Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 1, February 2009, pp. 81–100.

Torus quotients of homogeneous spaces of the general linear group and the standard representation of certain symmetric groups

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Dedicated to Professor C De Concini on the occasion of his 60th birthday

MS received 28 August 2007; revised 28 December 2007

Abstract. We give a stratification of the GIT quotient of the Grassmannian $G_{2,n}$ modulo the normaliser of a maximal torus of $SL_n(k)$ with respect to the ample generator of the Picard group of $G_{2,n}$. We also prove that the flag variety $GL_n(k)/B_n$ can be obtained as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable choice of an ample line bundle on $GL_{n+1}(k)/B_{n+1}$.

Keywords. GIT quotient; line bundle; simple reflection.

0. Introduction

Let k be an algebraically closed field. Consider the action of a maximal torus T of $SL_n(k)$ on the Grassmannian $G_{r,n}$ of r-dimensional vector subspaces of an n-dimensional vector space over k. Let N denote the normaliser of T in $SL_n(k)$. Let $\mathcal{L}_r$ denote the ample generator of the Picard group of $G_{r,n}$. Let $W = N/T$ denote the Weyl group of $SL_n(k)$ with respect to T.

In [7], it is shown that the semi-stable points of $G_{r,n}$ with respect to the T-linearised ample line bundle $\mathcal{L}_r$ is same as the stable points if and only if r and n are co-prime.

In this paper, we describe all the semi-stable points of $G_{r,n}$ with respect to $\mathcal{L}_r$. In this connection, we prove the following result:
First, we introduce some notation needed for the statement of the theorem.

Let $\mathfrak{h}_j$ be a Cartan subalgebra of $\mathfrak{sl}_{j+1}$, $\mathbb{P}(\mathfrak{h}_j)$ be the projective space and $R_j \subseteq \mathfrak{h}_j^*$ be the root system. Let V_j be the open subset of $\mathbb{P}(\mathfrak{h}_j)$ defined by

$$V_j := \{x \in \mathbb{P}(\mathfrak{h}_j) \colon \alpha(x) \neq 0, \forall \alpha \in R_j\}.$$

Here, the Weyl group of $\mathfrak{sl}_{j+1}$ is S_{j+1}, and $\mathfrak{h}_j$ is the standard representation of S_{j+1}.

With this notation, taking $m = \lceil \frac{n-1}{2} \rceil$ (for this notation, see Lemma 1.6) and $t = [\frac{n-3}{2}]$ we have the following.

Theorem. *$N \backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$ has a stratification $\bigcup_{i=0}^{t} C_i$ where $C_0 = {}_{S_{m+1}} \backslash \mathbb{P}(\mathfrak{h}_m)$, and $C_i = {}_{S_{i+m+1}} \backslash V_{i+m}$.*

On the other hand, the GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for any ample line bundle on $GL_{n+1}(k)/B_{n+1}$ and $GL_n(k)/B_n$ are both birational varieties. So, it is a natural question to ask whether the flag variety $GL_n(k)/B_n$ can be obtained as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable choice of an ample line bundle on $GL_{n+1}(k)/B_{n+1}$. We give an affirmative answer to this question. For a more precise statement, see Theorem 5.2. In this connection, we also prove that the action of the Weyl group S_{n+1} on the quotient is given by the standard representation. For a more precise statement, see Corollary 5.4.

Section 1 consists of preliminary notation and some combinatorial lemmas about minuscule weights. In §2, we describe all Schubert cells in $G_{r,n}$ admitting semi-stable points. In §3, we describe the action of the Weyl group W on $T\backslash\backslash G^{ss}_{r,n}(\mathcal{L}_r)$. In §4, we describe a stratification of $N\backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$. In §5, we obtain $GL_n(k)/B_n$ as a GIT quotient of $GL_{n+1}(k)/B_{n+1}$ modulo a maximal torus of $SL_{n+1}(k)$ for a suitable line bundle on $GL_{n+1}(k)/B_{n+1}$.

1. Preliminary notation and some combinatorial lemmas

This section consists of preliminary notation and some combinatorial lemmas about minuscule weights. Let G be a reductive Chevalley group over an algebraically closed field k. Let T be a maximal torus of the commutator subgroup $[G, G]$, B a Borel subgroup of G containing T and U be the unipotent radical of B. Let N be the normaliser of T in $[G, G]$. Let $W = N/T$ be the Weyl group of $[G, G]$ with respect to T and R denote the set of roots with respect to T, R^+ positive roots with respect to B. Let U_α denote the one dimensional T-stable subgroup of G corresponding to the root α and let $S = \{\alpha_1, \ldots, \alpha_l\} \subseteq R^+$ denote the set of simple roots. For a subset $I \subseteq S$ denote $W^I = \{w \in W | w(\alpha) > 0,\ \alpha \in I\}$ and W_I is the subgroup of W generated by $s_\alpha, \alpha \in I$. Then every $w \in W$ can be uniquely expressed as $w = w^I.w_I$, with $w^I \in W^I$ and $w_I \in W_I$. We recall the notation $R(w) = \{\alpha \in R^+\colon w(\alpha) < 0\}$ from p. 142 of [10]. Let w_0 denote the longest element of W with respect to S. Let $X(T)$ (resp. $Y(T)$) denote the set of characters of T (resp. one parameter subgroups of T). Let $E_1 := X(T) \otimes \mathbb{R}$, $E_2 = Y(T) \otimes \mathbb{R}$. Let $\langle \cdot\, , \cdot \rangle\colon E_1 \times E_2 \longrightarrow \mathbb{R}$ be the canonical non-degenerate bilinear form. Choose λ_j's in E_2 such that $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ for all i. Let the Weyl chamber corresponding to B be denoted by $\overline{C}$. We recall that $\overline{C} := \{\lambda \in E_2 | \langle \lambda, \alpha \rangle \geq 0\ \forall \alpha \in R^+\}$. For more details, see p. 64 of [1]. Also, we recall that for each $\alpha \in R$, there is a homomorphism $SL_2 \xrightarrow{\phi_\alpha} G$ (see p. 19 of [2]). We have $\check{\alpha}\colon G_m \longrightarrow G$ defined by $\check{\alpha}(t) = \phi_\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)$. We also have $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$ for all $\alpha \in R$ and $\chi \in E_1$. Set $s_i = s_{\alpha_i}\ \forall i = 1, 2, \ldots, l$. Let $\{\omega_i\colon i = 1, 2, \ldots, l\} \subset E_1$ be the fundamental weights; i.e. $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$ for all $i, j = 1, 2, \ldots, l$. For a reference, see p. 180 of [1]. A dominant weight χ is said to be minuscule if $\langle \chi, \check{\alpha} \rangle \leq 1\, \forall \alpha \in R^+$.

In this section, we prove the following elementary properties of minuscule weights:

Let ω be a minuscule weight. Let $I := \{\alpha \in S\colon \langle \omega, \check{\alpha} \rangle = 0\}$. Then, we have the following:

1. Let $\alpha \in S$ and $\tau \in W$ such that $l(s_\alpha \tau) = l(\tau) + 1$ and $s_\alpha \tau \in W^I$, then $\tau \in W^I$ with $s_\alpha \tau(\omega) = \tau(\omega) - \alpha$.
2. For any $w \in W^I$, the number of times s_i, $1 \leq i \leq n - 1$ appearing in a reduced expression of w is equal to (coefficient of α_i in ω)$-$ (coefficient of α_i in $w(\omega)$) and hence it is independent of the reduced expression of w.

3. Let $w \in W^I$ and let $w = s_{i_1}.s_{i_2}\ldots s_{i_k} \in W^I$ be a reduced expression. Then $w(\omega) = \omega - \sum_{j=1}^{k}\alpha_{i_j}$ and $l(w) = ht(\omega - w(\omega))$.
4. Let $w = s_{i_1}s_{i_2}\ldots s_{i_k} \in W$ such that $ht(\omega - s_{i_1}s_{i_2}\ldots s_{i_k}(\omega)) = k$ then $w \in W^I$ and $l(w) = k$.
5. There is a unique minimal element $w \in W^I$ such that $w(n\omega) \leq 0$ for some positive integer n.
6. There is a unique maximal element $\tau \in W^I$ such that $\tau(n\omega) \geq 0$ for some positive integer n.

For the details of proof of the above properties, see Lemma 1.3, Corollary 1.4(1), Corollary 1.4(2), Lemma 1.5, Corollary 1.9 and Corollary 1.10 respectively.

For notation in this section, we refer to [9].

Lemma 1.1. *Let I be any nonempty subset of S, and let μ be a weight of the form $\sum_{\alpha_i\in I} m_i\alpha_i - \sum_{\alpha_i\notin I} m_i\alpha_i$, where $m_i \in \mathbb{Q}$ for all i, $1 \leq i \leq l$; $m_i > 0$ for all $\alpha_i \in I$ and $m_i \geq 0$ for all $\alpha_i \in S\backslash I$. Then there is an $\alpha \in I$ such that $s_\alpha(\mu) < \mu$.*

Proof. Since $s_\alpha(\mu) = \mu - \langle\mu, \check{\alpha}\rangle\alpha$, we need to find an $\alpha \in I$ such that $\langle\mu, \check{\alpha}\rangle > 0$. This follows because the Cartan matrix $(\langle\alpha_i, \check{\alpha_j}\rangle)_{i,j}$ is positive definite, so we can find an $\alpha \in I$ such that $\langle\sum_{\alpha_i\in I} m_i\alpha_i, \check{\alpha}\rangle > 0$. Now we know that for any $\alpha_i, \alpha_j \in S, i \neq j, \langle\alpha_i, \check{\alpha_j}\rangle \leq 0$. Hence, $\langle\sum_{\alpha_i\notin I} m_i\alpha_i, \check{\alpha}\rangle \leq 0$ for this $\alpha \in I$. Thus $\langle\mu, \check{\alpha}\rangle > 0$. This proves the lemma. □

Lemma 1.2. *Let λ be any dominant weight and let $I = \{\alpha \in S \colon \langle\lambda, \check{\alpha}\rangle = 0\}$. Let $w_1, w_2 \in W^I$ be such that $w_1(\lambda) = w_2(\lambda)$. Then $w_1 = w_2$.*

Proof. See [2] and [4]. □

In the rest of this section, ω will denote a minuscule weight and $I := \{\alpha \in S \colon \langle\omega, \check{\alpha}\rangle = 0\}$.

