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On the torus cobordant cohomology spheres

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Dedicated to Prof. Wu-Yi Hsiang on the occasion of his seventieth birthday

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Abstract. Let G be a compact Lie group. In 1960, P A Smith asked the following question: "Is it true that for any smooth action of G on a homotopy sphere with exactly two fixed points, the tangent G-modules at these two points are isomorphic?" A result due to Atiyah and Bott proves that the answer is 'yes' for $\mathbb{Z}_p$ and it is also known to be the same for connected Lie groups. In this work, we prove that two linear torus actions on S^n which are c-cobordant (cobordism in which inclusion of each boundary component induces isomorphisms in $\mathbb{Z}$-cohomology) must be linearly equivalent. As a corollary, for connected case, we prove a variant of Smith's question.

Keywords. Equivariant cohomology; integral weight; Serre spectral sequence; cobordism.

1. Introduction

Let G be a compact Lie group. By a real G-module we mean a finite dimensional real vector space V with a linear action of G. Let X be a smooth G manifold with non-empty fixed point set X^G. For any point $x \in X^G$, the tangent space $T_x X$ becomes a real G-module by taking the derivatives of transformations $g\colon X \to X, x \mapsto gx$ for all $g \in G$ at the point x. We refer to this G-module $T_x X$ as to the tangent G-module at x. Two real G-modules U and V are called Smith equivalent if there exists a smooth action of G on a homotopy sphere Σ with $\Sigma^G = \{x, y\}$ such that $T_x\Sigma \simeq U$ and $T_y\Sigma \simeq V$ as real G-modules.

In 1960, P A Smith (p. 406 of [23]), asked the following question:

Smith isomorphism question: Is it true that Smith equivalent G-modules are isomorphic?

There is a considerable list of references related to Smith's question. If G is of odd prime power order or if G acts semi-freely then Smith equivalent modules must be isomorphic, as has been shown by Atiyah and Bott [1] and by Milnor [17]. By a result of Sanchez [21] the same is true for cyclic groups of order pq where p and q are odd primes. Bredon [4] showed that this is also true for 2-groups if their dimensions are large in comparison to the order of the group. The first negative answers were established by Petrie [19] for odd order abelian groups with at least four noncyclic Sylow subgroups. Subsequently, a number of papers have established a negative answer for even order groups. For some classes of cyclic groups of odd order non-isomorphic Smith equivalent modules were constructed by Dovermann and Petrie [10].

For additional work on Smith equivalent modules see the work of Siegel [22], Dovermann [9], Suh [25, 26], Cho [6, 7], Dovermann and Washington [13], Dovermann and Suh [12], the surveys by Masuda and Petrie [16], Cappel and Shaneson [5], Dovermann, Petrie and Schultz [11] and a book by Petrie and Randall [20] on this topic.

It is well-known that Smith equivalent modules are isomorphic for connected groups; a result due to Atiyah and Bott [1]. In this paper, for connected case, we show that homotopy sphere in Smith's question can be replaced by smooth integral cohomology sphere. Throughout this paper T denotes a torus. Let B_G be the classifying space of G and $X_G = (X \times E_G)/G$ be the balanced product where $E_G \to B_G$ is a universal principal G-bundle. Then the integral coefficient equivariant cohomology ring of X is defined by $H^*_G(X; \mathbb{Z}) = H^*(X_G; \mathbb{Z})$.

In this paper we will prove the following theorem.

Theorem 1.1. *Let G be a compact connected Lie group, Σ a smooth integral cohomology G-sphere with exactly two fixed points namely p and q. Then the tangent representations of G at $T_p\Sigma$ and $T_q\Sigma$ are linearly equivalent.*

Let a compact Lie group G act on a smooth integral cohomology sphere Σ with $\Sigma^G = \{p, q\}$. If T is a maximal torus of G, it is well-known that a real representation of G is determined by its restriction to T and hence by its linear weights. So proof of the theorem reduces to showing equality of linear weights at the boundary components. Also, we only need to prove the theorem in the case $\Sigma^T = \{p, q\}$ (otherwise Σ^T is connected by Smith theorems and it is an easy exercise to show that the tangential representations at any two points of a connected component of the fixed point set are equivalent).

