Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 1, February 2009, pp. 71–80.

On Kähler–Norden manifolds

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MS received 7 September 2007; revised 4 October 2007

Abstract. This paper is concerned with the problem of the geometry of Norden manifolds. Some properties of Riemannian curvature tensors and curvature scalars of Kähler–Norden manifolds using the theory of Tachibana operators is presented.

Keywords. Kähler–Norden manifold; Norden metric; twin metric; pure tensor; holomorphic tensor.

1. Introduction

Let M_{2n} be a Riemannian manifold with a neutral (Kleinian) metric, i.e. with a pseudo-Riemannian metric g of signature (n, n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

We say (M_{2n}, φ) is an almost complex manifold if M_{2n} can be endowed with an affinor field $\varphi \in \Im_1^1(M_{2n})$ such that $\varphi^2 = -I$, where I is a field of identity endomorphisms. If M_{2n} is a manifold of class C^ω and the Nijenhuis tensor field $N_\varphi \in \Im_2^1(M_{2n})$ vanishes, then φ is a complex structure and moreover M_{2n} is a C-holomorphic manifold $X_n(C)$ whose transition functions are holomorphic mappings. $N_\varphi = 0$ is equivalent to the condition $\nabla\varphi = 0$, where ∇ is a torsion-free affine connection. Let (M_{2n}, φ) be an almost complex manifold. A metric g is a Norden metric [9] if

$$g(\varphi X, \varphi Y) = -g(X, Y) \tag{1.1}$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2n})$. Metrics of this kind have also been studied under the names: anti-Hermitian, pure and B-metrics (see [18, 7, 11, 6, 1–5, 8, 15, 10]). If (M_{2n}, φ) is an almost complex manifold with Norden metric g, we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold.

Let t^* be a complex tensor field on $X_n(C)$. The real model of such a tensor field is a tensor field on M_{2n} of the same order that is independent of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [7, 19, 12–14, 20]).

In particular, being applied to a $(0, q)$-tensor field ω, the purity means that for any $X_1, \ldots, X_q \in \Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \ldots = \omega(X_1, X_2, \ldots, \varphi X_q).$$

We define an operator

$$\phi_\varphi \colon \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to the pure tensor field ω by [16, 21]

$$\begin{aligned}(\phi_\varphi \omega)&(X, Y_1, Y_2, \ldots, Y_q)\\ &= (\varphi X)(\omega(Y_1, Y_2, \ldots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \ldots, Y_q))\\ &\quad + \omega((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q) + \cdots + \omega(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X),\end{aligned} \tag{1.2}$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a complex structure on M_{2n} and the tensor field $\phi_\varphi \omega$ vanishes, the complex tensor field ω^* on $X_n(C)$ is said to be holomorphic [7]. Thus a holomorphic tensor field ω^* on $X_n(C)$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = 0 \tag{1.3}$$

for any $X, Y_1, \ldots, Y_q \in \Im_0^1(M_{2n})$. Therefore such a tensor field ω on M_{2n} is also called holomorphic tensor field.

The main results of the present paper are as follows: Almost complex manifolds endowed with an almost holomorphic Riemannian metric g are the manifolds where the Levi-Civita connection of the metric parallelizes the almost complex structure (Theorem 2), i.e. the Kähler–Norden manifolds. In that case, the Levi-Civita connection is also the Levi-Civita connection of the twin metric G given by $G(X, Y) = g(\varphi X, Y)$ (Theorem 5). Moreover, in such a manifold, the Riemannian curvature tensor is pure and holomorphic (Theorems 6, 7), also the curvature scalar is locally holomorphic function (Theorem 8).

2. Kähler–Norden manifolds

In a Norden (an almost Norden) manifold a Norden metric g is called a *holomorphic* (*an almost holomorphic*) if

$$(\phi_\varphi g)(X, Y, Z) = 0.$$

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric g, we say that (M_{2n}, φ, g) is a *holomorphic Norden manifold*.

Now we establish a formula for the Norden metric for an almost Norden manifold. As a direct consequence of (1.1) and $\nabla g = 0$, we obtain

Theorem 1. *Let g be a Norden metric of almost Norden manifold. Then*

$$g(Z, (\nabla_Y \varphi)(X)) = g((\nabla_Y \varphi)(Z), X),$$

where ∇ denotes the operator of the Riemannian covariant derivative with respect to g.