Lemma 1.3. *Let $\alpha \in S$ and $\tau \in W$ such that $l(s_\alpha\tau) = l(\tau) + 1$ and $s_\alpha\tau \in W^I$, then $\tau \in W^I$ with $s_\alpha\tau(\omega) = \tau(\omega) - \alpha$.*

Proof. The proof of the first part of the lemma is clear. Now $s_\alpha\tau(\omega) = \tau(\omega) - \langle\tau(\omega), \check{\alpha}\rangle\alpha$. Since the pairing $\langle\cdot\,,\cdot\rangle$ is W-invariant, $\langle\tau(\omega), \check{\alpha}\rangle = \langle\omega, (\tau^{-1}\alpha)\check{}\rangle$. Again since $l(s_\alpha\tau) = l(\tau)+1$, we have $\tau^{-1}\alpha > 0$. Let $(\tau^{-1}\alpha)\check{} = \sum_{i=1}^{l} m_i\check{\alpha}_i, m_i \in \mathbb{Z}_{\geq 0}$. Now, if $\langle\omega, (\tau^{-1}\alpha)\check{}\rangle = 0$, then $m_i > 0 \Rightarrow \langle\omega, (\tau^{-1}\alpha_i)\check{}\rangle = 0$ for $1 \leq i \leq l$. This gives a contradiction, since $s_\alpha\tau \in W^I$ and $s_\alpha\tau(\tau^{-1}\alpha) = s_\alpha(\alpha) < 0$. Thus, $\langle\omega, (\tau^{-1}\alpha)\check{}\rangle = 1$. Hence the lemma is proved. □

COROLLARY 1.4

1. *For any $w \in W^I$, the number of times s_i, $1 \leq i \leq n-1$ appearing in a reduced expression of w = (coefficient of α_i in ω) − (coefficient of α_i in $w(\omega)$) and hence it is independent of the reduced expression of w.*
2. *Let $w \in W^I$ and let $w = s_{i_1}\cdot s_{i_2}\ldots s_{i_k} \in W^I$ be a reduced expression. Then $w(\omega) = \omega - \sum_{j=1}^{k}\alpha_{i_j}$ and $l(w) = ht(\omega - w(\omega))$.*

Proof. Follows from Lemma 1.3. □

Lemma 1.5. *Let $w = s_{i_1}s_{i_2}\ldots s_{i_k} \in W$ such that $ht(\omega - s_{i_1}s_{i_2}\ldots s_{i_k}(\omega)) = k$ then $w \in W^I$ and $l(w) = k$.*

Proof. This follows from Corollary 1.4. □

Lemma 1.6. *Let* $\omega = \sum_{i=1}^{l} m_i\alpha_i$, $m_i \in \mathbb{Q}_{\geq 0}$ *be a minuscule weight. Let* $I = \{\alpha \in S\colon \langle \omega, \check{\alpha}\rangle = 0\}$. *Then, there exists a unique* $w \in W^I$ *such that* $w(\omega) = \sum_{i=1}^{l}(m_i - \lceil m_i \rceil)\alpha_i$ *where for any real number* x,

$$\lceil x \rceil := \begin{cases} x & \text{if } x \text{ is an integer} \\ [x]+1 & \text{otherwise} \end{cases}.$$

Proof. Using Lemma 1.1 and the fact that ω is minuscule, we can find a sequence $s_{i_k}, s_{i_{k-1}}, \ldots, s_{i_1}$ of simple reflections in W such that for each j, $2 \leq j \leq k+1$, coefficient of α_{i_j} in $s_{i_{j-1}} \cdot s_{i_{j-2}} \ldots s_{i_1}(\omega_r)$ is positive and $(s_{i_k} \cdot s_{i_{k-1}} \ldots s_{i_1}(\omega_r)) = \omega_r - \sum_{j=1}^{k} \alpha_{i_j}$. The existence part of the lemma follows from here. The uniqueness follows from Lemma 1.2. □

Lemma 1.7. *Let* $\omega = \sum_{i=1}^{l} m_i\alpha_i$, $m_i \in \mathbb{Q}_{\geq 0}$ *be a minuscule weight. Let* $I = \{\alpha \in S\colon \langle \omega, \check{\alpha}\rangle = 0\}$. *Then, there exists a unique* $\tau \in W^I$ *such that* $\tau(\omega) = \sum_{i=1}^{l}(m_i - [m_i])\alpha_i$.

Proof. Proof is similar to that of Lemma 1.6. □

Now onwards, we say that for two elements w_1 and w_2 in W, $w_1 \leq w_2$ if $l(w_2) = l(w_1) + l(w_2 w_1^{-1})$.

Lemma 1.8. *Let* ω *and* I *be as in Lemma* 1.6 *and* $w_1, w_2 \in W^I$. *Then* $w_2(\omega) \leq w_1(\omega) \Leftrightarrow w_1 \leq w_2$.

Proof. Only the implication $\Rightarrow$ is to be proved. The proof is by induction on $ht(w_1(\omega) - w_2(\omega))$ which is a non-negative integer. By Lemma 1.2, the height may be assumed to be positive.

$ht(w_1(\omega) - w_2(\omega)) = 1$: This means $w_1(\omega) = w_2(\omega) + \alpha$ for some $\alpha \in S$. Applying s_α on both sides of this equation, we have

$$\begin{aligned} s_\alpha w_1(\omega) &= -\alpha + s_\alpha w_2(\omega) \\ &\Longrightarrow w_2(\omega) - \langle \omega, (w_1^{-1}\alpha)\check{}\rangle\alpha = -2\alpha + w_2(\omega) - \langle \omega, (w_2^{-1}\alpha)\check{}\rangle\alpha \\ &\Longrightarrow \langle \omega, (w_1^{-1}\alpha)\check{}\rangle = 2 + \langle \omega, (w_2^{-1}\alpha)\check{}\rangle. \end{aligned}$$

Since ω is minuscule, we get $\langle \omega, (w_1^{-1}\alpha)\check{}\rangle = 1$ and $\langle \omega, (w_2^{-1}\alpha)\check{}\rangle = -1$. This implies, by Lemma 1.5, that $l(s_\alpha w_1) = l(w_1) + 1$ and $s_\alpha w_1 \in W^I$. Now, we have $s_\alpha w_1(\omega) = w_2(\omega)$. Hence, by Lemma 1.2, we get $w_2 = s_\alpha w_1$ with $l(w_2) = l(w_1) + 1$. Thus the result follows in this case.

Let us assume that the result is true for $ht(w_1(\omega) - w_2(\omega)) \leq m - 1$.

$ht(w_1(\omega) - w_2(\omega)) = m$: Let $w_1(\omega) - w_2(\omega) = \sum_{\alpha_i \in J} m_i\alpha_i$ where $J \subseteq S$ and m_i's are positive integers. Since $\langle \sum_{\alpha_i \in J} m_i\alpha_i, \sum_{\alpha_i \in J} m_i\check{\alpha_i}\rangle \geq 0$, there exists an $\alpha_j \in J$ such that $\langle w_1(\omega) - w_2(\omega), \check{\alpha_j}\rangle > 0$. Hence, either $\langle w_1(\omega), \check{\alpha_j}\rangle > 0$ or $\langle w_2(\omega), \check{\alpha_j}\rangle < 0$.

Case I. Let us assume $\langle w_1(\omega), \check{\alpha_j}\rangle > 0$. Then $l(s_{\alpha_j} w_1) = l(w_1) + 1$ and $s_{\alpha_j} w_1 \in W^I$. Now $ht(s_{\alpha_j} w_1(\omega) - w_2(\omega)) = m - 1$. Hence, by induction hypothesis $w_2 = \phi_1 s_{\alpha_j} w_1$ with $l(w_2) = l(\phi_1) + l(s_{\alpha_j} w_1)$. Thus taking $\phi = \phi_1.s_{\alpha_j}$ and noting that $l(\phi) = l(\phi_1) + 1$, we are done in this case.

Case II. Let us assume $\langle w_2(\omega), \check{\alpha_j}\rangle < 0$. Then $l(s_{\alpha_j} w_2) = l(w_2) - 1$ and $s_{\alpha_j} w_2 \in W^I$. Since $w_1(\omega) - s_{\alpha_j} w_2(\omega) = m-1$, by induction hypothesis $s_{\alpha_j} w_2 = \phi_2 w_1$ with $l(s_{\alpha_j} w_2) = l(\phi_2) + l(w_1)$. Thus taking $\phi = s_{\alpha_j}\phi_2$ and noticing that $l(\phi) = 1 + l(\phi_2)$, we are done in this case also. This completes the proof. □

COROLLARY 1.9

Let ω, w and I be as in Lemma 1.6. *Let $\sigma \in W^I$ be such that $\sigma(n\omega) \leq 0$ for some positive integer n. Then, we have $w \leq \sigma$.*

Proof. The proof follows from Lemmas 1.6, 1.8 and the fact that ω is minuscule. □

COROLLARY 1.10

Let ω, τ and I be as in Lemma 1.7. *Let $\sigma \in W^I$ be such that $\sigma(n\omega) \geq 0$ for some positive integer n. Then, we have $\sigma \leq \tau$.*

Proof. The proof follows from Lemmas 1.7, 1.8 and the fact that ω is minuscule. □

2. Description of Schubert varieties in the Grassmannian having semi-stable points

In this section, we have the following notation. Let $G = GL_n(k)$ where characteristic of k is either zero or bigger than n. Let $r \in \{2, \ldots, n-2\}$. Consider the action of a maximal torus T of $SL_n(k)$ on the Grassmannian $G_{r,n}$. Let B be a Borel subgroup of G containing T. Let $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$ be the set of simple roots with respect to B arranged in the ordering of the vertices in the Dynkin diagram of type A_{n-1}. Let $I_r = S \backslash \{\alpha_r\}$. We first note that $G_{r,n}$ is the homogeneous space $GL_n(k)/P_r$ where $P_r = BW_{I_r}B$ is the maximal parabolic subgroup of $GL_n(k)$ containing B associated to the simple root α_r. Let ω_r be the fundamental weight associated to the simple root α_r and let $\mathcal{L}_r$ denote the line bundle on $GL_n(k)/P_r$ corresponding to ω_r. We describe all Schubert cells in $GL_n(k)/P_r$ admitting semi-stable points for the above-mentioned action of T with respect to the line bundle $\mathcal{L}_r$.

Some of the elementary facts about the combinatorics of W^{I_r} which are being used in this section can be found in [9]. For the convenience of the reader, we prove them here.