2. The integral weight systems for cobordant torus actions on spheres

Let X be a T-space and assume that $H^*(X; \mathbb{Z}) = H^*(S^n; \mathbb{Z})$ (which will be denoted by $X \sim_{\mathbb{Z}} S^n$ and X will be called a 'cohomology n-sphere'). The equivariant cohomology theory makes it possible to define a geometric weight system together with an 'integral content' $C \in \mathbb{Z}$ which is a generalization of linear weight system (see [15]). In this section, for smooth actions, we show that integral weight systems of $T - c$-cobordant cohomology spheres are equal. Our approach is to consider the cohomological behavior of fixed point sets of boundary components and cobordism under some non-connected subgroups actions.

Consider the family of p-groups $T_{p,\alpha} \subset T$, $\alpha \geq 1$, where $T_{p,\alpha} = \mathbb{Z}^r_{p^\alpha}$, $T = (S^1)^r$.

Lemma 2.1. *Let X be a compact $T - c$-cobordism (cobordism in which inclusion of each boundary component induces isomorphisms in $\mathbb{Z}$-cohomology) such that $\partial X = X_0 \sqcup X_1$ and $X \sim_{\mathbb{Z}} S^n$. Let p be a prime number and $\alpha \geq 1$. Then inclusions $X_i^{T_{p,\alpha}} \hookrightarrow X^{T_{p,\alpha}}$ induce isomorphisms $H^*(X^{T_{p,\alpha}}; \mathbb{Z}_p) \simeq H^*(X_i^{T_{p,\alpha}}; \mathbb{Z}_p)$, $i = 0, 1$.*

Proof. It suffices to consider the inclusion $X_0^{T_{p,\alpha}} \hookrightarrow X^{T_{p,\alpha}}$. We prove the lemma by double induction on r and α. Let $\alpha = 1$. We will show that inclusion $X_0^{\mathbb{Z}^r_p} \hookrightarrow X^{\mathbb{Z}^r_p}$ induces isomorphism, $H^*(X^{\mathbb{Z}^r_p}; \mathbb{Z}_p) \approx H^*(X_0^{\mathbb{Z}^r_p}; \mathbb{Z}_p)$. For $r = 1$, a standard inequality in Smith theory (see p. 144 of [2]) implies:

$$\sum_{i=0}^{n} \text{rank } H^i(X^{\mathbb{Z}_p}, X_0^{\mathbb{Z}_p}; \mathbb{Z}_p) \leq \sum_{i=0}^{n} \text{rank } H^i(X, X_0; \mathbb{Z}_p)$$

for all n. The right-hand side, however, is 0. Thus $H^*(X^{\mathbb{Z}_p}, X_0^{\mathbb{Z}_p}; \mathbb{Z}_p) = 0$, implying the desired isomorphism. Let us assume that $X_0^{\mathbb{Z}_p^r} \hookrightarrow X^{\mathbb{Z}_p^r}$ induces isomorphism $H^*(X^{\mathbb{Z}_p^r}; \mathbb{Z}_p) \approx H^*(X_0^{\mathbb{Z}_p^r}; \mathbb{Z}_p)$. Consider the action of $\mathbb{Z}_p^{r+1}/\mathbb{Z}_p^r \simeq \mathbb{Z}_p$ on $X^{\mathbb{Z}_p^r}$. Fixed point set of this action is $X^{\mathbb{Z}_p^{r+1}}$ and we get

$$\sum_{i=0}^{n} \text{rank}\, H^i(X^{\mathbb{Z}_p^{r+1}}, X_0^{\mathbb{Z}_p^{r+1}}; \mathbb{Z}_p) \leq \sum_{i=0}^{n} \text{rank}\, H^i(X^{\mathbb{Z}_p^r}, X_0^{\mathbb{Z}_p^r}; \mathbb{Z}_p),$$