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 2. *An almost Norden manifold of class C^ω is holomorphic Norden manifold if and only if the almost complex structure is parallel with respect to the Levi-Civita connection* ∇.

Proof. Putting $(g \circ \varphi)(X, Y) = g(\varphi X, Y)$, we get from (1.2)

$$\begin{aligned}(\phi_\varphi g)(X, Z_1, Z_2)\\ &= (L_{\varphi X} g - L_X(g \circ \varphi))(Z_1, Z_2) + g(Z_1, \varphi L_X Z_2) - g(\varphi Z_1, L_X Z_2)\\ &= (\varphi X)g(Z_1, Z_2) - Xg(\varphi Z_1, Z_2) - g(\nabla_{\varphi X} Z_1, Z_2) + g(\nabla_{Z_1} \varphi X, Z_2)\\ &\quad - g(Z_1, \nabla_{\varphi X} Z_2) + g(Z_1, \nabla_{Z_2} \varphi X) + g(\varphi(\nabla_X Z_1), Z_2)\\ &\quad - g(\varphi(\nabla_{Z_1} X), Z_2) + g(\varphi Z_1, \nabla_X Z_2) - g(Z_1, \varphi(\nabla_{Z_2} X)). \qquad (2.1)\end{aligned}$$

We find

$$\begin{aligned}&g(\nabla_{Z_1} \varphi X, Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(Z_1, \nabla_{Z_2} \varphi X) - g(Z_1, \varphi(\nabla_{Z_2} X))\\ &\quad = g((\nabla\varphi)(X, Z_1), Z_2) + g(Z_1, (\nabla\varphi)(X, Z_2)). \qquad (2.2)\end{aligned}$$

Substituting (2.2) into (2.1), (2.1) may be written as

$$\begin{aligned}(\phi_\varphi g)(X, Z_1, Z_2)\\ &= (\varphi X)g(Z_1, Z_2) - Xg(\varphi Z_1, Z_2) + g((\nabla\varphi)(X, Z_1), Z_2)\\ &\quad + g(Z_1, (\nabla\varphi)(X, Z_2)) - g(\nabla_{\varphi X} Z_1, Z_2) - g(Z_1, \nabla_{\varphi X} Z_2)\\ &\quad + g(\varphi(\nabla_X Z_1), Z_2) + g(\varphi Z_1, \nabla_X Z_2). \qquad (2.3)\end{aligned}$$

On the other hand, with respect to the Levi-Civita connection ∇, we have

$$(\varphi X)g(Z_1, Z_2) - g(\nabla_{\varphi X} Z_1, Z_2) - g(Z_1, \nabla_{\varphi X} Z_2) = (\nabla_{\varphi X} g)(Z_1, Z_2) = 0 \qquad (2.4)$$

and

$$-Xg(\varphi Z_1, Z_2) + g(\varphi(\nabla_X Z_1), Z_2) + g(\varphi Z_1, \nabla_X Z_2) = -g((\nabla_X \varphi) Z_1), Z_2). \qquad (2.5)$$

By virtue of (2.4) and (2.5), (2.3) reduces to

$$\begin{aligned}(\phi_\varphi g)(X, Z_1, Z_2)\\ &= -g((\nabla_X \varphi) Z_1, Z_2) + g((\nabla_{Z_1} \varphi) X, Z_2) + g(Z_1, (\nabla_{Z_2} \varphi) X). \qquad (2.6)\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&(\phi_\varphi g)(Z_2, Z_1, X) \\
&\quad = -g((\nabla_{Z_2}\varphi)Z_1, X) + g((\nabla_{Z_1}\varphi)Z_2, X) + g(Z_1, (\nabla_X\varphi)Z_2). \qquad (2.7)
\end{aligned}
$$

The sufficiency follows easily from (2.6) (or (2.7)).

By virtue of Theorem 1, we find

$$(\phi_\varphi g)(X, Z_1, Z_2) + (\phi_\varphi g)(Z_2, Z_1, X) = 2g(X, (\nabla_{Z_2}\varphi)Z_2). \qquad (2.8)$$

Now, putting $\phi_\varphi g = 0$ in (2.8), we find $\nabla\varphi = 0$ from which the necessity follows. Thus Theorem 2 is proved.

COROLLARY 1

The almost complex structure φ on almost Norden manifold of class C^ω is integrable iff $\phi_\varphi g = 0$.

