Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 1, February 2009, pp. 119–135.

On upper bounds for the growth rate in the extended Taylor–Goldstein problem of hydrodynamic stability

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MS received 9 May 2007; revised 30 October 2007

Abstract. For the extended Taylor–Goldstein problem of hydrodynamic stability governing the stability of shear flows of an inviscid, incompressible but density stratified fluid in sea straits of arbitrary cross-section a new estimate for the growth rate of an arbitrary unstable normal mode is given for a class of basic flows. Furthermore the Howard's conjecture, namely, the growth rate $kc_i \to 0$ as the wave number $k \to \infty$ is proved for two classes of basic flows.

Keywords. Shear flows; hydrodynamic stability; sea straits; variable bottom.

1. Introduction

The Taylor–Goldstein problem (or TGP for short) of hydrodynamic stability deals with the stability of shear flows of an inviscid, incompressible but density stratified fluid to infinitesimal normal mode disturbances and this problem has been extensively studied (see, for example, [5] and [17]). Recently Pratt *et al* [13] have studied an extended Taylor–Goldstein problem (or ETGP for short) which considers a flow domain with variable bottom as this is necessary for the study of shear instabilities in sea straits. The ETGP problem is given by the second order ordinary differential equation

$$W'' + \left[\frac{N^2}{(U_0-c)^2} - \frac{U_0''}{(U_0-c)} - k^2\right]W + \frac{1}{(U_0-c)}[T(U_0-c)W]' = 0, \tag{1}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{2}$$

Here a prime denotes differentiation with respect to the vertical coordinate z varying over [0, D] and the real part of $W(z)e^{ik(x-ct)}$ is the vertical velocity of a normal mode disturbance, $k > 0$ is the wave number, $c = c_r + ic_i$ is the complex phase velocity, $U_0(z)$ is the basic velocity, $N^2 \geq 0$ is the stratification parameter and $T(z)$ is the topography.

An equivalent form of equation (1) is

$$\left[\frac{(bW)'}{b}\right]' + \left[\frac{N^2}{(U_0-c)^2} - \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c)} - k^2\right]W = 0, \tag{3}$$

where $b(z)$ is the width function and $T(z) = \frac{b'(z)}{b(z)} = [\log b(z)]'$. When the width function is a constant we have $T(z) \equiv 0$ and the ETGP reduces to the standard TGP.

For the ETGP the following results are already known:

(i) A sufficient condition for stability is that the minimum Richardson number $J_0 = \min\left[\frac{N^2}{(U_0')^2}\right]$ is greater than or equal to one quarter (cf. Deng *et al* [4]).
(ii) The instability region is a semi-circle in the upper half of $c_r - c_i$ plane whose diameter is given by the range of the basic velocity profile (cf. Deng *et al* [4]).
(iii) The instability region is a semi-ellipse lying inside the semicircle of Deng *et al* [4], whose major axis is same as the range of the basic velocity profile while the minor axis depends on J_0 [19].
(iv) The instability region lies inside a semi-ellipse type region in the upper half of $c_r - c_i$ plane which depends on the depth of the fluid layer in addition to J_0. Moreover, for monotonic velocity profiles, this region reduces to the line $c_i = 0$ as $J_0 \to (1/4)$ [8].
(v) When $N^2 = 0$, it has been proved that the growth rate tends to zero as the wave number tends to ∞ [18].

For the TGP, Howard [11] obtained an estimate for the growth rate of an unstable mode, given by

$$k^2 c_i^2 \leq \max\left[\frac{(U_0')^2}{4} - N^2\right].$$

In the absence of stratification this reduces to the Höiland's estimate, namely,

$$k^2 c_i^2 \leq \max\left[\frac{(U_0')^2}{4}\right].$$

In the paper of Howard [11] it is stated that 'this estimate is not usually sharp' (cf. page 511, line 15 of [11]). Furthermore it is stated that from this estimate it follows that $c_i \to 0$ as $k \to \infty$. 'But there is a likelihood that in fact $kc_i \to 0$ as $k \to \infty$'. This statement which has come to be known as the Howard's conjecture has been proved for the Rayleigh problem in [3] and for two classes of basic flows in the TGP (cf. [15] and [1]).

A different type of estimate has been obtained by Sattinger [14] for the Rayleigh problem, namely,

$$k^2 c_i^2 \leq |U_0''|_{\max}[U_{0\max} - U_{0\min}].$$

This estimate for the growth rate has been generalized to the TGP in [16].

For the Rayleigh–Kuo problem, Hickernell [10] has extended both types of estimates for the growth rate. It is found in Hickernell [10] that the Sattinger type estimate is sharper than that of the Höiland type estimate for large values of the Coriolis parameter $|\beta|$. But both estimates are found to be weak compared to the growth rates found by numerical computation for a basic flow. Hence, Hickernell [10] concluded that these estimates are not tight (see page 92 of [10]).

