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Clean elements in abelian rings

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Abstract. Let R be a ring with identity. An element in R is said to be clean if it is the sum of a unit and an idempotent. R is said to be clean if all of its elements are clean. If every idempotent in R is central, then R is said to be abelian. In this paper we obtain some conditions equivalent to being clean in an abelian ring.

Keywords. Clean; topologically boolean; abelian ring.

1. Introduction

Let R be a ring with identity. An element in R is said to be clean if it is the sum of a unit and an idempotent. R is said to be clean if all of its elements are clean. The notion of a clean ring was introduced by Nicholson [3]. A compilation of characterizations of commutative clean rings can be found in [2]. In this paper we consider rings (not necessarily commutative) with central idempotents and obtain some conditions equivalent to being clean. A ring is said to be abelian if all of its idempotents are central. Among the conditions that are shown to be equivalent to being clean for an abelian ring is 'topologically boolean'. In line with [1] we say that a ring R (not necessarily commutative) is *right* (resp. *left*) *topologically boolean*, or a *right* (resp. *left*) *tb-ring* for short, if for every pair of distinct maximal right (resp. left) ideals of R there is an idempotent in exactly one of them. We say that R is a tb-ring if it is both left and right tb. Throughout this paper all rings considered are associative with identity.

2. Preliminaries

In this section we only discuss right ideals. The left analogue of the definitions and results in this section also hold. For a ring R, a proper right ideal P of R is prime if $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. Let $\mathrm{Spec}_r(R)$ be the set of all prime right ideals of R. It has been shown in Corollary 2.8 of [5], that if R is not a right quasi-duo ring, then $\mathrm{Spec}_r(R)$ is a space with the weak Zariski topology but not with the Zariski topology. The weak Zariski topology on $\mathrm{Spec}_r(R)$ has also been studied in [4, 6].

For a right ideal I of R, let $\mathcal{U}_r(I) = \{P \in \mathrm{Spec}_r(R) | P \not\supseteq I\}$ and $\mathcal{V}_r(I) = \mathrm{Spec}_r(R) \backslash \mathcal{U}_r(I)$. Let τ be the collection

$$\tau = \{\mathcal{U}_r(I) | I \text{ is a right ideal of } R\}.$$

Then τ contains the empty set and $\mathrm{Spec}_r(R)$. In general, τ is just a subbase of the weak Zariski topology on $\mathrm{Spec}_r(R)$. For any element $a \in R$, let $\mathcal{U}_r(a) = \mathcal{U}_r(aR)$ and $\mathcal{V}_r(a) = \mathcal{V}_r(aR)$. Then $\mathcal{U}_r(a) = \{P \in \mathrm{Spec}_r(R) | a \notin P\}$ and $\mathcal{V}_r(a) = \{P \in \mathrm{Spec}_r(R) | a \in P\}$.

Let $\mathrm{Max}_r(R)$ be the set of all maximal right ideals of R. Since maximal right ideals are prime right ideals (see [4]), $\mathrm{Max}_r(R)$ inherits the weak Zariski topology on $\mathrm{Spec}_r(R)$. Let $U_r(I) = \mathrm{Max}_r(R) \cap \mathcal{U}_r(I)$ and $V_r(I) = \mathrm{Max}_r(R) \cap \mathcal{V}_r(I)$ for any right ideal I of R.

A clopen set in a topological space is a set which is both open and closed. A topological space is said to be zero-dimensional if it has a base consisting of clopen sets. It will be shown here that $\mathrm{Max}_r(R)$ is a compact Hausdorff space.

Lemma 2.1. *Let R be a ring. Then* $\mathrm{Max}_r(R)$ *is a compact T_1-space.*

Proof. Let $\cup U_r(I_\alpha)$ be an open cover of $\mathrm{Max}_r(R)$ consisting of subbasic open sets $U_r(I_\alpha)$. Then $\mathrm{Max}_r(R) = \cup U_r(I_\alpha) = \cup(\mathrm{Max}_r(R) \backslash V_r(I_\alpha)) = \mathrm{Max}_r(R) \backslash (\cap V_r(I_\alpha))$. This implies that $\cap V_r(I_\alpha) = \emptyset$. Now, $V_r(\sum I_\alpha) = \cap V_r(I_\alpha) = \emptyset$ and hence, $\sum I_\alpha = R$. It follows that $R = I_1 + \cdots + I_n$ for some $I_i \in \{I_\alpha\}$ $(i = 1, \ldots, n)$. Then $V_r(I_1 + \cdots + I_n) = \emptyset$ and hence, $\mathrm{Max}_r(R) = U_r(I_1) \cup \cdots \cup U_r(I_n)$. It follows by Alexander's subbase theorem that $\mathrm{Max}_r(R)$ is compact.