Recall that a Kähler–Norden manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} of class C^ω endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian: $g(\varphi X, Y) = g(X, \varphi Y)$. Therefore, there exist a one-to-one correspondence between Kähler–Norden manifolds and complex Riemannian manifolds with a *holomorphic metric* as they were defined in [7] (see also [19, 11, 2, 3]). Let (M_{2n}, φ, g) be a Kähler–Norden manifold. Since in dimension 2 such a manifold is flat (p. 113 of [19]), we assume in the sequel that $\dim M \geq 4$, i.e. $n \geq 2$.

Remark. An almost Norden manifold of class C^∞ with Kähler–Norden metric is called a pseudo-Kähler–Norden manifold. In pseudo-Kähler–Norden manifold the metric g is almost holomorphic (see [11]).

3. Twin Norden metrics

Let (M_{2n}, φ, g) be an almost Norden manifold. The associated Norden metric of almost Norden manifold is defined by

$$G(X, Y) = (g \circ \varphi)(X, Y) \qquad (3.1)$$

for all vector fields X and Y on M_{2n}. One can easily prove that G is a metric, which is also called the twin (or dual) metric of g and it plays a role similar to the Kähler form in Hermitian geometry. We shall now apply the Tachibana operator to the pure Riemannian metric G:

$$
\begin{aligned}
&(\phi_\varphi G)(X, Y, Z) \\
&\quad = (L_{\varphi X}G - L_X(G \circ \varphi))(Y, Z) + G(Y, \varphi L_X Z) - G(\varphi Y, L_X Z) \\
&\quad = (\phi_\varphi g)(X, \varphi Y, Z) + g(N_\varphi(X, Y), Z). \qquad (3.2)
\end{aligned}
$$

Thus (3.2) implies the following:

Theorem 3. *In an almost Norden manifold, we have*

$$\phi_\varphi G = (\phi_\varphi g) \circ \varphi + g \circ (N_\varphi).$$

COROLLARY 2

In a Norden manifold the following conditions are equivalent:

(a) $\phi_\varphi g = 0$,
(b) $\phi_\varphi G = 0$.

From Theorems 2 *and* 3 *we have the following.*

Theorem 4. *Almost Norden manifold of class* C^ω *with conditions* $\phi_\varphi G = 0$ *and* $N_\varphi \neq 0$, (*i.e. analogues of the almost Kähler manifolds with closed Kähler form*) *do not exist.*

We denote by ∇_g the covariant differentiation of Levi-Civita connection of Norden metric g. Then, we have

$$\nabla_g G = (\nabla_g g) \circ \varphi + g \circ (\nabla_g \varphi) = g \circ (\nabla_g \varphi)$$

which implies $\nabla_g G = 0$ by virtue of Theorem 2. Therefore we have the following.

Theorem 5. *Let* (M_{2n}, φ, g) *be a Kähler–Norden manifold. Then the Levi-Civita connection of Norden metric* g *coincides with the Levi-Civita connection of twin Norden metric* G.

4. Curvature tensors in Kähler–Norden manifolds

Let R and S be the curvature tensors formed by g and G respectively. Then for the Kähler–Norden manifold we have $R = S$ by means of Theorem 5. Applying Ricci's identity to φ, we get

$$\varphi(R(X, Y)Z) = R(X, Y)\varphi Z, \tag{4.1}$$

by virtue of $\nabla\varphi = 0$. Hence $R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4)$ is pure with respect to X_3 and X_4 and also pure with respect to X_1 and X_2:

$$\begin{aligned} R(X_1, X_2, \varphi X_3, X_4) &= g(R(X_1, X_2)\varphi X_3, X_4) \\ &= g(\varphi(R(X_1, X_2)X_3), X_4) \\ &= g(R(X_1, X_2)X_3, \varphi X_4) \\ &= R(X_1, X_2, X_3, \varphi X_4). \end{aligned}$$

On the other hand, S being the curvature tensor formed by twin metric G, if we put $S(X_1, X_2, X_3, X_4) = G(S(X_1, X_2)X_3, X_4)$, then we have

$$S(X_1, X_2, X_3, X_4) = S(X_3, X_4, X_1, X_2) \tag{4.2}$$

Taking account of (1.2), (3.1), (4.1) and $R = S$, we find that

$$\begin{aligned} S(X_1, X_2, X_3, X_4) &= G(S(X_1, X_2)X_3, X_4) \\ &= g(\varphi(S(X_1, X_2)X_3), X_4) \\ &= g(S(X_1, X_2)X_3, \varphi X_4) \\ &= g(R(X_1, X_2)X_3, \varphi X_4) \\ &= R(X_1, X_2, X_3, \varphi X_4) \end{aligned}$$

and

$$\begin{aligned} S(X_3, X_4, X_1, X_2) &= G(S(X_3, X_4)X_1, X_2) \\ &= g(\varphi(S(X_3, X_4)X_1), X_2) \\ &= g(S(X_3, X_4)X_1, \varphi X_2) \\ &= g(R(X_3, X_4)X_1, \varphi X_2) \\ &= R(X_3, X_4, X_1, \varphi X_2) \\ &= R(X_1, \varphi X_2, X_3, X_4). \end{aligned}$$