From the above works, namely, Howard [11] and Hickernell [10] it is clear that sharper bounds on the growth rate of unstable modes should be obtained.

In our paper we have found a new estimate for the growth rate of an unstable mode for a class of basic flows of the ETGP, i.e., for flows with topography satisfying $T' \leq 0$. This estimate is applicable to the standard TGP also and this estimate is shown to be sharper than that of [16].

Furthermore we obtain the proof of Howard's conjecture for two classes of basic flows of the ETGP. In the first class the basic flow is of the Garcia type and the topography satisfies $T'(z) \leq 0$ and we extend the result of Banerjee *et al* [1]. However it should be noted that the flow domain in ETGP is bounded in the z-direction unlike the case of the TGP where the domain may be bounded or unbounded.

In the second class we consider basic flow with weak stratification as in [15] and $T' \leq 0$. The condition on T includes flow for which T is equal to any constant which means that the width function $b(z)$ is an exponential function and also the flow with $T = 1/z$ which is the topography appropriate for the sea straits like Bab al Mandab as shown in Deng *et al* [4].

2. Preliminary results

In Lemmas 1–4 we obtain some preliminary results that are needed for the proof of the main results presented later. In what follows all the integrals are integrals over $(0, D)$ only so that the upper and lower limits are dropped for simplicity.

Lemma 1.

$$\mathrm{Re}\left[\int TW^*W'\mathrm{d}z\right] = \frac{-1}{2}\int T'|W|^2\mathrm{d}z \tag{4}$$

and

$$\mathrm{Im}\left[\int TW^*W'\mathrm{d}z\right] = c_i\int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}|W|^2\mathrm{d}z, \tag{5}$$

where W^* is the complex conjugate of W. Re *and* Im *denote the real and imaginary parts respectively.*

Proof. Let $I = \int TW^*W'\mathrm{d}z$. Using integration by parts and applying the boundary conditions (2), we get

$$\int TW^*W'\mathrm{d}z = -\int T'|W|^2\mathrm{d}z - \int TW(W^*)'\mathrm{d}z,$$

$$\int T[W^*W' + W(W^*)']\mathrm{d}z = -\int T'|W|^2\mathrm{d}z.$$

Therefore,

$$\mathrm{Re}\left[\int TW^*W'\mathrm{d}z\right] = \frac{-1}{2}\int T'|W|^2\mathrm{d}z$$

and

$$\mathrm{Im}\left[\int TW^*W'\mathrm{d}z\right] = c_i\int \frac{b\left(\frac{U_0'}{b}\right)'}{|U_0-c|^2}|W|^2\mathrm{d}z.$$

This completes the proof of the lemma. □

Lemma 2.

$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z \geq \frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2} \int |W|^2 \mathrm{d}z. \tag{6}$$

Proof. We have

$$\int \left[\frac{(bW)'}{b} \right]' (bW)^* \mathrm{d}z = -\int \frac{|(bW)'|^2}{b} \mathrm{d}z,$$

$$\begin{aligned} \int \frac{|(bW)'|^2}{b} \mathrm{d}z &\leq \left| \int \left[\frac{(bW)'}{b} \right]' (bW)^* \mathrm{d}z \right| \\ &\leq \int \left| \left[\frac{(bW)'}{b} \right]' \right| |(bW)^*| \mathrm{d}z \\ &\leq \left[\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z \right]^{\frac{1}{2}} \left[\int |(bW)|^2 \mathrm{d}z \right]^{\frac{1}{2}} \end{aligned}$$

(by Cauchy–Schwartz inequality).

Using the well-known Rayleigh–Ritz inequality, we have

$$\int \frac{|(bW)'|^2}{b} \mathrm{d}z \geq \frac{\pi^2}{D^2 b_{\max}} \int |(bW)|^2 \mathrm{d}z,$$

and consequently, we have

$$\frac{\pi^2}{D^2 b_{\max}} \int |(bW)|^2 \mathrm{d}z \leq \left[\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z \right]^{\frac{1}{2}} \left[\int |(bW)|^2 \mathrm{d}z \right]^{\frac{1}{2}}$$

and hence,

$$\frac{\pi^2}{D^2 b_{\max}} \left[\int |(bW)|^2 \, \mathrm{d}z \right]^{\frac{1}{2}} \leq \left[\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z \right]^{\frac{1}{2}}.$$

Now, squaring on both sides, we get

$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z \geq \frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2} \int |W|^2 \mathrm{d}z.$$

This completes the proof of the lemma. □

When $b(z)$ is a positive constant our inequality (6) reduces to that of Lemma 1 in Banerjee *et al* [2].