applying the same inequality. Because of induction hypothesis the right-hand side is 0. Thus $H^*(X^{\mathbb{Z}_p^{r+1}}, X_0^{\mathbb{Z}_p^{r+1}}; \mathbb{Z}_p) = 0$. Let us assume that $X_0^{T_{p,\alpha}} \hookrightarrow X^{T_{p,\alpha}}$ induces the isomorphism $H^*(X^{T_{p,\alpha}}; \mathbb{Z}_p) \approx H^*(X_0^{T_{p,\alpha}}; \mathbb{Z}_p)$. In this case just replace (X, X_0) by $(X^{T_{p,\alpha}}, X_0^{T_{p,\alpha}})$ with the induced action of $T_{p,\alpha+1}/T_{p,\alpha} \simeq \mathbb{Z}_p^r$ which has fixed point sets $X^{T_{p,\alpha+1}}$ and $X_0^{T_{p,\alpha+1}}$ respectively. ■

The actions of abelian groups are usually studied using the Borel construction. Let X be a T-space and $E_T \to B_T$ be the universal principal T-bundle. Let $X_T = (X \times E_T)/T$ where T acts on $X \times E_T$ via $t \cdot (x, e) = (t \cdot x, t \cdot e)$. Then $X_T \to B_T$ is a fiber bundle with fiber X. The Leray–Serre spectral sequence of a fiber bundle (more generally of a fibration) $E \to B$ (B having the weak homotopy type of a CW complex) with fiber F, is a spectral sequence $\{E_n^{p,q}, d_n\}$ where $E_2^{p,q} \cong H^p(B, H^q(F))$ (usually and in our case $H^p(B, H^q(F)) \cong H^p(B) \otimes H^q(F))$ converging to $H^*(E)$. The Serre spectral sequence is obtained by filtration of the base space and the differentials of E_2 term are essentially the connecting homomorphisms of the cohomology long exact sequence of a pair.

Let X be a T-space and assume that $H^*(X; \mathbb{Z}) = H^*(S^n; \mathbb{Z})$. The equivariant cohomology theory makes it possible to define a geometric weight system together with an ‘integral content’ $C \in \mathbb{Z}^+$ which is a generalization of linear weight system (see [15]). When $H^*(X; \mathbb{Z}) \cong H^*(S^n; \mathbb{Z})$ then $H^*(X^T; \mathbb{Z}) \cong H^*(S^r; \mathbb{Z})$ (where $n - r$ is even) by a generalization of Smith’s theorems. The equivariant Euler class $E_T(X)$, its integral content C and integral weights of an action on a cohomology sphere are defined as follows (see [14] for details): Let $X \sim_{\mathbb{Z}} S^n$, $X^T \sim_{\mathbb{Z}} S^r$, $\alpha \in H^n(X; \mathbb{Z}) \cong H^n(X, X^T; \mathbb{Z})$, $\beta \in H^{r+1}(X, X^T; \mathbb{Z})$ be generators. In the Leray–Serre spectral sequence of the fiber bundle pair $(X_T, B_T \times X^T)$ with projection $X_T \to B_T$ and fibre (X, X^T), E_k $(k \geq 2)$ term has two non-zero lines: $E_k^{*,n}$ and $E_k^{*,r}$. The ‘transgression’ of α

$$\mathrm{d}\alpha = E_T(X) \otimes \beta \in H^{n-r}(B_T; \mathbb{Z}) \otimes H^{r+1}(X, X^T; \mathbb{Z}).$$

The equivariant Euler class $E_T(X)$ splits into a product of linear factors:

$$E_T(X) = \pm C \prod_{i=1}^{s} \omega_i^{m_i}, \; C \in \mathbb{Z}^+ \; (\omega_i \in H^2(B_T; \mathbb{Z}) \text{ are determined up to a sign}).$$

ω_i are called the integral weights of the action and m_i is called the multiplicity of the weight ω_i.

By repeating the same argument with X replaced by $X^{\omega_i^\perp}$, we obtain

$$E_T(X^{\omega_i^\perp}) = \pm C_i \omega_i^{m_i}, \; C_i \in \mathbb{Z}^+, \quad i = 1, 2, \ldots, s.$$

It is proven in [24] that $C = \prod_{i=1}^{s} C_i$.