Thus eq. (4.2) becomes

$$R(X_1, X_2, X_3, \varphi X_4) = R(X_1, \varphi X_2, X_3, X_4),$$

which shows that $R(X_1, X_2, X_3, X_4)$ is pure with respect to X_2 and X_4. Therefore $R(X_1, X_2, X_3, X_4)$ is pure.

Thus we get the following.

Theorem 6. *In a Kähler–Norden manifold the Riemannian curvature tensor of Norden metric is pure.*

If a torsion-free affine connection ∇ preserving the structure φ ($\nabla\varphi = 0$) satisfies the condition $\nabla_{\varphi X} Y = \varphi(\nabla_X Y)$ then ∇ is called a *holomorphic connection* (p. 185 of [19]) (see also [7, 1–3]). The purity of the curvature tensor field of a connection ∇ is a necessary and sufficient condition for its holomorphy [7, 19] (see also [11]). Therefore, from Theorem 6 we have as follows.

COROLLARY 3

In a Kähler–Norden manifold the Levi-Civita connection of Norden metric is holomorphic.

Since the Riemannian curvature tensor R is pure, we can apply the ϕ-operator to R. Using $\nabla\varphi = 0$ (see [12]),

$$\begin{aligned} &(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) \\ &\quad = (\nabla_{\varphi X} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3, Y_4). \end{aligned} \tag{4.3}$$

Using (4.1) and applying the Bianchi's 2nd identity to (4.3), we get

$$\begin{aligned}(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3), Y_4)\\ &= g((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4)\\ &= g(-(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) - (\nabla_{Y_2} R)(\varphi X, Y_1, Y_3)\\ &\quad - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4). \end{aligned} \tag{4.4}$$

On the other hand, using $\nabla\varphi = 0$, we find

$$\begin{aligned}(\nabla_{Y_2} R)(\varphi X, Y_1, Y_3) &= \nabla_{Y_2}(R(\varphi X, Y_1, Y_3)) - R(\nabla_{Y_2}(\varphi X), Y_1, Y_3)\\ &\quad - R(\varphi X, \nabla_{Y_2} Y_1, Y_3) - R(\varphi X, Y_1, \nabla_{Y_2} Y_3)\\ &= (\nabla_{Y_2}\varphi)(R(X, Y_1, Y_3)) + \varphi(\nabla_{Y_2} R(X, Y_1, Y_3))\\ &\quad - R((\nabla_{Y_2}\varphi)X + \varphi(\nabla_{Y_2} X), Y_1, Y_3)\\ &\quad - R(\varphi X, \nabla_{Y_2} Y_1, Y_3) - R(\varphi X, Y_1, \nabla_{Y_2} Y_3)\\ &= \varphi(\nabla_{Y_2} R(X, Y_1, Y_3)) - \varphi(R(\nabla_{Y_2} X, Y_1, Y_3))\\ &\quad - \varphi(R(X, \nabla_{Y_2} Y_1, Y_3)) - \varphi(R(X, Y_1, \nabla_{Y_2} Y_3))\\ &= \varphi((\nabla_{Y_2} R)(X, Y_1, Y_3)). \end{aligned} \tag{4.5}$$

Similarly

$$(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) = \varphi((\nabla_{Y_1} R)(Y_2, X, Y_3)) \tag{4.6}$$

Substituting (4.5) and (4.6) in (4.4) and using again the Bianchi's 2nd identity, we obtain

$$\begin{aligned}(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g(-\varphi((\nabla_{Y_1} R)(Y_2, X, Y_3)) - \varphi((\nabla_{Y_2} R)(X, Y_1, Y_3))\\ &\quad - \varphi((\nabla_X R)(Y_1, Y_2, Y_3)), Y_4)\\ &= -g(\varphi(\sigma\{(\nabla_X R)(Y_1, Y_2\}, Y_3)), Y_4)\\ &= 0, \end{aligned}$$

where σ denotes the cyclic sum with respect to X, Y_1 and Y_2. Therefore we have the following.