Lemma 3. *A necessary condition for the existence of non-trivial solutions with $c_i > 0$ is that the integral relation*

$$\int |W'|^2 \mathrm{d}z + k^2 \int |W|^2 \mathrm{d}z + \int \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z$$

$$- \int \frac{N^2[(U_0 - c_r)^2 - c_i^2]}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z - \frac{1}{2} \int T' |W|^2 \mathrm{d}z = 0 \quad (7)$$

and

$$- \operatorname{Im}\left[\int T W^* W' \mathrm{d}z\right] + \int \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z$$

$$- 2 \int \frac{N^2 (U_0 - c_r)}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z = 0, \quad (8)$$

are true.

Proof. Multiplying (1) by W^* (complex conjugate of W), integrating the resulting equation over $[0, D]$ and using (2), we get

$$\int |W'|^2 \mathrm{d}z + k^2 \int |W|^2 \mathrm{d}z - \int \frac{N^2}{(U_0 - c)^2} |W|^2 \mathrm{d}z$$

$$+ \int \frac{U_0''}{U_0 - c} |W|^2 \mathrm{d}z - \int T' |W|^2 \mathrm{d}z - \int T W' W^* \mathrm{d}z$$

$$- \int \frac{T U_0'}{U_0 - c} |W|^2 \mathrm{d}z = 0.$$

The real part of the above gives (using (4) of Lemma 1)

$$\int |W'|^2 \mathrm{d}z + k^2 \int |W|^2 \mathrm{d}z + \int \frac{(U_0'' - T U_0')(U_0 - c_r)}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z$$

$$- \int \frac{N^2[(U_0 - c_r)^2 - c_i^2]}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z - \frac{1}{2} \int T' |W|^2 \mathrm{d}z = 0.$$

Since $U_0'' - T U_0' = b(U_0'/b)'$, we can rewrite the above equation as

$$\int |W'|^2 \mathrm{d}z + k^2 \int |W|^2 \mathrm{d}z - \int \frac{N^2[(U_0 - c_r)^2 - c_i^2]}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z$$

$$+ \int \frac{b\left(\frac{U_0'}{b}\right)'(U_0 - c_r)}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z - \frac{1}{2} \int T' |W|^2 \mathrm{d}z = 0.$$

Equating imaginary parts, using (5) and the fact that $c_i > 0$, we get

$$
\begin{aligned}
&-\operatorname{Im}\left[\int TW^*W'\mathrm{d}z\right] + \int \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z \\
&\quad -2\int \frac{N^2(U_0-c_r)}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z = 0.
\end{aligned}
$$

This completes the proof of the lemma. □

When $b(z)$ is a positive constant this result reduces to Lemma 1 of Banerjee *et al* [1].

Lemma 4. *A necessary condition for the existence of non-trivial solutions with* $c_i > 0$ *is the integral relation*

$$
\begin{aligned}
&\int \left|\left[\frac{(bW)'}{b}\right]'\right|^2 \mathrm{d}z + k^2\int |W'|^2\mathrm{d}z \\
&\quad -\frac{k^2}{2}\int T'|W|^2\mathrm{d}z + 2\int \frac{N^2 b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z \\
&\quad -\int \frac{N^4}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z + k^2\int \frac{N^2[(U_0-c_r)^2-c_i^2]}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z \\
&\quad -\int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z - k^2\int \frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z = 0
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&\int \left|\left[\frac{(bW)'}{b}\right]'\right|^2 \mathrm{d}z + 2k^2\int |W'|^2\mathrm{d}z + k^4\int |W|^2\mathrm{d}z \\
&\quad -k^2\int T'|W|^2\mathrm{d}z + 2\int \frac{N^2 b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{\left[(U_0-c_r)^2+c_i^2\right]^2}|W|^2\mathrm{d}z \\
&\quad -\int \frac{N^4}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z - \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z = 0,
\end{aligned}
\tag{10}
$$

are true.

Proof. Multiplying (3) by $[(bW^*)'/b]'$, integrating over $[0, D]$ and using (2), we get

$$\int \left|\left[\frac{(bW)'}{b}\right]'\right|^2 \mathrm{d}z + \int \left[\frac{N^2}{(U_0-c)^2} - \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c)}\right] W \left[\frac{(bW^*)'}{b}\right]' \mathrm{d}z$$

$$+ k^2 \int |W'|^2 \mathrm{d}z + k^2 \int T W^* W' \mathrm{d}z = 0. \tag{11}$$

From (3) and by taking the complex conjugate, we get

$$\left[\frac{(bW^*)'}{b}\right]' = \left[k^2 + \frac{b\left(\frac{U_0'}{b}\right)'}{(U_0-c^*)} - \frac{N^2}{(U_0-c^*)^2}\right] W^*. \tag{12}$$