Lemma 2.1. *Let $w \in W^I$, $w \neq id$. Then there exists an $i \in \mathbb{N}$, $i \leq r$ and a sequence of positive integers $\{a_j\}$, $j = 1, 2, \ldots, r$ such that the following holds.*

(a) $a_j \geq j$ *for all* j, $i \leq j \leq r$.
(b) $w = (s_{a_i} \cdot s_{a_i - 1} \ldots s_i)(s_{a_{i+1}} \cdot s_{a_{i+1}-1} \ldots s_{i+1}) \ldots (s_{a_r} \cdot s_{a_r - 1} \ldots s_r)$ *with* $l(w) = \sum_{j=i}^{r}(a_j - j + 1)$.

Proof. Let i be the least positive integer such that $s_{\alpha_i} \leq w$. The rest of the proof follows from braid relations in W. □

Lemma 2.2. *Let $w, \tau \in W^I$. Write $w = (s_{a_i} \cdot s_{a_i-1} \ldots s_i)(s_{a_{i+1}} \cdot s_{a_{i+1}-1} \ldots s_{i+1}) \ldots (s_{a_r} \cdot s_{a_r-1} \ldots s_r)$ and $\tau = (s_{b_k} \cdot s_{b_k-1} \ldots s_k)(s_{b_{k+1}} \cdot s_{b_{k+1}-1} \ldots s_{k+1}) \ldots (s_{b_r} \cdot s_{b_r-1} \ldots s_r)$ be as in Lemma* 2.1. *Then $w \leq \tau \Leftrightarrow k \leq i$ and $b_j \geq a_j$ for all j, $i \leq j \leq r$.*

Proof. The proof follows from Lemma 1.8 and the fact that $w(\omega_r) \geq \tau(\omega_r) \Leftrightarrow k \leq i$ and $b_j \geq a_j$ for all $j, i \leq j \leq r$. □

Now, write $n = qr + t$ with $1 \leq t \leq r$ and let $\tau_r \in W^{I_r}$ be the unique element as in Lemma 1.6 for the case when $\omega = \omega_r$. Then, τ_r must be of the form $\tau_r = (s_{a_1} \ldots s_1) \ldots (s_{a_r} \ldots s_r)$ where

$$a_i = \begin{cases} i(q+1) & \text{if } i \leq t-1 \\ iq + (t-1) & \text{if } t \leq i \leq r \end{cases}.$$

Let $\tau^{(n-r)} \in W^{I_{n-r}}$ be the unique element as in Lemma 1.7 for the case $\omega = \omega_r$. Let $w_0^{I_r}$ denote the minimal representative of the longest element w_0 of W in W^{I_r}. Then, we have $\tau_r = \tau^{(n-r)} w_0^{I_r}$ and $l(w_0^{I_r}) = l(\tau_r) + l(\tau^{(n-r)})$.

Let $w \in W^{I_r}$ be such that $w(n\omega_r) \leq 0$.

Then, we have

Lemma 2.3. $\tau_r \leq w$ *and* $w\tau_r^{-1} \leq (\tau^{(n-r)})^{-1}$.

Proof. Proof follows from Corollary 1.9 and Corollary 1.10. □

For any such w, we describe the set $R(w^{-1})$.

Lemma 2.4. $R(w^{-1})$ *consists of roots of the form* $\alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}$ *for* $1 \leq i \leq r$ *where* $j \neq a_k + 1$ *for any* $k < i$.

Proof. We have $w^{-1} = (s_r \ldots s_{a_r}) \ldots (s_2 \ldots s_{a_2}).(s_1 \ldots s_{a_1})$, which is a reduced expression. Thus the elements of $R(w^{-1})$ are

$$\beta_{i,j-i+1} = (s_{a_1} \ldots s_1) \cdot (s_{a_2} \ldots s_2) \ldots (s_{a_i} \ldots s_{j+1} \cdot \hat{s_j} \cdot \hat{s_{j-1}} \ldots \hat{s_i})(\alpha_j)$$

where $i \leq j \leq a_i$, $1 \leq i \leq r$, $\hat{}$ denotes omission of the symbols. We have

$$(s_{a_i} \ldots s_{j+1} \cdot \hat{s_j} \cdot \hat{s_{j-1}} \ldots \hat{s_i})(\alpha_j) = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}$$

Since $a_1 < a_2 < \cdots < a_r$, each $\beta_{i,j}$ is of the form

$$\alpha_j + \alpha_{j+1} + \cdots + \alpha_{a_i}.$$

Now $j \neq a_k + 1$ for any $k < i$ follows from the fact that $l(w)$ is the same as the cardinality of $R(w^{-1})$. □

Remark 2.5. From the lemma it follows that the elements of $R(w^{-1})$ can be written in an array as follows:

$$\begin{array}{ccccccccccccc}
\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,a_1} \\
\beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,a_1} & \beta_{2,a_1+1} & \beta_{2,a_1+2} & \cdots & \beta_{2,a_2-1} \\
\beta_{3,1} & \beta_{3,2} & \cdots & \beta_{3,a_1} & \beta_{3,a_1+1} & \beta_{3,a_1+2} & \cdots & \beta_{3,a_2-1} & \beta_{3,a_2} & \cdots & \beta_{3,a_3-2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\beta_{r,1} & \beta_{r,2} & \cdots & \beta_{r,a_1} & \beta_{r,a_1+1} & \beta_{r,a_1+2} & \cdots & \beta_{r,a_2-1} & \beta_{r,a_2} & \cdots & \beta_{r,a_3-2} & \cdots & \beta_{r,a_r-r+1}
\end{array}$$

where the array has r rows, and the length of the i-th row is $a_i - (i-1)$. Note that $\beta_{1,a_1} = \alpha_{a_1}$, and for $2 \le i \le r$, $\beta_{i,a_i-i+1} = \alpha_{a_i}$, only if $a_i \ge a_{i-1}+2$. In this case, for all j, $i \le j \le r$, $\beta_{j,a_{i-1}-i+2} = \beta_{i-1,a_{i-1}-i+2} + \alpha_{a_{i-1}+1} + \alpha_{a_{i-1}+2} + \cdots + \alpha_{a_j}$ and $\beta_{j,a_{i-1}-i+3} = \alpha_{a_{i-1}+2} + \alpha_{a_{i-1}+3} + \cdots + \alpha_{a_j}$. If $a_i = a_{i-1}+1$, then $a_i - i + 1 = a_{i-1} - (i-1) + 1$, therefore, the $(i-1)$-th and i-th rows have same length. In this case for all j, $i \le j \le r$, $\beta_{j,a_i-i+1} = \beta_{i-1,a_i-i+1} + \alpha_{a_{i-1}+1} + \alpha_{a_{i-1}+2} + \cdots + \alpha_{a_j}$.

For any $w \in W^{I_r}$, let $X(w) := \overline{BwP_r/P_r}$ denote the Schubert variety in $GL_n(k)/P_r$.

We recall $BwP_r/P_r = U_w w P_r$, where U_w is the product $\prod_{\alpha \in R(w^{-1})} U_\alpha$ of the root groups U_α, and we describe below the ordering of roots in which the product is taken.

Consider the open set

$$V := \left\{ \prod_{\beta_{ij} \in R(w^{-1})} u_{\beta_{ij}}(x_{\beta_{ij}}) w P_r \colon x_{\beta_{ij}} \ne 0,\ \forall \beta_{ij} \in R(w^{-1}) \right\}$$

of $X(w)$ in $GL_n(k)/P_r$ where the order in which the product is taken is as follows: Put a partial order on $R(w^{-1})$ by declaring $\beta_{ij} \le \beta_{kl}$ if either $i = k$ and $j \ge l$ or if $i < k$. Now we take the product so that whenever $\beta_{ij} \le \beta_{kl}$, $u_{\beta_{ij}}(x_{\beta_{ij}})$ appears on the right-hand side of $u_{\beta_{kl}}(x_{\beta_{kl}})$. Note that $u_{\beta_{ij}}(x_{\beta_{ij}})$'s commute with each other, since $\beta_{i_1,j_1}, \beta_{i_2,j_2} \in R(w^{-1})$ implies $\beta_{i_1,j_1} + \beta_{i_2,j_2}$ is not a root. This follows from the fact that no element of $R(w^{-1})$ starts or ends with α_{a_k+1}, for any k, $1 \le k \le r-1$ (i.e. for all $\beta_{i,j} \in R(w^{-1})$ and $1 \le k \le r-1$, $\beta_{i,j} - \alpha_{a_k+1} \ne 0$ is not a root.)

Now the natural action of the maximal torus T on $GL_n(k)/P_r$, induces an action of T on V.

Lemma 2.6. *Consider the torus $T' = \prod_{\beta \in R(w^{-1})} G_{m,\beta}$ where $G_{m,\beta} = G_m$ for each $\beta \in R(w^{-1})$. We have a natural action of T on T' through the homomorphism of algebraic groups $\Psi\colon T \to T'$ defined by $\Psi(t) = (t^\beta)$ for all $t \in T$. The map $V \to T'$ defined by $\prod u_\beta(x_\beta) w \cdot P \mapsto (x_\beta)_\beta$ is a T-equivariant isomorphism of varieties.*

Proof. Proof is easy. □

Now, we recall the definition of the Hilbert–Mumford numerical function and definition of the semistable points from [5]. We also refer to [6] for notation in geometric invariant theory.

1. Let X be a projective variety with an action of a reductive group G. Let λ be a one-parameter subgroup of G. Let $\mathcal{L}$ be a G-linearised very ample line bundle on X. Let $x \in \mathbb{P}(H^0(X, \mathcal{L})^*)$ and $\hat{x}$ be a point in the cone over X which lies on x. Write $\hat{x} = \sum_{i=1}^r v_i$, where each v_i is a weight vector of λ of weight m_i.
   Then, we have
   $$\mu^{\mathcal{L}}(x, \lambda) = -\min\{m_i | i = 1, \ldots, r\}.$$
2. A point $x \in X(w)$ is said to be semi-stable with respect to the T-linearised line bundle $\mathcal{L}$ if there is a positive integer $m \in \mathbb{N}$, and a T-invariant section $s \in H^0(X(w), \mathcal{L}^m)$ with $s(x) \ne 0$. We denote by $X(w)^{ss}_T(\mathcal{L})$, the set of all points semi-stable points in $X(w)$ with respect to the T linearised line bundle $\mathcal{L}$.

We now describe all the Schubert varieties admitting semi-stable points.

Let $n = qr + t$, with $1 \le t \le r$ and let $w \in W^{I_r}$.