The reduced integral weight system $\Omega'(X)$ and unreduced integral weight system $\Omega(X)$ are defined as follows:

$$\Omega'(X) = \{\pm C_i \omega_i^{m_i}\}_{i=1}^{s}$$

and

$$\Omega(X) = \Omega'(X) \cup \{0 \text{ weight with multiplicity } \tfrac{1}{2}(n-r)\}.$$

From now on all cohomologies are sheaf cohomology and $X \sim_{\mathbb{Z}} Y$ (resp. $X \sim_p Y$) will mean that X is a compact $\mathbb{Z}$-cohomology manifold with $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) cohomology ring is isomorphic to that of Y. If $X \sim_{\mathbb{Z}} S^n$ (resp. $X \sim_p S^n$) we will write $\dim_{\mathbb{Z}} X = n$ (resp. $\dim_p X = n$). The following theorem will be useful to compare cohomological behavior of fixed point sets of boundary components under the actions of the family of p-groups, $T_{p,\alpha} \subset T$.

Theorem 2.2 [14]. *Let X be a T space and $X \sim_{\mathbb{Z}} S^n$. Suppose that for any prime p and $\alpha \geq 1$, $X^{T_{p,\alpha}}$ has finitely generated integral cohomology. Let $C_X \in \mathbb{Z}^+$ be the integral content of the Euler class $E_T(X)$ and for a fixed prime p write $C_X = p^{e_p} C'$ where $p \nmid C'$ then*

$$\sum_{\alpha \geq 1} [\dim_p X^{T_{p,\alpha}} - \dim_{\mathbb{Z}} X^T] = 2e_p.$$

Golber formula shows that integral content of a toral action on a sphere derived from equivariant cohomology is related to dimensions of the fixed point sets of various non-connected subgroups.

Recall that a pair (X, A) is totally nonhomologous to zero in (X_G, A_G) with respect to rational cohomology if $H^*(X_G, A_G; \mathbb{Q}) \to H^*(X, A; \mathbb{Q})$ is surjective. Note that if G is a compact connected Lie group, (X, A) is a G space and $H^*(X, A; \mathbb{Q})$ is zero in odd degrees then the Leray–Serre spectral sequence associated to $(X, A) \to (X_G, A_G) \to B_G$ degenerates for formal reasons. In addition, since G acts trivially on $H^*(X, A; \mathbb{Q})$, (II.10.6 of [3], cf. II.11.11 of [3]) (X, A) is totally nonhomologous to zero in (X_G, A_G).

Theorem 2.3. *Let T be a torus and X is a compact T−c-cobordism such that $\partial X = X_0 \sqcup X_1$ and $X \sim_{\mathbb{Z}} S^n$. Then X_0, X_1 and X have the same integral weight systems, and therefore have the same equivariant Euler class and the same integral contents.*

Proof. First observe that the pair (X, X_k) is totally nonhomologous to zero in $(X_T, (X_k)_T)$, $k = 0, 1$. Therefore the inclusion $X_k^T \hookrightarrow X^T$ induces an isomorphism $H^*(X^T; \mathbb{Q}) \approx H^*((X_k)^T; \mathbb{Q})$, $k = 0, 1$. This is essentially Chapter VII, Theorem 1.6 of [2]. The proof in [2] is for the case where $T = S^1$ or $\mathbb{Z}_p$. But one gets the result for higher rank tori and p-tori using induction. We must assume finitistic orbit space for S^1 action but this is now known to be true by Deo and Tripathi [8]. Thus $H^*(X, X^T; \mathbb{Q}) \to H^*(X_k, (X_k)^T; \mathbb{Q})$ (induced by the inclusion) is an isomorphism for $k = 0, 1$.

Let us consider the fiber bundle map between the fiber bundle pairs $((X_0)_T, (X_0^T)_T)$ and $(X_T, (X^T)_T)$ induced by the inclusion $X_0 \to X$. The homomorphism between the Leray–Serre spectral sequences $\{E_k^{p,q}\}$ and $\{F_k^{p,q}\}$ (with rational coefficients) is an isomorphism along $E_2^{0,*}$ and also along $E_2^{*,0}$, therefore it is an isomorphism $E_k^{p,q} \to F_k^{p,q}$ for every $k \geq 2$. Therefore $E_T(X) = E_T(X_0)$ with rational coefficients. Since $H^n(B_T; \mathbb{Z})$

is free each n, using the universal coefficient theorem, we see that integral weights and their multiplicities are the same for X and X_0.