Substituting (12) in (11), we get

$$\int \left|\left[\frac{(bW)'}{b}\right]'\right|^2 \mathrm{d}z + k^2 \int |W'|^2 \mathrm{d}z$$

$$+ k^2 \int T W^* W' \mathrm{d}z + \int \frac{N^2 b\left(\frac{U_0'}{b}\right)'}{(U_0-c)^2(U_0-c^*)} |W|^2 \mathrm{d}z$$

$$- \int \frac{N^4}{(U_0-c)^2(U_0-c^*)^2} |W|^2 \mathrm{d}z$$

$$+ k^2 \int \frac{N^2}{(U_0-c)^2} |W|^2 \mathrm{d}z - \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0-c)(U_0-c^*)} |W|^2 \mathrm{d}z$$

$$+ \int \frac{N^2 b\left(\frac{U_0'}{b}\right)'}{(U_0-c)(U_0-c^*)^2} |W|^2 \mathrm{d}z - k^2 \int \frac{b\left(\frac{U_0'}{b}\right)'}{U_0-c} |W|^2 \mathrm{d}z = 0.$$

Taking real part and using (4), we get

$$\int \left|\left[\frac{(bW)'}{b}\right]'\right|^2 \mathrm{d}z + k^2 \int |W'|^2 \mathrm{d}z$$

$$- \frac{k^2}{2} \int T' |W|^2 \mathrm{d}z + 2 \int \frac{N^2 b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{[(U_0-c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z$$

$$-\int \frac{N^4}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z + k^2\int \frac{N^2[(U_0-c_r)^2-c_i^2]}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z$$

$$-\int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z - k^2\int \frac{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z = 0.$$

Multiplying (7) by k^2 and adding the resultant equation to (9), we get

$$\int \left|\left[\frac{(bW)'}{b}\right]'\right|^2 \mathrm{d}z + 2k^2\int |W'|^2\mathrm{d}z + k^4\int |W|^2\mathrm{d}z$$

$$-k^2\int T'|W|^2\mathrm{d}z + 2\int \frac{N^2 b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z$$

$$-\int \frac{N^4}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z - \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z = 0.$$

This completes proof of the lemma. □

When $b(z)$ is a positive constant our result reduces to Lemma 2 of Banerjee *et al* [1].

3. New estimates for growth rate

Now we shall present the new estimates for the growth rate.

Theorem 1. *For the class of flows satisfying $T'(z) \le 0$ in the flow domain an estimate for the growth rate of an arbitrary unstable mode is given by*

$$kc_i \le \left[(N^2_{\max})^2 + \left|b\left(\frac{U_0'}{b}\right)'\right|^2_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2 + N^2_{\max}\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]\right]^{\frac{1}{4}}. \tag{13}$$

Proof. In equation (10) of Lemma 4, the first two terms are non-negative while the fourth term is non-negative for the class of flows under consideration. Hence by dropping these non-negative terms, we obtain

$$k^4\int |W|^2\mathrm{d}z + 2\int \frac{N^2 b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z - \int \frac{N^4}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z - \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0-c_r)^2+c_i^2}|W|^2\mathrm{d}z \le 0. \tag{14}$$

Since $(U_0 - c_r)^2 + c_i^2 \geq 2(U_0 - c_r)c_i$ and $\frac{1}{(U_0-c_r)^2+c_i^2} \leq \frac{1}{c_i^2}$, it follows that

$$k^4 \int |W|^2 \mathrm{d}z + \int \left[\frac{N^2 b \left(\frac{U_0'}{b}\right)'}{c_i^3} - \frac{N^4}{c_i^4} - \frac{\left[b \left(\frac{U_0'}{b}\right)' \right]^2}{c_i^2} \right] |W|^2 \,\mathrm{d}z \leq 0;$$

i.e.,

$$k^4 \leq \frac{N_{\max}^2 \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max} c_i + (N_{\max}^2)^2 + \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max}^2 c_i^2}{c_i^4}.$$

Hence it follows, on using the fact $c_i \leq \left[\frac{U_{0\max} - U_{0\min}}{2}\right]$ that,

$$k^4 c_i^4 \leq \left[(N_{\max}^2)^2 + \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max}^2 \left[\frac{U_{0\max} - U_{0\min}}{2} \right]^2 + N_{\max}^2 \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max} \left[\frac{U_{0\max} - U_{0\min}}{2} \right] \right],$$

$$kc_i \leq \left[(N_{\max}^2)^2 + \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max}^2 \left[\frac{U_{0\max} - U_{0\min}}{2} \right]^2 + N_{\max}^2 \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max} \left[\frac{U_{0\max} - U_{0\min}}{2} \right] \right]^{\frac{1}{4}}.$$

This completes the proof of the Theorem. □

Now we can use the appropriate estimates found earlier for the two dropped terms in (10) to obtain the following sharper estimate for the growth rate which also depends on the depth D of the fluid layer.

