. Then

- (1) M is an S -p.q.-Baer module if and only if $M[x]$ is an $S[x]$ -p.q.-Baer module.
- (2) M is an S -Baer module if and only if $M[x]$ is an $S[x]$ -Baer module.
- (3) M is an S -q.-Baer module if and only if $M[x]$ is an $S[x]$ -q.-Baer module.

Proof. Let M be an S -semicommutative module. By Lemma 2.1, S is semicommutative and so an abelian ring.

(1) \Rightarrow . Assume that M is an S -p.q.-Baer module. Let $m(x) = \sum_{i=0}^k m_i x^i \in M[x]$, $f(x) = \sum_{j=0}^t f_j x^j \in S[x]$ satisfy $f(x)m(x) = 0$. Let $l_S(m_i) = Se_i$ where $e_i^2 = e_i \in S$ ($i = 0, 1, 2, \dots, k$). Since S is abelian, $f_i m_j = 0$ implies $f_i e_j = e_j f_i = f_i$ for all $i = 0, 1, 2, \dots, t$ and $j = 0, 1, 2, \dots, k$. Let $e = e_0 e_1 e_2 \dots e_k$. Then e is a central idempotent in S . We prove $l_{S[x]}(m(x)) = S[x]e$. Let $f(x) = \sum f_j x^j \in l_{S[x]}(m(x))$, then $f_j e = f_j$ and so $f(x)e = f(x)$. Hence $f(x) \in S[x]e$ and so $l_{S[x]}(m(x)) \leq S[x]e$. Let $g(x) \in S[x]e$. Since S is abelian, $em(x) = 0$ and $g(x)em(x) = 0$. Hence $S[x]e \leq l_{S[x]}(m(x))$.

\Leftarrow . Suppose that $M[x]$ is an $S[x]$ -p.q.-Baer module. Let $m \in M$. Then $l_{S[x]}(m) = S[x]e$ for some $e^2 = e \in S[x]$. By Lemma 2.19, $e \in S$. Clearly $(S[x]e) \cap S = Se$. Hence $l_S(m) = Se$.

(2) \Rightarrow . Assume that M is an S -Baer module. Let A be any $R[x]$ -submodule of $M[x]$. We will prove that there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(A) = S[x]e$. Let A^* be the right R -submodule of M generated by the coefficients of elements of A . By assumption $l_S(A^*) = Se$ for some $e^2 = e \in S$. Then $S[x]e \leq l_{S[x]}(A)$ is clear. To prove reverse inclusion, let $g(x) = c_0 + c_1 x + \dots + c_n \in l_{S[x]}(A)$. Then $g(x)A = 0$ and so $g_i A = 0$. By Lemma 2.1, S is semicommutative and so abelian. Hence S is Armendariz, that is, $g_i A^* = 0$, $g_i \in l_S(A^*) = Se$ and $g_i e = g_i$ for all $0 \leq i \leq n$. So $g(x)e = g(x) \in S[x]e$. $l_{S[x]}(A) \leq S[x]e$. Therefore $l_{S[x]}(A) = S[x]e$.

\Leftarrow . Assume that $M[x]$ is an $S[x]$ -Baer-module. Let A be any submodule of M . Then $l_{S[x]}(A[x]) = S[x]e$ for some $e^2 = e \in S[x]$. By Lemma 2.19, $e \in S$. Then $(S[x]e) \cap S = Se$. Hence M is an S -Baer module.

(3) Similar to proof of (2). □

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