Let p be a prime and p^a (respectively p^b) be the largest power of p that divides C_X (respectively C_{X_0}). Since $\dim_p X^{T_{p,\alpha}} = \dim_p X_0^{T_{p,\alpha}}$ by Lemma 2.1 and $\dim_{\mathbb{Z}} X^T = \dim_{\mathbb{Z}} X_0^T$, applying Golber's formula for T actions on X and X_0 we find $a = b$. Thus $C_X = C_{X_0}$. Therefore equivariant Euler classes of X and X_0 are equal. Let $\Omega'(X) = \{\pm C_i \omega_i^{m_i}\}_{i=1}^s$ and $\Omega'(X_0) = \{\pm C_i' \omega_i^{m_i}\}_{i=1}^s$ be the reduced integral weight systems of X and X_0 respectively. Applying a similar argument to T-spaces $(X^{\omega_i^{\perp}}, X_0^{\omega_i^{\perp}})$ in the discussion above, we get $C_i = C_i'$, $i = 1, 2, \ldots, s$. Therefore X and X_0 have the same integral weight systems. Similarly X and X_1 have the same integral weight systems. ■

COROLLARY 2.4

Let X be a compact $T - c$-cobordism such that $\partial X = X_0 \sqcup X_1$ and $X \sim_{\mathbb{Z}} S^n$. Then $H^*(X^T; \mathbb{Z}) \to H^*(X_k^T; \mathbb{Z})$, $k = 0, 1$, induced by inclusions, are isomorphisms.

Proof. Let us denote the spectral sequences $(X_0, X_0^T) \to ((X_0)_T, B_T \times X_0^T) \to B_T$ and $(X, X^T) \to (X_T, B_T \times X^T) \to B_T$ by $E_*^{*,*}$ and $D_*^{*,*}$ respectively. If α is a generator of $H^n(X_0; \mathbb{Z}) = H^n(X_0, X_0^T; \mathbb{Z})$ then $d_{n-r}(\alpha) = E_T(X_0) \otimes \beta$, β a generator of $H^{r+1}(X_0, X_0^T; \mathbb{Z})$. Consider the following commutative diagram induced by the fiber bundle map $(X_0)_T \to X_T$ (which is induced by the inclusion $X_0 \hookrightarrow X$).

$$
\begin{array}{ccc}
D_{n-r}^{0,n} = H^n(X;\mathbb{Z}) & \xrightarrow{\approx} & E_{n-r}^{0,n} = H^n(X_0, X_0^T;\mathbb{Z}) \\
\Big\downarrow d'_{n-r} & & \Big\downarrow d_{n-r} \\
D_{n-r}^{n-r,r+1} = H^{n-r}(B_T;\mathbb{Z}) \otimes H^{r+1}(X, X^T;\mathbb{Z}) & \xrightarrow{\approx} & E_{n-r}^{n-r,r+1} = H^{n-r}(B_T;\mathbb{Z}) \otimes H^{r+1}(X_0, X_0^T;\mathbb{Z}) \\
\Big\uparrow 1\otimes\delta \; \approx & & \Big\uparrow 1\otimes\delta \; \approx \\
H^{n-r}(B_T;\mathbb{Z}) \otimes \widetilde{H}^r(X^T;\mathbb{Z}) & \xrightarrow{\approx} & H^{n-r}(B_T;\mathbb{Z}) \otimes \widetilde{H}^r(X_0^T;\mathbb{Z})
\end{array}
$$

It is clear that homomorphism $\widetilde{H}^r(X^T; \mathbb{Z}) \to \widetilde{H}^r(X_0^T; \mathbb{Z})$, takes a generator to a generator, since $E_T(X_0) = E_T(X)$. Similarly, $\widetilde{H}^r(X^T; \mathbb{Z}) \to \widetilde{H}^r(X_1^T; \mathbb{Z})$, induced by the inclusion $X_1 \hookrightarrow X$, is an isomorphism. ■