Theorem 2. *For the class of flows satisfying $T'(z) \leq 0$ in the flow domain an estimate for the growth rate of an arbitrary unstable mode is given by*

$$kc_i \leq \frac{\left[(N_{\max}^2)^2 + \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max}^2 \left[\frac{U_{0\max} - U_{0\min}}{2} \right]^2 + N_{\max}^2 \left| b \left(\frac{U_0'}{b}\right)' \right|_{\max} \left[\frac{U_{0\max} - U_{0\min}}{2} \right] \right]^{\frac{1}{4}}}{\left[1 + \frac{\pi^2 b_{\min}}{k^2 D^2 b_{\max}} \right]^{\frac{1}{2}}}. \tag{15}$$

Proof. Use of Lemma 2 (eq. (6)) and Rayleigh–Ritz inequality for the first two terms of eq. (10), gives

$$\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2} + 2\frac{k^2\pi^2 b_{\min}}{D^2 b_{\max}} + k^4\right]\int |W|^2 \mathrm{d}z + 2\int \frac{N^2 b\left(\frac{U_0'}{b}\right)'(U_0 - c_r)}{[(U_0 - c_r)^2 + c_i^2]^2}|W|^2 \mathrm{d}z$$

$$-\int \frac{N^4}{[(U_0 - c_r)^2 + c_i^2]^2}|W|^2 \mathrm{d}z - \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0 - c_r)^2 + c_i^2}|W|^2 \mathrm{d}z \le 0.$$

Since $(U_0 - c_r)^2 + c_i^2 \ge 2(U_0 - c_r)c_i$ and $\frac{1}{(U_0-c_r)^2+c_i^2} \le \frac{1}{c_i^2}$, it follows that

$$\left[k^2 + \frac{\pi^2 b_{\min}}{D^2 b_{\max}}\right]^2 \int |W|^2 \mathrm{d}z$$

$$+\int \left[\frac{N^2 b\left(\frac{U_0'}{b}\right)'}{c_i^3} - \frac{N^4}{c_i^4} - \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{c_i^2}\right]|W|^2 \mathrm{d}z \le 0;$$

i.e.,

$$k^4 \le \frac{N_{\max}^2 \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max} c_i + (N_{\max}^2)^2 + \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}^2 c_i^2}{c_i^4\left[1 + \frac{\pi^2 b_{\min}}{k^2 D^2 b_{\max}}\right]^2}.$$

On using the fact that $c_i \le \left[\frac{U_{0\max} - U_{0\min}}{2}\right]$ on the numerator, it follows that

$$k^4 c_i^4 \le \frac{\left[(N_{\max}^2)^2 + \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}^2 \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2 + N_{\max}^2 \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\left[\frac{U_{0\max} - U_{0\min}}{2}\right]\right]}{\left[1 + \frac{\pi^2 b_{\min}}{k^2 D^2 b_{\max}}\right]^2}.$$

From this we get the following estimate for the growth rate:

$$kc_i \le \frac{\left[(N_{\max}^2)^2 + \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}^2 \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2 + N_{\max}^2 \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\left[\frac{U_{0\max} - U_{0\min}}{2}\right]\right]^{\frac{1}{4}}}{\left[1 + \frac{\pi^2 b_{\min}}{k^2 D^2 b_{\max}}\right]^{\frac{1}{2}}}.$$

This completes the proof of the Theorem. □

If $b_{\min} \ne 0$ and $D \ne \infty$ then we can conclude from (15) that $kc_i \to 0$ as $k \to 0$ and/or as $D \to 0$. Later it is proved that $kc_i \to 0$ as $k \to \infty$ though only for two classes of basic flows.

Remark. When $b(z)$ is a positive constant the above estimate for the growth rate reduces to the estimate

$$kc_i \le \frac{\left[(N^2_{\max})^2 + |U_0''|^2_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2 + N^2_{\max}|U_0''|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]\right]^{\frac{1}{4}}}{\left[1+\frac{\pi^2}{k^2D^2}\right]^{\frac{1}{2}}}. \tag{16}$$

This is a new estimate for the TGP and hence it is desirable to compare it with the already known estimate for the TGP, namely,

$$k^2c_i^2 \le \frac{|(\rho U_0')'|_{\max}[U_{0\max} - U_{0\min}] + (\rho N^2)_{\max}}{\rho_{\min}\left[1+\frac{\pi^2}{D^2k^2}\right]},$$

which is due to Subbiah and Jain [16]. First, let us consider the particular flow with $U_0(z) = 4z(1-z)$ in $0 \le z \le 1$, $b(z) \equiv 1$, $\rho(z) \equiv 1$ and $N^2(z) \equiv 1$. An elementary computation shows that our estimate gives

$$kc_i \le \frac{(21)^{\frac{1}{4}}}{\left(1+\frac{\pi^2}{k^2}\right)^{\frac{1}{2}}} \approx \frac{2.14}{\left(1+\frac{\pi^2}{k^2}\right)^{\frac{1}{2}}},$$

whereas the estimate of Subbiah and Jain [16] gives

$$kc_i \le \frac{3}{\left(1+\frac{\pi^2}{k^2}\right)^{\frac{1}{2}}}.$$

Thus it is clear that our estimate for growth rate is sharper than the already known one. Not only for this particular flow, but for any basic flow this is true, that is, our estimate (15) is sharper than the one given in Subbiah and Jain [16] as shown below.