Lemma 2.7. *The following are equivalent:*

(1) $X(w)^{ss}_T(\mathcal{L}_r)$ *is non-empty.*
(2) $\tau_r \leq w$ *and* $w\tau_r^{-1} \leq (\tau^{(n-r)})^{-1}$.
(3) $w = (s_{a_1}\cdots s_1)\cdots(s_{a_r}\cdots s_r)$, *where* $\{a_i: i = 1, 2, \ldots, r\}$ *is an increasing sequence of positive integers in* $\{1, 2, \ldots, n-1\}$ *such that* $a_i \geq i(q+1)$ $\forall i \leq t-1$ *and* $a_i = iq + (t-1)$ $\forall i$ *such that* $t \leq i \leq r$.

Proof. By Hilbert–Mumford criterion (Theorem 2.1 of [5]) a point $x \in G/P_r$ is semi-stable if and only if $\mu^L(\sigma x, \lambda) \geq 0$ for all $\lambda \in \overline{C}$ and for all $\sigma \in W$. By Lemma 2.1 of [8], this statement is equivalent to $\langle -w_\sigma(\omega), \lambda\rangle \geq 0$ for all $\lambda \in \overline{C}$ and for all $\sigma \in W$, where $w_\sigma \in W^{I_r}$ is such that $\sigma x \in U_{w_\sigma} w_\sigma P_r$. Thus, by Corollary 1.9 applied to the situation $\omega = \omega_r$, a point x is semi-stable if and only if x is not in the W-translates of $U_\tau \tau P_r$ with $\tau \in W^{I_r}$ and $\tau_r \not\leq \tau$.

Now, for a $w \in W^{I_r}$, $X(w)$ is not contained in the finite union $\bigcup_{\tau \not\geq \tau_r} U_\tau \tau P_r$ if and only if $\tau_r \leq w$. The second condition $w\tau_r^{-1} \leq (\tau^{(n-r)})^{-1}$ is an immediate consequence when $w \geq \tau_r$. This completes the proof of (2). Proof of (3) follows from Corollary 1.9 and the discussion after Lemma 2.2. □

PROPOSITION 2.8

Let $X_{i,j}$ *denote the regular function on* V *defined by* $\prod u_{\beta_{kl}}(x_{\beta_{kl}})w \cdot P \mapsto x_{\beta_{ij}}$ *for all* $1 \leq i \leq r-1$ *and* $1 \leq j \leq a_i - i + 1$; *and let* $Y_{i,j} := \frac{X_{i,a_i-i+1} \cdot X_{i+1,j}}{X_{i,j} \cdot X_{i+1,a_i-i+1}}$. *Then the ring of* T*-invariant regular functions is generated by* $Y_{i,j}, Y_{i,j}^{-1}$, *where* $1 \leq j \leq a_i - i$, *for each* i, *and* $1 \leq i \leq r-1$; $Y_{i,j}$ *are algebraically independent.*

Proof. Recall the map, $T \xrightarrow{\Psi} T'$ defined by $\Psi(t) = (t^\beta)$, $\beta \in R(w^{-1})$ as in Lemma 2.6. Proof of the proposition follows from the following claim.

Claim. $E_{i,a_i-(i-1)} - E_{i+1,a_i-(i-1)} - E_{i,j} + E_{i+1,j}$; $i = 1, 2, \ldots, r-1$ and $j = 1, 2, \ldots, a_i - i$ forms a basis for $\mathrm{Ker}(\Psi^*: X(T') \longrightarrow X(T))$, where $E_{i,k}$ is the matrix with 1 in the (i,k)-th place and 0 elsewhere.

Proof of the Claim. Now any character of T' is of the form $(t^\beta) \mapsto \prod t^{m_\beta \beta}$ where m_β are integers. Now such a character is T-invariant iff the sum $\sum_\beta m_\beta \beta$ is zero. Plugging in the expression of β's in terms of the simple roots α_k's and noting that they are linearly independent we get a set of linear equations over $\mathbb{Z}$, by equating to zero the coefficient of each α_k. Let us denote by $R(p)$, $1 \leq p \leq r$ the set of roots appearing in p-th row of the array in Remark 2.5. Also, let $C(q)$, $1 \leq q \leq a_r - (r-1)$ denote the set of roots appearing in the q-th column of the array in Remark 2.5.

Comparing the coefficient of α_1, we have $\sum_{\beta \in C(1)} m_\beta = 0$.

Comparing the coefficient of α_2, and using the above observation, we get $\sum_{\beta \in C(2)} m_\beta = 0$. Proceeding this way, we get

$$\sum_{\beta \in C(j)} m_\beta = 0 \quad \forall j,\ 1 \leq j \leq a_1.$$

Let k be the least positive integer such that $\alpha_k + \cdots + \alpha_{a_i}$ is the first root in the column $C(a_1+1)$. Comparing the coefficient of α_k, we get $\sum_{\beta \in C(a_1+1)} m_\beta = 0$. Proceeding this way, we get

$$\sum_{\beta \in C(j)} m_\beta = 0 \quad \forall j,\ 1 \le j \le a_r - r + 1.$$

Now, comparing the coefficient of α_{a_r}, we get $\sum_{\beta \in R(r)} m_\beta = 0$. Comparing the coefficient of $\{\alpha_j \colon j = a_{r-1}, 2 + a_{r-1}, \ldots, a_r\}$, we get

$$\sum_{\beta \in R(r-1)} m_\beta + \sum_{\beta \in R(r)} m_\beta = 0.$$

Thus we have

$$\sum_{\beta \in R(r-1)} m_\beta = 0.$$

Proceeding this way, we get

$$\sum_{\beta \in R(i)} m_\beta = 0 \quad \forall i,\ 1 \le i \le r.$$

□

3. Description of the action of the Weyl group on the quotient $T\backslash\backslash G^{ss}_{r,n}(\mathcal{L}_r)$

In this section, we describe the action of the Weyl group on the quotient $T\backslash\backslash G^{ss}_{r,n}(\mathcal{L}_r)$. We recall the definition of $\mu^L(x, \lambda)$ from [5].

We first write down the stabiliser of $X(w)$ in W. Let $w = (s_{a_1} \ldots s_1)(s_{a_2} \ldots s_2) \ldots (s_{a_r} \ldots s_r) \in W^{I_r}$ be such that $w \ge \tau_r$. Then, we have as follows.

Lemma 3.1. *The set* $\{s_i \colon s_i(X(w)) \subseteq X(w), i = 1, 2, \ldots, n-1\}$ *consists of*:

1. $\{s_j \colon 1 \le j \le a_1 - 2\}$.
2. $\{s_j \colon a_p + 2 \le j \le a_{p+1} - 1, p = 1, 2, \ldots, r-1\}$.
3. $\{s_j \colon a_r + 2 \le j \le n-1\}$.
4. $\{s_{a_p} \colon p = 1, 2, \ldots, r\}$.

Proof. We see that $s_j w \le w$ if and only if $s_j(X_w) \subseteq X(w)$.

(1) Now for $1 \le j \le a_1 - 2$, we have $s_j(w) = (s_{a_1} \ldots s_1)s_{j+1}(s_{a_2} \ldots s_2) \ldots (s_{a_r} \ldots s_r)$, since $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$.
Using this for $j+1, \ldots, j+r-1$, we get $s_j w = w s_{j+r}$.
Since $j + r \in I_r$, we must get $(s_j w)^{I_r} = w$.
(2) If $a_p + 2 \le j \le a_{p+1} - 1$, then $s_j w = w s_{j+r-p}$. Hence $(s_j w)^{I_r} = w$ as $j + r - p \in I_r$.
(3) For $a_r + 2 \le j \le n-1$, we have $s_j w = w s_j$. Hence, $(s_j w)^{I_r} = w$ as $j \in I_r$.
(4) If $p = 1$, there is nothing to prove.

If $p \ge 2$, we divide the proof into two cases.

Case I. If $a_p = p$, then $a_{p-1} = p - 1$, and $s_{a_p} w = w s_{p-1}$. Thus, $(s_{a_p} w)^{I_r} = w$ as $p - 1 \in I_r$.

Case II. If $a_p \ge p + 1$.

Subcase I. If $a_p \geq a_{p-1}+2$, then we have

$$s_{a_p}w = (s_{a_1}\ldots s_1)\ldots(s_{a_{p-1}}\ldots s_{p-1})(s_{a_p}-1\ldots s_p) \times (s_{a_{p+1}}\ldots s_{p+1})\ldots(s_{a_r}\ldots s_r) < w.$$

Subcase II. If $a_p = 1+a_{p-1}$, then

$$\begin{aligned} s_{a_p} &= (s_{a_1}\ldots s_1)\cdots(s_{a_{p-2}}\ldots s_{p-2})(s_{a_p}s_{a_p-1}s_{a_p}s_{a_{p-1}-1}\ldots s_{p-1}) \\ &\quad\times(s_{a_p-1}\ldots s_p)(s_{a_{p+1}}\ldots s_{p+1})\cdots(s_{a_r}\ldots s_r) \\ &= (s_{a_1}\ldots s_1)\cdots(s_{a_{p-1}}\ldots s_{p-1})(s_{a_p}\ldots s_p)s_{p-1}(s_{a_{p+1}}\ldots s_{p+1}) \\ &\quad\times\cdots(s_{a_r}\ldots s_r) = ws_{p-1}.\end{aligned}$$

Hence, $(s_{a_p}w)^{I_r} = w$, as $p-1 \in I_r$. □

We now explicitly describe the action of the stabilisers.