Theorem 7. *In a Kähler–Norden manifold, the Riemannian curvature tensor field is a holomorphic tensor field.*

5. Curvature scalars in Kähler–Norden manifolds

Let (M_{2n}, φ) be a complex manifold.

Lemma. A necessary and sufficient condition for an exact 1*-form* df, $f \in \Im_0^0(M_{2n})$ *to be a holomorphic, i.e.* $\phi_\varphi(df) = 0$, *is that an associated* 1*-form* $df \circ \varphi$ *be a closed, i.e.* $d(df \circ \varphi) = 0$.

Proof. Using

$$(d\omega)(X,Y)=\frac{1}{2}\{X(\omega(Y))-Y(\omega(X))-\omega([X,Y])\},$$
$$X,Y\in\Im^1_0(M_{2n}),\ \ \omega\in\Im^0_1(M_{2n})$$

for $(\omega\circ\varphi)(X)=\omega(\varphi(X))$, we have

$$\begin{aligned}(d\omega)(Y,\varphi X)&=\frac{1}{2}\{Y(\omega(\varphi X))-(\varphi X)(\omega(Y))-\omega([Y,\varphi X])\}\\&=\frac{1}{2}\{Y(\omega(\varphi X))-(\varphi X)(\omega(Y))+\omega([\varphi X,Y])\}\\&=\frac{1}{2}\{Y(\omega(\varphi X))-(\varphi X)(\omega(Y))+\omega([\varphi X,Y]\\&\quad-\varphi[X,Y])+\omega(\varphi[X,Y])\}\end{aligned}\tag{5.1}$$

From (1.2), we have

$$\begin{aligned}(\phi_\varphi\omega)(X,Y)&=(\varphi X)(\omega(Y))-X(\omega(\varphi Y))+\omega((L_Y\varphi)(X))\\&=(\varphi X)(\omega(Y))-X(\omega(\varphi Y))-\omega([\varphi X,Y]-\varphi[X,Y])\end{aligned}\tag{5.2}$$

Substituting (5.2) into (5.1), we obtain

$$\begin{aligned}&(d\omega)(Y,\varphi X)\\&=\frac{1}{2}\{-(\phi_\varphi\omega)(X,Y)+Y(\omega(\varphi X))-X(\omega(\varphi Y))+\omega(\varphi[X,Y])\}\\&=-\frac{1}{2}\{(\phi_\varphi\omega)(X,Y)+Y((\omega\circ\varphi)(X))\\&\quad-X((\omega\circ\varphi)(Y))-(\omega\circ\varphi)([Y,X])\}\\&=-\frac{1}{2}(\phi_\varphi\omega)(X,Y)+(d(\omega\circ\varphi)(Y,X)).\end{aligned}$$

From this we see that equation $\phi_\varphi\omega=0$ is equivalent to

$$(d(\omega\circ\varphi))(Y,X)=(d\omega)(Y,\varphi X).\tag{5.3}$$

For $\omega=df$, eq. (5.3) turns into the following simple form:

$$(d(df\circ\varphi))(Y,X)=(d^2f)(Y,\varphi X)=0,\ \text{ i.e.}$$
$$d(df\circ\varphi)=0.\tag{5.4}$$

Thus the lemma is proved.

If there exist a function g in a Kähler–Norden manifold such that $df\circ\varphi=dg$ for a function f, then we shall call f a *holomorphic (analytic) function* and g its associated function [17]. If such a function f is defined locally, then we call it a locally holomorphic function.

We notice that eq. (5.4) is equivalent to $df \circ \varphi = dg$ only locally. Hence the condition for f to be locally holomorphic ($\varphi_i^m \partial_m f = \partial_i g$) is given by

$$(\phi_\varphi df)_{ij} = \varphi_i^m \partial_m \partial_j f - \partial_i(\varphi_j^m \partial_m f) + (\partial_j \varphi_i^m)\partial_m f = 0.$$