Our estimate is

$$k^4c_i^4 \le \frac{\left[(N^2_{\max})^2 + \left|b\left(\frac{U_0'}{b}\right)'\right|^2_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2 + N^2_{\max}\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]\right]}{\left[1+\frac{\pi^2 b_{\min}}{k^2D^2b_{\max}}\right]^2}.$$

Multiplying the second and third term of RHS by 4, we get

$$k^4c_i^4$$

$$\le \frac{\left[(N^2_{\max})^2 + 4\left|b\left(\frac{U_0'}{b}\right)'\right|^2_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]^2 + 4N^2_{\max}\left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}\left[\frac{U_{0\max}-U_{0\min}}{2}\right]\right]}{\left[1+\frac{\pi^2 b_{\min}}{k^2D^2b_{\max}}\right]^2}$$

$$= \frac{\left[N^2_{\max} + \left|b\left(\frac{U_0'}{b}\right)'\right|_{\max}[U_{0\max} - U_{0\min}]\right]^2}{\left[1+\frac{\pi^2 b_{\min}}{k^2D^2b_{\max}}\right]^2}.$$

Hence we have

$$k^2 c_i^2 \le \frac{\left[N_{\max}^2 + \left| b\left(\frac{U_0'}{b}\right)' \right|_{\max} [U_{0\max} - U_{0\min}] \right]}{\left[1 + \frac{\pi^2 b_{\min}}{k^2 D^2 b_{\max}} \right]},$$

which is same as the Subbiah and Jain [16] estimate.

Hence it is clear that the new estimate for the growth rate obtained in this paper is sharper than the one obtained in Subbiah and Jain [16].

Now we shall consider some specific examples of basic flows in the TGP and compare the values of growth rates obtained by numerical computation with values obtained from the bounds of our paper.

Example 1. $U_0 = \tanh z$, $N^2 = J$, $-\infty < z < \infty$ where $J > 0$ is a constant.

Our estimate (13) gives $kc_i \le (J^2 + 1 + J)^{\frac{1}{4}}$. From this we can see that maximum growth rate decreases with decreasing values of J. This is in conformity with the results of the numerical computations for the above example (cf. figure 7 on page 700 of [6]). Furthermore, Hazel [9] has found from numerical computation that the presence of boundaries decreases the growth rate. This follows from our estimate (13) also as we can see that $D \to 0$ implies $kc_i \to 0$.

Example 2. Consider the basic flow $U_0 = \sin z$, $N^2 = J$, $-\pi \le z \le \pi$.

The instability of this flow has been shown analytically in Huppert [12] and numerical computation of Hazel [9] gives $kc_i \le 0.1565$ corresponding to $J = 0$ and $k = 0.58$.

Our estimate (16) gives $kc_i \le 0.7574$. It is seen that Hazel's estimate is sharper than ours, but our estimate is

$$kc_i \le \frac{(J^2 + 1 + J)^{\frac{1}{4}}}{\left(1 + \frac{1}{4k^2}\right)^{\frac{1}{2}}}.$$

From this we can see that estimate for kc_i decreases with decreasing J.

Example 3. Consider the basic flow $U_0 = z$, $N^2 = Jz^2$, $-\pi \le z \le \pi$.

The instability of this flow has been established in [12] (see also [7]).

No estimates for growth rate has been computed so far but from numerical computations Huppert [12] has given a stability diagram (figure 2, page 367 of [12]). Instability is found for values of J ranging from 0 to 12 and k ranging from 0 to 3.0. But, our inequality (16) gives

$$kc_i \le \frac{\pi^2 J^{\frac{1}{2}}}{\left(1 + \frac{1}{4k^2}\right)^{\frac{1}{2}}}.$$

From this we see that $kc_i \to 0$ as $J \to 0$ and as $k \to 0$. Moreover, we see that $kc_i = 0$ when $J = 0$, i.e., the basic flow is stable in the homogeneous case.

4. Proof of Howard's conjecture

Now we shall give the proof of Howard's conjecture.

For the TGP, Banerjee *et al* [1] has proved the validity of the Howard's conjecture for Garcia type flows. In such flows the flow domain is of infinite extent in the vertical direction. But for the ETGP the flow domain is of finite extent in the vertical direction and the significance of such flows for ETGP is unknown. However the analysis of Banerjee *et al* [1] can be extended to the ETGP as shown below.

We consider basic flows for which $U_0'' - TU_0' = 0$ at a point $z = z_s$ in the flow domain. Let us denote $U_0(z_s)$ by U_{0s}. Consider unstable disturbances with complex phase velocity $c = c_r + ic_i$ with $c_r = U_{0s}$ and $c_i > 0$. Then we have the following result.