PROPOSITION 3.2

Description of the action of the s_i's:

(1) *s_j interchanges $Y_{i,j}$ and $Y_{i,j+1}$ for $i = 1,2,\ldots,r-1$, and keeps all other $Y_{i,k}$'s fixed.*

(2) *s_j interchanges $Y_{i,j-p}$ and $Y_{i,j-p+1}$ for $j+1 \leq i \leq r-1$, and keeps all other $Y_{i,k}$'s fixed.*

(3a) *If $2 \leq p \leq r$, then s_{a_p-1} fixes all the $Y_{i,k}$, $1 \leq i \leq p-1$.*

(3b) *If $p \leq i \leq r-1$, $a_p-p = a_i-i$ and $1 \leq k < a_p-p$, then $s_{a_p-1}(Y_{i,a_p-p}) = Y^{-1}_{i,a_p-p}$, and $s_{a_p-1}(Y_{i,k}) = Y_{i,k}\cdot Y^{-1}_{i,a_p-p}$.*

(3c) *If $p+1 \leq i \leq r-1$, $a_i - i \gneq a_p - p$, then $s_{a_p-1}(Y_{i,a_p-p}) = Y_{i,a_p-p+1}$, and keeps all other $Y_{i,k}$'s fixed.*

(4a) *$2 \leq p \leq r-1$, and $a_p = a_{p-1}+1$.*

- (i) *If $3 \leq p \leq r$ and $1 \leq k \leq a_{a_{p-2}-p+2}$, then $s_{a_p}(Y_{p-2,k}) = Y_{p-2,k}\cdot Y_{p-1,k}\cdot Y^{-1}_{p-1,a_{p-2}-p+3}$.*
- (ii) *If $1 \leq k \leq a_p-p$ then $s_{a_p}(Y_{p-1,k}) = Y^{-1}_{p-1,k}$ and $s_{a_p}(Y_{p,k}) = Y_{p,k}\cdot Y_{p-1,k}$.*
- (iii) *$Y_{i,k}$'s are fixed for $i \neq p-2, p-1, p$ and $1 \leq k \leq a_i - i$.*

(4b)

- (i) *If $1 \leq i \leq p-1$ or $a_p-p+1 \leq k \leq a_r$, $Y_{i,k}$'s are fixed.*
- (ii) *If $i = p$ and $1 \leq k \leq a_p-p$ then $s_{a_p}(Y_{p,k}) = 1 - Y_{p,k}$.*
- (iii) *If $p+1 \leq i \leq r-1$ and $1 \leq k \leq a_p-p$, then,*
  $s_{a_p}(Y_{i,k}) = \frac{1-\prod_{m=p}^{i}(Y_{m,k}/Y_{m,a_p-p+1})}{1-\prod_{m=p}^{i-1}(Y_{m,k}/Y_{m,a_p-p+1})}\times Y_{i,a_p-p+1}$.

(4c) *Action of s_{a_r}.*

- (i) *If $a_r = a_{r-1}+1$, then*
  $s_{a_r}(Y_{r-2,k}) = Y_{r-2,k}\cdot Y_{r-1,k}\cdot Y^{-1}_{r-1,a_{r-2}-r+3}$, for $1 \leq k \leq a_{r-2}-r+2$ and $s_{a_r}(Y_{r-1,k}) = Y^{-1}_{r-1,k}$, for $1 \leq k \leq a_r - r$.
- (ii) *If $a_{r-1}+2 \leq a_r$, then $Y_{r,k}$'s are fixed for $1 \leq k \leq a_r-r+1$.*

Proof. Proof is essentially based on the following properties of groups with BN-pair and commutator relations:

(i)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 0 & \frac{-1}{x} \end{pmatrix},$$

and

(ii)

$$[u_\alpha(x_\alpha), u_\beta(x_\beta)] = \begin{cases} u_{\alpha+\beta}(x_\alpha \cdot x_\beta) & \text{if } \alpha = \epsilon_i - \epsilon_j \text{ and } \beta = \epsilon_j - \epsilon_k, i < j < k; \\ u_{\alpha+\beta}(-x_\alpha \cdot x_\beta) & \text{if } \alpha = \epsilon_i - \epsilon_j \text{ and } \beta = \epsilon_k - \epsilon_i, k < i < j. \end{cases}$$

We first consider the action of W on the $X_{j,k}$'s and then describe resulting action on the $Y_{j,k}$'s. If $1 \leq i \leq a_1 - 2$, then, s_i interchanges $X_{j,i}$ and $X_{j,i+1}$ for all j, $1 \leq j \leq r$. Therefore, it follows that s_i interchanges $Y_{j,i}$ and $Y_{j,i+1}$ for all j, $1 \leq j \leq r-1$ and keeps all other $Y_{j,k}$'s fixed. Similarly for $p \geq 2$ and $a_p + 2 \leq a_{p+1}$, if $a_p + 2 \leq i \leq a_{p+1} - 2$, s_i interchanges $X_{j,i-p}$ and $X_{j,i-p+1}$. Thus s_i interchanges $Y_{j,i-p}$ and $Y_{j,i-p+1}$ for all j, $i+1 \leq j \leq r-1$ and keeps all other $Y_{j,k}$'s fixed. Now, we compute the actions of s_{a_i-1}, s_{a_i} and s_{a_i+1}.

Action of s_{a_i+1} for each i, $1 \leq i \leq r-1$

Case I. $a_i + 2 \leq a_{i+1}$. In this case we have

$$\begin{aligned} s_{a_i+1}w &= s_{a_1+1} \cdot (s_{a_1} \dots s_1) \cdot (s_{a_2} \dots s_2) \cdots (s_{a_r} \dots s_r) \\ &= (s_{a_1} \dots s_1) \cdots (s_{a_i+1} \cdot s_{a_i} \dots s_i) \cdots (s_{a_r} \dots s_r) \end{aligned}$$

which is a reduced expression and $s_{a_i+1} \cdot w \in W^I$ by Lemma 1.12. Now Lemma 1.13 implies that $s_{a_i+1} \cdot w \geq w$. Hence, $X(w)$ is not stable under the action of s_{a_i+1}.

Case II. $a_i + 1 = a_{i+1}$. In this case $s_{a_i+1} = s_{a_{i+1}}$ and the action will be described in the later part of this paragraph. In fact we see that in this case $(s_{a_i+1}w)^I = w$. Hence $X(w)$ is stable under the action of s_{a_i+1}.

Action of s_{a_i-1}

In case $i = 1$, we may assume that $a_1 \neq 1$, and for $i \geq 2$, $a_{i-1} \neq a_i - 1$. Now s_{a_i-1} interchanges the $(a_i - i)$-th and $(a_i - i + 1)$-th columns of each of the j-th row of the array of roots $R(w^{-1})$, for $i \leq j \leq r$; thus s_{a_i-1} interchanges X_{j,a_i-i} and X_{j,a_i-i+1} for each j, $i \leq j \leq r$. Therefore, the action of s_{a_i-1} is as follows:

(1) s_{a_i-1} fixes all the $Y_{j,k}$, for $1 \leq j \leq i-1$, for $i \geq 2$.
(2) For $j \geq i \leq r-1$ and $a_i - i = a_j - j$, $Y_{j,a_i-i} \mapsto Y^{-1}_{j,a_i-i}$, and for $Y_{j,k} \mapsto Y_{j,k}.Y^{-1}_{j,a_i-i}$ for $1 \leq k < a_i - i$.
(3) For $i+1 \leq j \leq r-1$ if $a_j - j > a_i - i$, then s_{a_i-1} interchanges Y_{j,a_i-i} and Y_{j,a_i-i+1} and keeps all other $Y_{j,k}$'s fixed.

Action of s_{a_i} for $1 \leq i \leq r$

Let us show that $X(w)$ is stable under the action of each of the s_{a_i}. Let

$$w = (s_{a_1} \dots s_1) \cdot (s_{a_2} \dots s_2) \cdots (s_{a_r} \dots s_r).$$

Thus

$$s_{a_i} w = (s_{a_1} \dots s_1) \cdots (s_{a_{i-2}} \dots s_{i-2}) \cdot s_{a_i} \cdot (s_{a_{i-1}} \dots s_{i-1}) \cdot (s_{a_i} \dots s_i) \cdots (s_{a_r} \dots s_r).$$

Case 1. $i = 1$ or $a_{i-1} + 2 \leq a_i$ for $i \geq 2$. In this case it is clear that

$$s_{a_i} w = (s_{a_1} \dots s_1) \cdots (s_{a_{i-2}} \dots s_{i-2}) \cdot (s_{a_{i-1}} \dots s_{i-1}) \cdot (s_{a_i - 1} \dots s_i) \cdots (s_{a_r} \dots s_r)$$

which, by Lemmas 2.1 and 2.2, is in W^{I_r} and $s_{a_i} w \leq w$.

Case 2. $a_{i-1} + 1 = a_i$. Note that,

$$w_1 = (s_{a_{i-1}} \dots s_{i-1}) \cdot (s_{a_i} \dots s_i) \in W^J,$$

where $J = S \backslash \{\alpha_i\}$. Now,

$$w_1(\omega_i) = \omega_i - \sum_{j=i-1}^{a_{i-1}} \alpha_j - \sum_{j=i}^{a_i} \alpha_j$$
$$\Rightarrow s_{a_i} w_1(\omega_i) = s_{a_i}(\omega_i) - \sum_{j=i-1}^{a_{i-1}} \alpha_j - \sum_{j=i}^{a_i} \alpha_j.$$

Now, if $a_i = i$, then $a_{i-1} = i - 1$; so $s_{a_i} w_1 = s_i \cdot s_{i-1} \cdot s_i = s_{i-1} \cdot s_i \cdot s_{i-1} = w_1 \cdot s_{i-1}$. Otherwise, $a_i \neq i$. This implies that $s_{a_i}(\omega_i) = \omega_i$. Therefore, $s_{a_i} w_1(\omega_i) = w_1(\omega_i)$. Hence, by Lemma 1.3, we get $s_{a_i} w_1 = w_1 \cdot s_\alpha$ for some $\alpha \in J$. This gives $w_1^{-1} s_{a_i} w_1 = s_{w_1^{-1}(\alpha_{a_i})} = s_\alpha$. Now it follows that $w_1^{-1}(\alpha_{a_i}) = \alpha_{i-1}$. Hence, $s_{a_i} w_1 = w_1 \cdot s_{i-1}$. Therefore, in both the sub-cases $s_{a_i} \cdot w = w \cdot s_{i-1}$; in particular $(s_{a_i} \cdot w)^{I_r} = w$. Now we shall compute the action of s_{a_i}, for $1 \leq i \leq r$.

Case I. $2 \leq i \leq r - 1$ and $a_i = a_{i-1} + 1$. In this case, s_{a_i} interchanges $X_{i,k}$ and $X_{i-1,k}$ for $1 \leq k \leq a_i - i + 1$ and keeps all other $X_{j,k}$'s fixed. Hence, the action of s_{a_i} on $Y_{j,k}$'s is as follows:

(1) If $i \geq 3$, $Y_{i-2,k} \mapsto Y_{i-2,k} \cdot Y_{i-1,k} \cdot Y^{-1}_{i-1,a_{i-2}-i+3}$ for $1 \leq k \leq a_{i-2} - i + 2$
(2) $Y_{i-1,k} \mapsto Y^{-1}_{i-1 \cdot k}$ for $1 \leq k \leq a_i - i$.
(3) $Y_{i,k} \mapsto Y_{i,k} \cdot Y_{i-1,k}$ for $1 \leq k \leq a_i - i$.
(4) $Y_{j,k}$ is fixed for $1 \leq k \leq a_j - j$ for each $j \neq i - 2, i - 1, i$.