To show that $\mathrm{Max}_r(R)$ is a T_1-space, let $M_1, M_2 \in \mathrm{Max}_r(R)$ with $M_1 \neq M_2$. Then there exists $a \in M_1$, $a \notin M_2$ and hence $U_r(a)$ is an open set in $\mathrm{Max}_r(R)$ such that $M_2 \in U_r(a)$ and $M_1 \notin U_r(a)$.

Let $\mathrm{Id}(R)$ denote the set of all idempotents of R and let $\xi = \{U_r(e) | e \in \mathrm{Id}(R)\}$. We begin with the following lemma, the proof of which is straightforward.

Lemma 2.2. *Let R be an abelian ring and N a maximal right ideal of R. If $e^2 = e \in R$ and $e \notin N$, then $1 - e \in N$.*

Note that Lemma 2.2 is not necessarily true if R is not an abelian ring (for example, take R to be the ring of 2×2 matrices over $\mathbb{Z}_2$).

Lemma 2.3. *Let R be an abelian ring. Then for $e, f \in \mathrm{Id}(R)$,*

(a) $U_r(e) \cap U_r(f) = U_r(ef) = U_r(fe)$;
(b) $U_r(e) \cup U_r(f) = U_r(e + f - ef)$;
(c) $U_r(e) = V_r(1 - e)$.

In particular, every set in ξ is clopen.

Proof. Since idempotents are central in R, it is straightforward to prove parts (a) and (b). For part (c), we have by Lemma 2.2 that $U_r(e) = \mathrm{Max}_r(R) \backslash U_r(1 - e) = V_r(1 - e)$. Thus every set in ξ is clopen.

Remark. In Theorem 3.5 of [6], it has been shown that the set $\{\mathcal{U}_r(e) | e \in \mathrm{Id}(R)\}$ consists of all the clopen sets in $\mathrm{Spec}_r(R)$.

PROPOSITION 2.4

Let R be an abelian clean ring. Then R is a right tb-ring.

Proof. Suppose that M and N are distinct maximal right ideals of R. Let $a \in M \backslash N$. Then $N + aR = R$ and hence, $1 - ax \in N$ for some $x \in R$. Let $r = ax$. Then $1 - r \in N$

and $r \in M \backslash N$. Since R is clean, there exist an idempotent e and a unit u in R such that $r = e + u$. If $e \in M$, then $u = r - e \in M$ from which it follows that $M = R$; a contradiction since M is a maximal right ideal of R. Thus $e \notin M$. If $e \notin N$, then $1 - e \in N$ (by Lemma 2.2) and hence, $u = r - e = (r - 1) + (1 - e) \in N$. It follows that $N = R$ which is also not possible since N is a maximal right ideal of R. We thus have that e is an idempotent belonging to N only.

PROPOSITION 2.5

Let R be an abelian ring. If R is a right tb-ring, then ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$. *In particular,* $\mathrm{Max}_r(R)$ *is a compact, zero-dimensional Hausdorff space.*

Proof. Notice that the last statement follows from the second. Next, note that if M_1, M_2 are two distinct maximal right ideals of R, then since R is a right tb-ring, there exists $e \in \mathrm{Id}(R)$ such that $e \notin M_1$, $e \in M_2$ (that is, $M_1 \in U_r(e)$, $M_2 \notin U_r(e)$). The points in $\mathrm{Max}_r(R)$ can therefore be separated by clopen sets belonging to ξ. Hence $\mathrm{Max}_r(R)$ is Hausdorff. By Lemma 2.1, we know that $\mathrm{Max}_r(R)$ is compact.