3. Integral liftings and proof of Theorem 1.1

For a n-cohomology sphere X with (reduced) integral weight system $\{\pm C_i \omega_i^{m_i}\}_{i=1}^s$, we can find m_i integral weights $w_{i,j} = C_{i,j}\omega_i$ with $\prod C_{i,j} = C_i$. Then the resulting set $\Gamma = \{\pm C_{i,1}\omega_i, \pm C_{i,2}\omega_i, \ldots, \pm C_{i,m_i}\omega_i\}_{i=1}^s$ is called an integral lifting of $\Omega'(X)$. Such a choice may not be unique and perhaps not meaningful. If T acts on S^n linearly without a fixed point then the linear weight system of the representation is a lifting of the integral weight system.

Let X be a compact differentiable T space with nonempty fixed point set. Then for all $K \subset T$ (locally, near a fixed point x),

$$\dim_{\mathbb{Z}} X^K = \sharp\{i\colon \pm w_i|_K = 0\} + m_0,$$

where $\{\pm w_1, \pm w_2, \ldots, \pm w_k\}$ are the (nonzero) linear weights of the representation at the fixed point x and m_0 is multiplicity of the zero weight. But no-fixed point case is more subtle in the topological category. There exist actions where the integral lifting is unique and still the dimension formula above fails. An example of such an action is given in [24].

Proof of Theorem 1.1. It suffices to prove the theorem for connected torus $T = (S^1)^r$. We choose a T invariant Riemannian metric on Σ so that T is a group of isometries. Let S_p and S_q be slices at p and q which can be chosen to be T-invariant open balls. Consider the $T - c$-cobordism $X = \Sigma - (S_p \cup S_q)$ (the fact that this is a c-cobordism follows from an easy Mayer–Vietoris sequence argument). Boundary components of this cobordism are representation spheres at p and q. In other words, $\partial X = S(T_p\Sigma) \sqcup S(T_q\Sigma) = X_0 \sqcup X_1$ is a union of two spheres with T acting linearly on both. Then integral weight systems of these representation spheres are equal by Theorem 2.3. Let us denote this common integral weight system by $\Omega = \Omega(X) = \{\pm C_i\omega_i^{m_i}\}_{i=1}^s$. Let $\Gamma = \{\pm w_i : i = 1, 2, \ldots, k\}$ (respectively $\Gamma' = \{\pm w'_i : i = 1, 2, \ldots, k\}$) be the linear weight system of the action of T on X_0 (respectively X_1). We have $E_T(X_0) = \pm\prod_{i=1}^k w_i = E_T(X_1) = \pm\prod_{i=1}^k w'_i = E_T(X)$. Both Γ and Γ' are integral liftings of $\Omega(X)$. Let us write (for $w, \omega \in H^2(B_T; \mathbb{Z})$) $w \backsim \omega$ if $w = c\omega$, for some $c \in \mathbb{Z}$. Applying a similar argument to $(X^{\omega_i^\perp}, X_0^{\omega_i^\perp})$ and $(X^{\omega_i^\perp}, X_1^{\omega_i^\perp})$, we obtain $C_i\omega_i^{m_i} = \pm\prod_{w_j \backsim \omega_i} w_j = \pm\prod_{w'_j \backsim \omega_i} w'_j$ for each $i = 1, 2, \ldots, s$. Let us fix an i. Let $\Gamma_i = \{w \in \Gamma : w \backsim \omega_i\} = \{C_{i,1}\omega_i, \ldots, C_{i,m_i}\omega_i\}$ and $\Gamma'_i = \{w' \in \Gamma' : w' \backsim \omega_i\} = \{C'_{i,1}\omega_i, \ldots, C'_{i,m_i}\omega_i\}$. We have $C_i = \prod_{j=1}^{m_i} C_{i,j} = \prod_{j=1}^{m_i} C'_{i,j}$. We will first prove $\Gamma_i = \Gamma'_i$. Observe that $X^{\omega_i^\perp}$ is a (differentiable) $T - c$-cobordism (T action has ineffective kernel $\omega_i^\perp$) with $\partial X^{\omega_i^\perp} = X_0^{\omega_i^\perp} \sqcup X_1^{\omega_i^\perp}$ where $X_j^{\omega_i^\perp}$ $(j = 0, 1)$ are spheres with T acting linearly with weight systems Γ_i and Γ'_i respectively. Let n be the largest among $C_{i,j}, C'_{i,j}$. Let $K \subset T$ be such that $K/\omega_i^\perp = \mathbb{Z}_n$. Then X^K is a differentiable manifold with boundary on which K acts differentiably (with ineffective kernel $\omega_i^\perp$). Since $X_0^K = (X_0^{\omega_i^\perp})^{\mathbb{Z}_n}$ and $X_1^K = (X_1^{\omega_i^\perp})^{\mathbb{Z}_n}$ are the boundary components and they have the same dimension we must have