Case II. $a_i \geq a_{i-1} + 2$ for $2 \leq i \leq r - 1$, or $i = 1$. In this case s_{a_i} changes only the i-th row and the $(a_i - i + 1)$-th column of the array of roots $R(w^{-1})$. The resulting i-th row turns out to be

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{a_i - 1},\ \alpha_2 + \cdots + \alpha_{a_i - 1},\ \dots, \alpha_{a_1} + \cdots + \alpha_{a_i - 1},$$
$$\alpha_{a_1 + 2} + \cdots + \alpha_{a_i - 1}, \dots, \alpha_{a_i - 1},\ -\alpha_{a_i}$$

and the transpose of the $(a_i - i + 1)$-th column turns out to be

$$-\alpha_{a_i},\ \alpha_{a_i+1} + \cdots + \alpha_{a_{i+1}},\ \alpha_{a_i+1} + \cdots + \alpha_{a_{i+2}},\ \ldots, \alpha_{a_i+1} + \cdots + \alpha_{a_r}.$$

Let $\beta_{j,k}$ be any root which is fixed under the action of s_{a_i} and let $\beta_{p,q}$ be any root of the i-th row or the $(a_i - i + 1)$-th column, i.e. either $p = i$ or $q = a_i - i + 1$. We claim that $u_{\beta_{i,j}}(X_{i,j})$ and $u_{s_{a_i}\beta_{p,q}}(X_{p,q})$ commute. This follows from the fact that $\beta_{j,k} - \alpha_{a_i} \notin R(w^{-1})$ and the observation that for any root $\beta \in R(w^{-1})$ and $1 \leq m \leq r$, $\beta - \alpha_{a_m+1} \notin R^+$. Let us denote by M the sub-array consisting of $\beta_{k,l}$ where $k \geq i$ and $1 \leq l \leq a_i - i + 1$. Then, by the above discussion, s_{a_i} acts trivially on the roots in M.

We also have $s_{a_i} \cdot u_{\beta_{k,l}}(X_{k,l}) = u_{s_{a_i}(\beta_{k,l})}(X_{k,l})$.

Thus, the action of s_{a_i}, in this case is as follows:

$$X_{i,a_i-i+1} \mapsto X^{-1}_{i,a_i-i+1};\quad X_{i,k} \mapsto X_{i,k} \cdot X^{-1}_{i,a_i-i+1} \text{ for } k \leq a_i - i,$$

$$X_{j,k} \mapsto X_{j,k} - \frac{X_{j,a_i-i+1} \cdot X_{i,k}}{X_{i,a_i-i+1}} \text{ for } i + 1 \leq j \leq r \text{ and } 1 \leq k \leq a_i - i,$$

$$X_{j,a_i-i+1} \mapsto -X_{j,a_i-i+1}/X_{i,a_i-i+1} \text{ for } i + 1 \leq j \leq r.$$

From this the resulting action on $Y_{j,k}$ turns out to be as follows:

(1) s_{a_i} fixes $Y_{j,k}$'s provided $j \leq i - 1$ or $k \geq a_i - i + 1$. We now make the convention that $Y_{j,k} := 1$ if $k \geq a_j - j + 1$ or if $j \geq r$.

(2) Here, for $k \leq a_i - i$,

$$Y_{i,k} = \frac{X_{i,a_i-i+1} \cdot X_{i+1,k}}{X_{i+1,a_i-i+1} \cdot X_{i,k}},$$

$$\therefore\quad s_{a_i}(Y_{i,k}) = \frac{X^{-1}_{i,a_i-i+1} \cdot \left(X_{i+1,k} - \frac{X_{i+1,a_i-i+1} \cdot X_{i,k}}{X_{i,a_i-i+1}}\right)}{X_{i,k} \cdot X^{-1}_{i,a_i-i+1} \cdot (-X_{i+1,a_i-i+1}/X_{i,a_i-i+1})}$$

$$= 1 - Y_{i,k}.$$

(3) For $i+1 \leq j \leq r-1$ and $1 \leq k \leq a_i - i$, define $Y'_{j,k} = (X_{i,a_i-i+1} \cdot X_{j,k})/(X_{j,a_i-i+1} \cdot X_{i,k})$. Then, we have $s_{a_i}(Y_{j,k}) = 1 - Y_{j,k}$. It follows that $Y_{j,k} = Y'_{j+1,k} \cdot {Y'_{j,k}}^{-1} \cdot Y_{j,a_i-i+1}$. Hence, $s_{a_i}(Y_{j,k}) = \frac{1-Y'_{j+1,k}}{1-Y'_{j,k}} \cdot Y_{j,a_i-i+1}$. Now,

$$Y'_{j,k} = \prod_{m=i}^{j-1} \frac{X_{m,a_i-i+1} \cdot X_{m+1,k}}{X_{m+1,a_i-i+1} \cdot X_{m,k}}$$

$$= \prod_{m=i}^{j-1} \left\{ \left(\frac{X_{m,a_m-m+1} \cdot X_{m+1,k}}{X_{m+1,a_m-m+1} \cdot X_{m,k}} \right) \times \left(\frac{X_{m,a_m-m+1} \cdot X_{m+1,a_i-i+1}}{X_{m+1,a_m-m+1} \cdot X_{m,a_i-i+1}} \right)^{-1} \right\}$$

$$= \prod_{m=i}^{j-1} (Y_{m,k}/Y_{m,a_i-i+1}).$$

Thus we have,

$$s_{a_i}(Y_{j,k}) = \frac{1 - \prod_{m=i}^{j}(Y_{m,k}/Y_{m,a_i-i+1})}{1 - \prod_{m=i}^{j-1}(Y_{m,k}/Y_{m,a_i-i+1})} \times Y_{j,a_i-i+1}.$$

Case III. Action of s_{a_r}:

(1) If $a_r = a_{r-1} + 1$, then s_{a_r} interchanges $X_{r-1,k}$ and $X_{r,k}$, $1 \leq k \leq a_r - r + 1$. A straightforward checking proves as in Case I above, that in this case the action of s_{a_r} is as follows:

$$Y_{r-2,k} \mapsto Y_{r-2,k} \cdot Y_{r-1,k} \cdot Y^{-1}_{r-1,a_{r-2}-r+3}, \qquad \text{for } 1 \leq k \leq a_{r-2} - r + 2$$

$$Y_{r-1,k} \mapsto Y^{-1}_{r-1\cdot k}, \quad \text{for } 1 \leq k \leq a_r - r.$$

(2) If $a_r \geq a_{r-1} + 2$, s_{a_r} changes only $X_{r,k}$'s for $1 \leq k \leq a_r - r + 1$, as follows:

$$X_{r,k} \mapsto X_{r,k} \cdot X^{-1}_{r,a_r-r+1} \quad \text{for } 1 \leq k \leq a_r - r$$

$$X_{r,a_r-r+1} \mapsto X^{-1}_{r,a_r-r+1}.$$

It can be easily checked from here that the $Y_{i,j}$'s are all fixed by s_{a_r}. □

4. A stratification of $N\backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$

In this section, we give a stratification of $N\backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$.

Lemma 4.1. *Let* $w \in W^{I_2}$. *Let* $x \in U_w w P_2^{ss}$ *be such that* x *is not in the* W*-translate of* X_τ, $\tau < w$. *If* $\sigma(x) \in U_w w P_2$, *then* $\sigma \in$ *Stabiliser of* $X(w)$ *in* W.

Proof. If the lemma does not hold, then, there exists a $\sigma \in W$ such that $\sigma x \in U_w w P_2$ with σ not in the stabilizer of $X(w)$. Since $X(w)^{ss}$ is nonempty, by Lemma 2.7, we can write $w = (s_m \ldots s_1)(s_{n-1} \ldots s_2)$ with $m \geq \lceil \frac{n-1}{2} \rceil$. Since σ is not in the stabiliser of $X(w)$, $\exists$ a positive integer $t \in \mathbb{N}$ and a $\sigma_1 \in W$ such that $\sigma = \sigma_1 s_{m+t} s_{m+t-1} \ldots s_{m+1}$ with $l(\sigma) = l(\sigma_1) + t$.

We proceed with the proof by considering two cases.

Case 1. $t = 1$. As $l(\sigma) = l(\sigma_1) + t$, $\sigma_1(\alpha_{m+1}) > 0$. Also, as $\sigma x \in U_w w P_2$, σ_1 must be of the form $\sigma_1 = \phi s_{m+1} s_m$ with $l(\sigma_1) = l(\phi) + 2$. As σ is minimal for this choice of t, $\phi = id$.

Hence

$$\begin{aligned} \sigma x &= s_{m+1} s_m s_{m+1} x. \\ &= s_{m+1} u' s_{m+1} w P_2 \end{aligned}$$

with $u'_{\alpha_{m+1}} \neq id$. Thus, $\sigma x \in U s_{m+1} w P_2$, a contradiction.

Case 2. $t \geq 2$. Now, $s_{m+t} \ldots s_{m+1} x \in U(s_{m+t} \ldots s_{m+1} w P_2)$. Hence, σ_1 must be of the form

$$\sigma_1 \neq \phi s_{m+t} s_{m+t-1}, \text{ with } l(\phi) + 2 = l(\sigma_1).$$

Thus,

$$\begin{aligned}\sigma &= \sigma_1(s_{m+t}\dots s_{m+1})\\ &= \phi(\underbrace{s_{m+t}s_{m+t-1}s_{m+t}s_{m+t-1}}s_{m+t-2}\dots s_{m+1})\\ &= \phi(s_{m+t-1}s_{m+t}s_{m+t-2}\dots s_{m+1}).\end{aligned}$$

Hence, we have $l(\sigma)\leq l(\phi)+t=l(\sigma_1)+t-2<l(\sigma)$, which is a contradiction. Hence the lemma is proved. □

The longest element of W^{I_2} is

$$w^{I_2}=(s_{n-2}\cdot s_{n-3}\dots s_1)\cdot(s_{n-1}\cdot s_{n-2}\dots s_2)$$

and the unique minimal element τ_2 of W^{I_2} such that $\tau_2(n\omega_2)\leq 0$ is

$$\tau_2=(s_{\lceil\frac{n-1}{2}\rceil}\cdot s_{\lceil\frac{n-1}{2}\rceil-1}\dots s_1)\cdot(s_{n-1}\cdot s_{n-2}\dots s_2).$$

Therefore any element $w\in W^{I_2}$ such that $X(w)^{ss}_T(\mathcal{L}_2)\neq\emptyset$ is of the form

$$w=(s_m\cdot s_{m-1}\dots s_{\lceil\frac{n-1}{2}\rceil}\cdot s_{\lceil\frac{n-1}{2}\rceil-1}\dots s_1)\cdot(s_{n-1}\cdot s_{n-2}\dots s_2)$$

with $m\geq\lceil\frac{n-1}{2}\rceil$.