Let (M_{2n}, φ, g) be a Kähler–Norden manifold with Norden metric g. Then from Theorems 6, 7 and (4.3) we find that in Kähler–Norden manifolds the covariant derivative of the curvature tensor field ∇R is also pure. Now, the covariant derivative of the Ricci tensor $R_{ji} = R_{sji}^s = g^{ts} R_{tjis}$ is pure in all its indices and hence

$$\varphi_t^s \nabla_s R_{ji} = \varphi_j^s \nabla_t R_{si}.$$

Transvecting this equation with contravariant Norden metric g^{ji}, we find

$$\varphi_t^s \nabla_s R = g^{ji} \varphi_j^s \nabla_t R_{si} = \nabla_t (G^{si} R_{si}) = \nabla_t \overset{*}{R}, \tag{5.5}$$

where $R = g^{ij} R_{ij}$ and $\overset{*}{R} = G^{ij} R_{ij}$ are curvature scalars of Norden and twin Norden metrics respectively.

From (5.5) we have the following.

Theorem 8. *In a Kähler–Norden manifold, the curvature scalar R is a locally holomorphic function.*

Acknowledgements

We are very grateful to Professor V V Vishnevskii for useful discussions. This paper is supported by the Scientific and Technological Research Council of Turkey (108T590).

References

[1] Blažić N and Bokan N, Invariance theory and affine differential geometry, Differential geometry and applications (Brno) (1995) pp. 249–260 (Brno: Masarky Univ.) (1996)

[2] Borowiec A, Francaviglia M and Volovich I, Anti-Kählerian manifolds, *Differential Geom. Appl.* **12(3)** (2000) 281–289

[3] Borowiec A, Ferraris M, Francaviglia M and Volovich I, Almost complexes and almost-product Einstein manifolds from a variational principle, *J. Math. Phys.* **40(7)** (1999) 3446–3464

[4] Dragomir S and Francaviglia M, On Norden metrics which are locally conformal to anti-Kählerian metrics, *Acta Appl. Math.* **60(2)** (2000) 115–135

[5] Etayo F and Santamaria R, ($J^2 = \pm 1$)-metric manifolds, *Publ. Math. Debrecen* **57(3–4)** (2000) 435–444

[6] Ganchev G T and Borisov A V, Note on the almost complex manifolds with a Norden metric, *C. R. Acad. Bulgarie Sci.* **39(5)** (1986) 31–34

[7] Kruchkovich G I, Hypercomplex structure on a manifold, I, *Tr. Sem. Vect. Tens. Anal. Moscow Univ.* **16** (1972) 174–201

[8] Manev M, Classes of real time-like hypersurfaces of a Kaehler manifold with B-metric, *J. Geom.* **75(1–2)** (2002) 113–122

[9] Norden A P, On a certain class of four-dimensional A-spaces, *Izv. Vuzov. Mat.* **4** (1960) 145–157

[10] Oproiu V and Papaghiuc N, Some classes of almost anti-Hermitian structures on the tangent bundle, *Mediterr. J. Math.* **1(2)** (2004) 269–282

[11] Salimov A A, Almost analyticity of a Riemannian metric and integrability of a structure, *(Russian) Trudy Geom. Sem. Kazan. Univ.* **15** (1983) 72–78

[12] Salimov A A, Generalized Yano-Ako operator and the complete lift of tensor fields, *Tensor (N.S.)* **55(2)** (1994) 142–146

[13] Salimov A A, Lifts of poly-affinor structures on pure sections of a tensor bundle, *Russian Math. (Iz. VUZ)* **40(10)** (1996) 52–59

[14] Salimov A A, Iscan M and Etayo F, Paraholomorphic B-manifold and its properties *Topology and its Application* **154** (2007) 925–933

[15] Sluka K, On the curvature of Kähler-Norden manifolds, *J. Geom. Phys.* **54(2)** (2005), 131–145

[16] Tachibana S, Analytic tensor and its generalization, *Tôhoku Math. J.* **12(2)** (1960) 208–221

[17] Tachibana S and Kotô S, On almost-analytic functions, tensors and invariant subspaces, *Tôhoku Math. J.* **14(2)** (1962) 177–186

[18] Vishnevskii V V, Affinor structures of affine connection spaces, *Izv. Vuzov. Mat.* **1** (1970) 12–23

[19] Vishnevskii V V, Shirokov A P and Shurygin V V, Spaces over algebras (Kazan: Kazan Gos. University) (1985) Russian

[20] Vishnevskii V V, Integrable affinor structures and their plural interpretations, *J. Math. Sci.* **108(2)** (2002) 151–187

[21] Yano K and Ako M, On certain operators associated with tensor fields, *Kodai Math. Sem. Rep.* **20** (1968) 414–436