$$\sharp\{C_{i,j} : n | C_{i,j}\} = \sharp\{C'_{i,j} : n | C'_{i,j}\}.$$

Therefore the largest of the numbers $C_{i,j}$ must be equal to the largest of $C'_{i,j}$ and they must have the same multiplicity. Repeating this argument for all $C_{i,j}$ and $C'_{i,j}$ we reach the conclusion $\Gamma_i = \Gamma'_i$ for each $i = 1, \ldots, s$ hence $\Gamma = \Gamma'$. ■

Let a compact Lie group G act differentiably on a connected manifold X. Let G/H be the principal orbit type of the action. Then $\dim X - \dim(G/H)$ is called the cohomogeneity of the action. Let us denote the cohomogeneity of the action (G, X) by $\mathrm{ch}(G, X)$. Using Theorem 1.1, we can show topological similarity of differentiable $T - c$-cobordant actions on spheres with nonempty fixed point sets and having cohomogeneity less than 3.

Lemma 3.1. *Let G be a compact Lie group, X a compact, connected, differentiable G-cobordism with $\partial X = X_0 \sqcup X_1$. Then* $\mathrm{ch}(G, X_0) = \mathrm{ch}(G, X_1) = \mathrm{ch}(G, X) - 1$.

Proof. We will prove that the principal orbit type of the action on X is the same as the principal orbit types of the restricted actions on X_0 and X_1. By the equivariant collaring theorem (Theorem 1.5, Ch. V of [2]), there is a neighborhood of $B = X_0 \sqcup X_1$ which is equivariantly homeomorphic to $B \times [0, 1)$ where the action on $[0, 1)$ is trivial. Since the principal orbits of X must be dense in X/G, the principal orbits of the actions on X_0, X_1 and X are all equal. Therefore $\mathrm{ch}(G, X_1) = \mathrm{ch}(G, X_0) = \mathrm{ch}(G, X) - 1$. ■

In 1956, Montgomery, Samelson and Yang [18] proved that if any compact connected Lie group G acts on an Euclidean n-space with cohomogeneity less than 3, then the action is equivalent to a linear action (i.e. there is an equivariant homeomorphism between $(G, \mathbb{R}^n)$ and a linear G space $\mathbb{R}^n$). As a consequence, they proved that actions with nonempty fixed point set are equivalent to orthogonal actions.

COROLLARY 3.2

Let X be a compact differentiable T − c-cobordism such that $\partial X = X_0 \sqcup X_1 = S^n \sqcup S^n$ and $\mathrm{ch}(T, X) < 4$. *Then* (T, X_0) *and* (T, X_1) *are topologically similar transformation groups.*

Proof. Let φ_0, φ_1 and φ denote actions of T on X_0, X_1 and X such that restricted actions of φ on X_0 and X_1 are φ_0 and φ_1. Since, by Lemma 3.1, cohomogeneity of φ_0 and $\varphi_1 \le 2$, these actions are topologically similar to orthogonal ones, say Ψ_0 and Ψ_1. So we can replace actions φ_0 and φ_1 by actions Ψ_0 and Ψ_1. It follows from Theorem 1.1 that Ψ_0 and Ψ_1 are linearly equivalent. This implies that φ_0 and φ_1 are topologically similar. ■

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