PROPOSITION 4.2

Let $r=2$, $w=(s_m\dots s_1)(s_{n-1}\dots s_2)$, $\lceil\frac{n-1}{2}\rceil\leq m\leq n-2$. We can arrange the Y_{ij}'s as $Y_1,Y_2,\dots,Y_{m-1}$ with

$$\begin{aligned}s_i(Y_i)&=Y_{i+1},\\ s_i(Y_j)&=Y_j\ \ if\ \ j\neq i,i+1\ \ and\ \ i=1,2,\dots,m-2,\\ s_{m-1}(Y_i)&=Y_i\cdot Y_{m-1}^{-1},\ \ if\ \ i\leq m-2,\\ s_{m-1}(Y_{m-1})&=Y_{m-1}^{-1},\\ s_m(Y_i)&=1-Y_i\ \ for\ \ i=1,2,\dots,m-1.\end{aligned}$$

Further, we have

$$s_i(Y_j)=Y_j\quad\forall i=m+2,\dots,n-1,\ \ when\ m\leq n-3$$

and

$$s_{n-1}(Y_j)=Y_j^{-1}\quad\forall j\ when\ m=n-2.$$

Proof. Proof follows from Proposition 3.2. □

Let w be as in Proposition 4.2. Now, let T_{m-1} be a maximal torus of $\mathbb{P}GL_m$, R_m is the root system of $\mathbb{P}GL_m$. Here, the Weyl group is S_m, the symmetric group on m symbols.

Let $U=\{t\in T\colon e^{\alpha}(t)\neq 1,\ \ \alpha\in R_m\}$. Clearly, U is S_m-stable. On the other hand, S_m stabilises $(U_w wP_2/P_2)^{ss}_T(\mathcal{L}_2)$. Let $Y(w)=T\backslash\backslash(U_w wP_2)^{ss}_T(\mathcal{L}_2)$. Then, we have

COROLLARY 4.3

There is a S_m-equivariant isomorphism $\Psi_1\colon Y(w) \xrightarrow{\sim} U$ such that

$$\Psi_1^*(e^{\alpha_i+\cdots+\alpha_{m-1}}) = Y_i, \;\; 1 \le i \le m-1.$$

Proof. Proof follows from Proposition 4.2. □

Let $\mathfrak{h}_m$ be a Cartan subalgebra of $\mathfrak{sl}_{m+1}$, $\mathbb{P}(\mathfrak{h}_m)$ be the projective space and $R_m \subseteq \mathfrak{h}_m^*$ be the root system. Let V_m be the open subset of $\mathbb{P}(\mathfrak{h}_m)$ defined by

$$V_m := \{x \in \mathbb{P}(\mathfrak{h}_m)\colon \alpha(x) \neq 0, \quad \forall \alpha \in R_m\}.$$

Clearly V_m is S_{m+1}-stable. For a recent study of quotients of flag varieties modulo a maximal torus, see [3].

COROLLARY 4.4

Let $w = (s_m \dots s_1)(s_{n-1} \dots s_2)$, $\lceil \frac{n-1}{2} \rceil \le m \le n-2$. Then, there is a S_{m+1}-equivariant isomorphism $\Psi_2\colon Y(w) \xrightarrow{\sim} V_m$ of affine varieties.

Proof. For $i = 1, 2, \dots, m-1$, take $Z_i = \frac{\alpha_i+\cdots+\alpha_m}{\alpha_m}$ and define Ψ_2 such that $\Psi_2^*(Z_i) = Y_i$. □

With notations as above and taking $t = \left[\frac{n-3}{2}\right]$ and $m = \left\lceil \frac{n-1}{2} \right\rceil$ we have the following.

Theorem. *$N\backslash\backslash G^{ss}_{2,n}(\mathcal{L}_2)$ has a stratification $\bigcup_{i=0}^{t} C_i$ where $C_0 = {}_{S_{m+1}}\backslash\mathbb{P}(\mathfrak{h}_m)$, and $C_i = {}_{S_{i+m+1}}\backslash V_{i+m}$.*

Proof. Proof follows from Lemma 4.1, Proposition 4.2 and Corollary 4.4. □

5. Flag variety as a GIT quotient of flag variety of higher dimension

Let $G = GL_{n+1}(k)$. Let T be a maximal torus of $SL_{n+1}(k)$. Let B_{n+1} be a Borel subgroup of G containing T. Let $S = \{\alpha_i\colon i = 1, 2, \dots, n\}$ denote the set of simple roots with respect to B_{n+1} and let $W = S_{n+1}$ be the Weyl group. Let s_i be the simple reflection corresponding to the simple root α_i. Let $I := S\backslash\{\alpha_n\}$, and let W_I be the subgroup of W generated by $\{s_i\colon i \in I\}$ and $w_{0,I}$ denote the longest element of W_I.

Lemma 5.1. *Let $\chi = \sum_{i=1}^{n} m_i\alpha_i$ be a regular dominant character, where $m_i \in \mathbb{N}$, $m_{i+1} > m_i$ for $1 \le i \le n-1$. Let $w \in W$. Then $w(\chi) \le 0 \Leftrightarrow w = s_1 \cdot s_2 \dots s_n \cdot \tau$ for some $\tau \in W_I$.*

Proof.
$\Rightarrow$: Since χ is dominant and $\tau \le w_{0,I}$, for all $\tau \in W_I$, we have $\tau(\chi) \ge w_{0,I}(\chi)$; using the fact that $w_{0,I}(\alpha_i) = -\alpha_{n-i}$ for $i = 1, \dots, n-1$ and $w_{0,I}(\alpha_n) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ we have $w_{0,I}(\chi) = \sum_{i=1}^{n-1}(m_n - m_{n-i})\alpha_i + m_n \cdot \alpha_n$. Therefore, $\tau(\chi) = \sum_{i=1}^{n-1} a_i\alpha_i + m_n\alpha_n$, $a_i > 0$. Now, let $w = \phi\tau$ with $\phi \in W^I$, $\tau \in W_I$. Therefore, $w(\chi) = \phi(\tau(\chi)) = \phi(\sum_{i=1}^{n-1} a_i\alpha_i + m_n\alpha_n)$. Thus $w(\chi) \le 0$ implies that $\phi = s_1 \cdot s_2 \dots s_n$.

$\Leftarrow$: Let $w = s_1 \cdot s_2 \dots s_n \cdot \tau$, $\tau \in W_I$. Now,

$$\begin{aligned} w(\chi) &= s_1 \cdot s_2 \dots s_n \tau(\chi) \\ &= s_1 \cdot s_2 \dots s_n \left(\sum_{i=1}^{n-1} a_i \alpha_i + m_n \alpha_n \right) \\ &= -m_n \alpha_1 + \sum_{i=2}^{n} (a_{i-1} - m_n) \alpha_i . \end{aligned}$$

Since χ is a dominant weight we have $\chi - \tau(\chi) \geq 0$. Hence we have $a_i \leq m_i \leq m_n$. Thus $w(\chi) \leq 0$. This completes the proof. □

Consider $GL_n(k)$ as a subgroup of $GL_{n+1}(k)$ given by the inclusion $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Let $B_n = B_{n+1} \bigcap GL_n(k)$ as a Borel subgroup with I as the simple roots.

Let χ be a regular dominant character as in Lemma 5.1.

Theorem 5.2. *We have an isomorphism*

$$\Psi\colon T\backslash\backslash(GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) \xrightarrow{\sim} GL_n(k)/B_n .$$

Proof. Proof uses cellular decomposition of both homogeneous spaces $GL_{n+1}(k)/B_{n+1}$ and $GL_n(k)/B_n$. First, we fix a total order on the set of positive roots of B_{n+1} such that $\sum_{i=1}^{n} \alpha_i > \sum_{i=1}^{n-1} \alpha_i > \cdots > \alpha_1 > \sum_{i=2}^{n} \alpha_i > \cdots > \alpha_2 > \sum_{i=3}^{n} \alpha_i > \cdots > \alpha_3 > \cdots > \alpha_{n-1} + \alpha_n > \alpha_n$. Now any GL_{n+1}/B_{n+1} (resp. GL_n/B_n) is the union of cells $U_w w B_{n+1}$ (resp. $U_\tau \tau B_n$) with $w \in W$ (resp. $\tau \in W_I$). Using the total order above we can write each element $x \in U_w$ as a product of u_α in the decreasing order from left to right. Let X_α (resp. Y_β) be the co-ordinate function on $U_w w B_{n+1}$ (resp. $U_\tau \tau B_n$) corresponding to the root α (resp. β).

With these notations we proceed with the proof.

Let $\tau \in W_I$. Let $w := s_1 s_2 \dots s_n \tau$ and let $V_\tau^0 := \{x \in U_w w B_{n+1} \colon X_\alpha(x) \neq 0 \forall \alpha \geq \alpha_1\}$. Set $V^0 := \bigcup_{\tau \in W_I} V_\tau^0$.

Step 1. We prove that $(GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) \subset V^0$.

This can be seen as follows:

By Hilbert–Mumford criterion (see Theorem 2.1 of [5]), a point $x \in GL_{n+1}/B_{n+1}$ is semi-stable $\Leftrightarrow \mu^L(x, \lambda) \geq 0$ for all 1-parameter subgroups λ of T $\Leftrightarrow \mu^L(\sigma x, \lambda) \geq 0$ for all one-parameter subgroups $\lambda \in \overline{C}$ and for all $\sigma \in W$. By Lemma 2.1 of [8], this statement is equivalent to $\langle -w_\sigma \chi, \lambda \rangle \geq 0$ for all $\lambda \in \overline{C}$ where $\sigma x \in U_{w_\sigma} w_\sigma B$. But this is equivalent to $w_\sigma(\chi) \leq 0$. By Lemma 5.1, this is equivalent to w_σ being of the form $(s_1 \dots s_n) \cdot \tau_1$ for some $\tau_1 \in W_I$.