Theorem 3. *Howard's conjecture, namely,* $kc_i \to 0$ *as* $k \to \infty$, *is true for the class of flows satisfying*:

(i) $\frac{N^2}{(U_0''-TU_0')(U_0-U_{0s})}$ *is bounded in the flow domain and*

(ii) $T' \leq 0$ *in the flow domain.*

Proof. It follows from eq. (14) that

$$k^4\int |W|^2\mathrm{d}z \leq$$

$$\int \frac{\left[\left[b\left(\frac{U_0'}{b}\right)'\right]^2(U_0-c_r)^2 - 2N^2 b\left(\frac{U_0'}{b}\right)'\times(U_0-c_r)+N^4+\left[b\left(\frac{U_0'}{b}\right)'\right]^2 c_i^2\right]}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z;$$

i.e.,

$$k^4\int |W|^2\mathrm{d}z \leq \int \frac{\left[\left[b\left(\frac{U_0'}{b}\right)'(U_0-c_r)-N^2\right]^2+\left[b\left(\frac{U_0'}{b}\right)'\right]^2 c_i^2\right]}{[(U_0-c_r)^2+c_i^2]^2}|W|^2\mathrm{d}z;$$

which implies that

$$k^4\int |W|^2\mathrm{d}z \leq$$

$$\int \frac{\left[\left[b\left(\frac{U_0'}{b}\right)'\right]^2[(U_0-c_r)^2+c_i^2]\left[1-\frac{N^2}{b\left(\frac{U_0'}{b}\right)'(U_0-c_r)}\right]^2+\left[b\left(\frac{U_0'}{b}\right)'\right]^2[(U_0-c_r)^2+c_i^2]\right]}{[(U_0-c_r)^2+c_i^2]^2}$$

$$\times |W|^2\mathrm{d}z.$$

Canceling the non-zero term $(U_0 - c_r)^2 + c_i^2$, we get

$$k^4 \int |W|^2 \mathrm{d}z \leq \int \frac{\left[\left[b\left(\frac{U_0'}{b}\right)'\right]^2\left[1 - \frac{N^2}{b\left(\frac{U_0'}{b}\right)'(U_0 - c_r)}\right]^2 + \left[b\left(\frac{U_0'}{b}\right)'\right]^2\right]}{[(U_0 - c_r)^2 + c_i^2]} |W|^2 \mathrm{d}z.$$

Since $c_r = U_{0s}$ and $\frac{1}{(U_0 - c_r)^2 + c_i^2} \leq \frac{1}{c_i^2}$, we get

$$k^4 \int |W|^2 \mathrm{d}z \leq \int \frac{\left[\left[b\left(\frac{U_0'}{b}\right)'\right]^2\left[1 - \frac{N^2}{b\left(\frac{U_0'}{b}\right)'(U_0 - U_{0s})}\right]^2 + \left[b\left(\frac{U_0'}{b}\right)'\right]^2\right]}{c_i^2} |W|^2 \mathrm{d}z.$$

Since $\frac{N^2}{b\left(\frac{U_0'}{b}\right)'(U_0 - U_{0s})}$ is bounded and letting $\lambda^2 = \left[1 - \frac{N^2}{b\left(\frac{U_0'}{b}\right)'(U_0 - U_{0s})}\right]^2 + 1$, we get

$$k^4 \int |W|^2 \mathrm{d}z \leq \left[\frac{\left[\left|b\left(\frac{U_0'}{b}\right)'\right|^2 \lambda^2\right]_{\max}}{c_i^2}\right] \int |W|^2 \mathrm{d}z.$$

This implies that

$$k^4 c_i^2 \leq \left[\left|b\left(\frac{U_0'}{b}\right)'\right|^2 \lambda^2\right]_{\max},$$

i.e.,

$$kc_i \leq \frac{\left[\max\left[\left|b\left(\frac{U_0'}{b}\right)'\right|^2 \lambda^2\right]\right]^{\frac{1}{2}}}{k}$$

which shows that

$$\lim_{k \to \infty} kc_i = 0.$$

This completes the proof of the theorem. □

Consider basic flows with $(U_0')^2_{\min} \neq 0$, i.e., monotonic flows. Then we define the local Richardson number as $J(z) = \frac{N^2(z)}{(U_0')^2}$ which is a non-dimensional variable. By a result of Deng *et al* [4] it is necessary for instability that $J(z) < \frac{1}{4}$ atleast once in the flow domain. Now let us consider only flows with weak stratification, i.e., $J(z) \ll 1$ so that $J^2(z)$ can be ignored in comparison to $J(z)$.

For the TGP, Howard's conjecture has been proved for flows with weak stratification. Now we shall do the same for ETGP.

Theorem 4. *For flows with* $(U_0')^2_{\min} \neq 0$ *and* $J(z) \ll 1$ *we have* $kc_i \to 0$ *as* $k \to \infty$.