To show that ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$, let $K \subseteq \mathrm{Max}_r(R)$ be a closed subset and take $M \notin K$. For each $N \in K$, since $N \neq M$, there exists a clopen set $U_r(e_N) \in \xi$ separating M and N, say $N \in U_r(e_N)$. The collection $\{U_r(e_N) | N \in K\}$ is therefore an open cover of the set K. Since K is compact, it has a finite subcover, that is, K is contained in a finite cover of sets of the form $U_r(e_N)$ with $N \in K$. By Lemma 2.3, there exists a clopen set $C \in \xi$ separating M from K. Hence ξ forms a base for the weak Zariski topology on $\mathrm{Max}_r(R)$.

PROPOSITION 2.6

Let R be an abelian ring. If R is a right tb-ring, then for any $a \in R$, there exists an idempotent $e \in R$ such that $e \notin M$ for every $M \in V_r(a)$ and $e \in N$ for every $N \in V_r(a-1)$.

Proof. Since $V_r(a)$, $V_r(a-1)$ are disjoint closed sets and $\mathrm{Max}_r(R)$ is a compact T_1-space (by Lemma 2.1), we have by Proposition 2.5 that there is a clopen set $U_r(e) \in \xi$ separating the sets $V_r(a)$, $V_r(a-1)$, say $V_r(a) \subseteq U_r(e)$ and $V_r(a-1) \subseteq V_r(e)$. Hence $e \in \mathrm{Id}(R)$ such that $e \notin M$ for every $M \in V_r(a)$ and $e \in N$ for every $N \in V_r(a-1)$.

Recall that a ring R is said to be directly finite if $uv = 1$ implies that $vu = 1$ for any $u, v \in R$. The following lemma is known but we prove it for completeness sake.

Lemma 2.7. *An abelian ring is directly finite.*

Proof. Let R be an abelian ring and suppose that $uv = 1$ where $u, v \in R$. Then $(vu)^2 = vuvu = vu$ and so vu is an idempotent and thus central. Then $u = (uv)u = u(vu) = vuu$ and it follows that $1 = uv = vuuv = vu$.

3. Main result

The main result in this paper is as follows:

Theorem 3.1. *Let R be an abelian ring. The following conditions are equivalent.*

(a) *R is clean.*
(b) *R is an exchange ring.*

(c) *R is a right tb-ring.*
(d) *The collection* $\xi = \{U_r(e) | e \in \mathrm{Id}(R)\}$ *forms a base for the weak Zariski topology on* $\mathrm{Max}_r(R)$.
(e) *For every* $a \in R$ *there is an* $e \in \mathrm{Id}(R)$ *such that* $V_r(a) \subseteq U_r(e)$ *and* $V_r(a-1) \subseteq V_r(e)$.
(f) *R is a left tb-ring.*
(g) *The collection* $\xi = \{U_l(e) | e \in \mathrm{Id}(R)\}$ *forms a base for the weak Zariski topology on* $\mathrm{Max}_l(R)$.

Proof.

(a) ⇔ (b): This is known by an earlier work of Nicholson [3].
(a) ⇒ (c): This follows by Proposition 2.4.
(c) ⇒ (d) ⇒ (e): This follows by Propositions 2.5 and 2.6.
(e) ⇒ (a): Let $a \in R$. By the hypothesis, there exists $e \in \mathrm{Id}(R)$ such that $V_r(a) \subseteq U_r(e)$ and $V_r(a-1) \subseteq V_r(e)$. We claim that $a - e$ has a right inverse. Let M be a maximal right ideal of R. Note that if $a \in M$, then $a - e \notin M$ since $e \notin M$. Next suppose that $a \notin M$. If $a - e \in M$, then $e \notin M$ and hence, $1 - e \in M$ (by Lemma 2.2). Then since $(a-1) + (1-e) = a - e \in M$, it follows that $a - 1 \in M$ and hence, $e \in M$ (because $V_r(a-1) \subseteq V_r(e)$); a contradiction. Thus $a - e \notin M$. We have therefore shown that $a - e \notin M$ for any maximal right ideal M of R. Hence $a - e$ has a right inverse. Since R is directly finite (by Lemma 2.7), it follows that $a - e$ is a unit. Hence, a is the sum of a unit and an idempotent. Since a is arbitrary in R, it follows that R must be clean.
(a) ⇔ (f) ⇔ (g): This follows from the left analogue of the arguments for (a) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (a). This completes the proof.

In the case when R is a commutative ring, the equivalence of the conditions in Theorem 3.1 together with several other conditions have been shown in Theorem 1.7 of [2].

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