Now let $x \in U_w w B_{n+1}$ with $w = (s_1 \dots s_n)\tau$, $\tau \in W_I$.

Now, let $X_\alpha(x) = 0$ for some $\alpha \geq \alpha_1$. Let $\alpha = \sum_{j=1,\dots,i} \alpha_j$. Then, we have $s_1 s_2 \dots s_i x = u' \phi B_{n+1}$ with $\phi \neq s_1 \dots s_n \tau$ for any $\tau \in W_I$. Hence, by the above discussion, x is not semi-stable.

Step 2. $(GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) = V^0$. This can be seen by the above discussion and from the following Claim.

Claim. V_0 is W-stable.

Proof of Claim. Let $\tau \in W_I$, $\tau' = s_1 s_2 \dots s_n \tau$ and $x \in U_{\tau'}\tau' B_{n+1}$, with $X_\alpha(x) \neq 0$ for all $\alpha \geq \alpha_1$. Then, we have $s_1 x \in U_{\tau'}\tau' B_{n+1}$ with $X_\alpha(s_1 x) = -\frac{X_\alpha(x)}{X_{\alpha_1}(x)}$ for $\alpha > \alpha_1$, and $X_{\alpha_1}(s_1 x) = \frac{1}{X_{\alpha_1}(x)}$. Hence, $s_1 x \in V^0$.

Now, let $i \neq 1$. If $X_{\alpha_i}(x) = 0$, then, $s_i x = u' s_1 s_2 \dots s_n s_{i-1} \tau B_{n+1}$ with $X_\alpha(s_i x) = X_{s_i(\alpha)}(x)$. Hence, $s_i(x) \in V^0$. Otherwise, we must have $s_i x \in U_{\tau'}\tau' B_{n+1}$ with $X_\alpha(s_i x) = X_\alpha(x)$ for all α such that $s_i(\alpha) = \alpha$, $X_\alpha(s_i x) = \frac{X_\alpha(x)}{X_{\alpha_i}(x)}$ for all α of the form $\alpha = \sum_{j=k}^{i} \alpha_j$ such that $k < i$, $X_{\alpha_i}(s_i x) = \frac{1}{X_{\alpha_i}(x)}$, and $X_\alpha(s_i x) = \frac{-X_\alpha(x)}{X_{\alpha_i}(x)}$ for all α of the form $\alpha = \sum_{j=i}^{k} \alpha_j$ such that $k > i$.

Hence $s_i V^0 \subset V^0$ for all $i = 1, \dots, n$. Thus, the Claim follows from the fact that W is generated by s_i's.

Step 3. Now, for each $\tau \in W_I$, we exhibit an isomorphism

$$\Psi_\tau \colon T\backslash\backslash V_\tau^0 \xrightarrow{\sim} U_\tau \tau B_n / B_n.$$

Let $\tau \in W_I$, consider the map $\pi_\tau \colon V_\tau^0 \longrightarrow (U_\tau \tau B_n)/B_n$ defined by $\pi_\tau(x) = y$ with $Y_{s_n \dots s_1(\beta)}(y) = \left(\frac{-X_\beta(x) X_{\beta'}(x)}{X_{\beta+\beta'}(x)}\right)$ where for each $\beta \in R(w^{-1})$ with $\beta \not\geq \alpha_1$, β' is the unique element of R^+ with $\beta' \geq \alpha_1$ such that $\beta + \beta' \in R^+$. Clearly this map is T-invariant. Thus the morphism π_τ gives rise to a morphism

$$\Psi_\tau \colon T\backslash\backslash V_\tau^0 \longrightarrow U_\tau \tau B_n / B_n.$$

Clearly Ψ_τ is surjective. We now prove that Ψ_τ is injective:

π_w is injective for each $w \in W$ of the form $w = s_1 \cdot s_2 \dots s_n \tau$, for some $\tau \in W_I$. Let x_1 and x_2 be two points of V_τ^0 such that $\pi_\tau(x_1) = \pi_\tau(x_2)$. Hence, $\frac{X_\beta(x_1) X_{\beta'}(x_1)}{X_{\beta+\beta'}(x_1)} = \frac{X_\beta(x_2) X_{\beta'}(x_2)}{X_{\beta+\beta'}(x_2)}$. Let $t \in T$ be such that $(\alpha_1 + \dots + \alpha_i)(t) = \frac{X_{\alpha_1+\dots+\alpha_i}(x_2)}{X_{\alpha_1+\dots+\alpha_i}(x_1)}$ for all i, $1 \leq i \leq n$. Then, it is easy to check that $t \cdot x_1 = x_2$. Thus Ψ_τ is bijective for each $\tau \in W_I$.

Step 4. Ψ_τ puts together to give an isomorphism

$$\Psi \colon T\backslash\backslash V^0 \xrightarrow{\sim} GL_n(k)/B_n.$$

Since the W-translates of $V_{w_{0,I}}^0$ is the whole of V^0, and W_I-translates of $U_{w_{0,I}} w_{0,I} B_n$ is the whole of GL_n/B_n, and there is an isomorphism from $W_{S\backslash\{\alpha_1\}}$ to W_I taking s_i to s_{i-1} for each $i = 2, \dots, n$, to prove the theorem, it is sufficient to prove that the T-invariant morphisms $\pi_\tau \colon V_\tau^0 \longrightarrow U_\tau \tau B_n / B_n$, and $\pi_{(s_{i-1}\tau)^{\tilde{}}} \colon V^0_{(s_{i-1}\tau)^{\tilde{}}} \longrightarrow U_{(s_{i-1}\tau)^{\tilde{}}} (s_{i-1}\tau)^{\tilde{}} B_n / B_n$ satisfy the following:

$$Y_\alpha(\pi_\tau(x)) = Y_\alpha(s_{i-1}(\pi_{(s_{i-1}\tau)^{\tilde{}}}(s_i x))) \text{ for each } \alpha \in R(\tau^{-1}).$$

(Here, the notation $(s_{i-1}\tau)^{\tilde{}} = \tau$ if $s_{i-1}\tau < \tau$ and $(s_{i-1}\tau)^{\tilde{}} = s_{i-1}\tau$ otherwise.)

We make use of the following observations using commutator relations:

$$X_\alpha(s_i(x)) = \begin{cases} \frac{-X_\alpha(x)}{X_{\alpha_i}(x)} & \text{if } \alpha = \alpha_i + \cdots + \alpha_k,\ i < k \text{ and } w^{-1}(\alpha_{i+1} + \cdots + \alpha_k) > 0, \\ \frac{1}{X_{\alpha_i}(x)} & \text{if } \alpha = \alpha_i, \\ X_{s_i(\alpha)}(x) & \text{otherwise.} \end{cases}$$

Let $\alpha \in R(\tau^{-1})$.

Case 1. $\alpha = \alpha_{k-1} + \cdots + \alpha_{i-1}, k < i, w^{-1}(\alpha_k + \cdots + \alpha_i) = \tau^{-1}(\alpha_{k-1} + \cdots + \alpha_{i-1}) > 0$ and $(s_{i-1}\tau)\tilde{} = \tau$.

In this case,

$$Y_\alpha(s_{i-1}(\pi_\tau(x))) = \frac{X_{\alpha_1+\cdots+\alpha_{k-1}}(x) X_{\alpha_k+\cdots+\alpha_i}(x)}{X_{\alpha_1+\cdots+\alpha_{i-1}}(x)} = Y_\alpha(\pi_\tau(s_i x)).$$

Case 2. $\alpha = \alpha_{i-1} + \cdots + \alpha_{k-1}, i < k$ and $w^{-1}(\alpha_i + \cdots + \alpha_k) = \tau^{-1}(\alpha_{i-1} + \cdots + \alpha_{k-1}) > 0$ and $(s_{i-1}\tau)\tilde{} = \tau$.

In this case,

$$Y_\alpha(s_{i-1}(\pi_\tau(x))) = -\frac{X_{\alpha_1+\cdots+\alpha_i}(x) X_{\alpha_i+\cdots+\alpha_k}(x)}{X_{\alpha_i}(x) X_{\alpha_1+\cdots+\alpha_k}(x)} = Y_\alpha(\pi_\tau(s_i x)).$$

Case 3. $\alpha = \alpha_{i-1}$.

$$Y_\alpha(s_{i-1}(\pi_\tau(x))) = \frac{X_{\alpha_1+\cdots+\alpha_i}(x)}{X_{\alpha_1+\cdots+\alpha_{i-1}}(x) X_{\alpha_i}(x)} = Y_\alpha(\pi_\tau(s_{i-1}(x))).$$

In all other cases, we have

$$Y_\alpha(s_{i-1}(\pi_{(s_{i-1}\tau)\tilde{}}(s_i x))) = \frac{X_{s_i s_1 \ldots s_n(\alpha)}(x) X_{s_i(\beta')}(x)}{X_{s_i(s_1 \ldots s_n(\alpha)+\beta')}(x)} = Y_\alpha(\pi_\tau(x)),$$

where β' is the unique root such that $\beta' \geq \alpha_1$ and $s_1 \ldots s_n(\alpha) + \beta'$ is a root.

This completes the proof. □

With Y_α's as in the proof of Theorem 5.2, we have the following.

COROLLARY 5.3

$$s_1(Y_\alpha) = \begin{cases} -(1 + Y_\alpha) & \textit{if } \alpha \geq \alpha_1 \\ Y_\alpha & \textit{otherwise}. \end{cases}$$

Proof. Proof follows from the fact that

$$X_\alpha(s_1 x) = \begin{cases} X_{\alpha_1} X_\alpha(x) + X_{\alpha_1+\alpha}(x) & \text{if } \alpha = \alpha_2 + \cdots + \alpha_i,\ i \geq 2, \\ \frac{-X_\alpha(x)}{X_{\alpha_1}(x)} & \text{if } \alpha = \alpha_1 + \cdots + \alpha_i,\ i \geq 2, \\ X_\alpha(x) & \text{if } \alpha = \alpha_3 + \cdots + \alpha_i,\ i \geq 3. \end{cases}$$

□

COROLLARY 5.4

Let $\mathfrak{h}_n$ be a Cartan subalgebra of $sl_{n+1}(k)$. Let χ be a regular dominant character as in Theorem 5.2. Then, the action of W on the GIT quotient

$$T\backslash\backslash(GL_{n+1}(k)/B_{n+1})^{ss}(L_\chi) \simeq GL_n(k)/B_n$$

is given by the n-dimensional representation $\mathfrak{h}_n$ of W.

Proof. Proof follows from Theorem 5.2 and Corollary 5.3. □

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