Proof. In eq. (10), the first and second terms are non-negative, while the fourth term is non-negative for the class of flows under consideration so they can be dropped to give

$$k^4 \int |W|^2 \mathrm{d}z + \int \frac{\left[2N^2 b\left(\frac{U_0'}{b}\right)'(U_0 - c_r) - N^4\right]}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z \leq \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z.$$

For flows under consideration we can rewrite the above inequality as

$$k^4 \int |W|^2 \mathrm{d}z + \int \frac{\left[2J(z)(U_0')^2 b\left(\frac{U_0'}{b}\right)'(U_0 - c_r) - J^2(z)(U_0')^4\right]}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z$$
$$\leq \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z$$

and the last term on the left hand side is ignored in comparison to the preceding term to get

$$k^4 \int |W|^2 \mathrm{d}z + \int \frac{\left[2J(z)(U_0')^2 b\left(\frac{U_0'}{b}\right)'(U_0 - c_r)\right]}{[(U_0 - c_r)^2 + c_i^2]^2} |W|^2 \mathrm{d}z$$
$$\leq \int \frac{\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{(U_0 - c_r)^2 + c_i^2} |W|^2 \mathrm{d}z.$$

Using the facts $(U_0 - c_r)^2 + c_i^2 \geq 2(U_0 - c_r)c_i$ and $\frac{1}{(U_0 - c_r)^2 + c_i^2} \leq \frac{1}{c_i^2}$, we get

$$k^4 \int |W|^2 \mathrm{d}z \leq \left[\frac{J_{\max}\left|(U_0')^2 b\left(\frac{U_0'}{b}\right)'\right|_{\max}}{c_i^3} + \frac{\left|b\left(\frac{U_0'}{b}\right)'\right|^2_{\max}}{c_i^2}\right] \int |W|^2 \mathrm{d}z,$$

$$k^3 c_i^3 \leq \frac{\left[J_{\max}\left|(U_0')^2 b\left(\frac{U_0'}{b}\right)'\right|_{\max} + \left|b\left(\frac{U_0'}{b}\right)'\right|^2_{\max} c_i\right]}{k},$$

i.e.,

$$k^3 c_i^3 \leq \frac{\left[J_{\max}\left|(U_0')^2 b\left(\frac{U_0'}{b}\right)'\right|_{\max} + \left|b\left(\frac{U_0'}{b}\right)'\right|^2_{\max}\left[\frac{U_{0\max} - U_{0\min}}{2}\right]\right]}{k}.$$

This implies that $kc_i \to 0$ as $k \to \infty$.

This completes the proof of the Theorem. □

5. Concluding remarks

In this paper we have obtained some analytical results on the extended Taylor–Goldstein problem (ETGP) of hydrodynamic stability dealing with stability of shear flows in sea straits of arbitrary cross-section. In this problem we study the stability of shear flows with velocity $U_0(z)$ and stratification parameter $N^2(z)$ as in the standard Taylor–Goldstein problem and with topography $T(z)$. We consider the special class of flows satisfying the condition $T'(z) \leq 0$. This condition includes the following special cases:

(i) $T(z) = 0$ (the standard TGP),

(ii) $T(z) =$ a constant, which means that the width function $b(z)$ is an exponential function and

(iii) topography given by $T(z) = 1/z$ which is the one appropriate for sea straits like Bab al Mandab studied in Pratt *et al* [13]. For this class of flows we have obtained a new estimate for the growth rate of an arbitrary unstable mode. This estimate reduces to a new estimate for the standard TGP also and, moreover, this new estimate is sharper than the one due to Subbiah and Jain [16]. We have compared our new bounds to growth rates computed numerically for some specific flows. An interesting example is the basic flow $U_0 = z$, $N^2 = Jz^2$ which is shown to be unstable on both sides of the neutral curve (cf. Huppert [12] and Engevik *et al* [7]). Our bound shows that the growth rate decreases with decreasing J and that the flow is stable when $J = 0$ follows from our bound. For some other flows it is found that the numerically computed estimates are sharper than our estimates. However, the computations of Hazel [9] show that the maximum growth rate decreases with increasing values of the stratification parameter J while the later computations of Drazin *et al* [6] show that the maximum growth rate decreases with decreasing values of J. Interestingly our estimate also has this property, that is, the growth rate bound decreases when the stratification parameter J decreases. Moreover, the numerical computations of Hazel [9] show that the growth rate bound decreases with decreasing values of the depth of the shear layer and this also follows from our bound. Furthermore, we have proved the Howard's conjecture, namely, the growth rate $kc_i \to 0$ as $k \to \infty$ for two classes of basic flows of the ETGP. These proofs are based on extending the corresponding proofs for the TGP due to Shandil *et al* [15] and Banerjee *et al* [1].

Acknowledgements

We are thankful to the referee for his suggestions to include a comparison of our bound with numerically computed growth rates.

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