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On equivariant embedding of Hilbert C^* modules

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Abstract. We prove that an arbitrary (not necessarily countably generated) Hilbert $G - \mathcal{A}$ module on a $G - C^*$ algebra $\mathcal{A}$ admits an equivariant embedding into a trivial $G - \mathcal{A}$ module, provided G is a compact Lie group and its action on $\mathcal{A}$ is ergodic.

Keywords. Hilbert modules; ergodic action; equivariant triviality; equivariant embedding.

1. Introduction

Let G be a locally compact group, $\mathcal{A}$ be a C^*-algebra, and assume that there is a strongly continuous representation $\alpha: G \to \text{Aut}(\mathcal{A})$. Following the terminology of [5], we introduce the concept of a Hilbert C^* $G - \mathcal{A}$-module as follows:

DEFINITION 1.1

A *Hilbert* C^* $G - \mathcal{A}$ *module* (or $G - \mathcal{A}$ *module* for short) is a pair (E, β) where E is a Hilbert C^* $\mathcal{A}$-module and β is a map from G into the set of $\mathbb{C}$-linear (caution: not $\mathcal{A}$-linear!) maps from E to E, such that $\beta_g \equiv \beta(g)$, $g \in G$ satisfies the following:

(i) $\beta_{gh} = \beta_g \circ \beta_h$ for $g, h \in G$, $\beta_e = \text{Id}$, where e is the identity element of G;
(ii) $\beta_g(\xi a) = \beta_g(\xi)\alpha_g(a)$ for $\xi \in E, a \in \mathcal{A}$;
(iii) $g \mapsto \beta_g(\xi)$ is continuous for each fixed $\xi \in E$;
(iv) $\langle \beta_g(\xi), \beta_g(\eta)\rangle = \alpha_g(\langle \xi, \eta\rangle)$ for all $\xi, \eta \in E$, where $\langle \cdot, \cdot \rangle$ denotes the $\mathcal{A}$-valued inner product of E.

When β is understood from the context, we may refer to E as a $G - \mathcal{A}$ module, without explicitly mentioning the pair (E, β). Given two $G - \mathcal{A}$ modules (E_1, β) and (E_2, γ), there is a natural G-action induced on $\mathcal{L}(E_1, E_2)$, given by $\pi_g(T)(\xi) := \gamma_g(T(\beta_{g^{-1}}(\xi)))$ for $g \in G, \xi \in E_1, T \in \mathcal{L}(E_1, E_2)$. $T \in \mathcal{L}(E_1, E_2)$ is said to be *G-equivariant* if $\pi_g(T) = T$ for all $g \in G$. It is clear that for each fixed $T \in \mathcal{L}(E_1, E_2)$ and $\xi \in E_1$, $g \mapsto \pi_g(T)\xi$ is continuous. We say that T is *G-continuous* if $g \mapsto \pi_g(T)$ is continuous with respect to the norm topology on $\mathcal{L}(E_1, E_2)$. We say that E_1 and E_2 are *isomorphic as* $G - \mathcal{A}$*-modules*, or that they are *equivariantly isomorphic* if there is a G-equivariant unitary map $T \in \mathcal{L}(E_1, E_2)$. We call a $(G - \mathcal{A})$ module of the form $(\mathcal{A} \otimes \mathcal{H}, \alpha_g \otimes \gamma_g)$ (where $\mathcal{H}$ is a Hilbert space) a trivial $G - \mathcal{A}$ module. We say that (E, β) is *embeddable* if there is an equivariant isometry from E to $\mathcal{A} \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$ with a G-action γ, or in other words, (E, β) is equivariantly isomorphic with a sub-$G - \mathcal{A}$ module of

$(\mathcal{A}\otimes\mathcal{H}, \beta\otimes\gamma)$. Note that $\mathcal{A}\otimes\mathcal{H}$ is the closure of $\mathcal{A}\otimes_{\text{alg}}\mathcal{H}$ under the norm inherited from $\mathcal{B}(\mathcal{H}_0, \mathcal{H}_0\otimes\mathcal{H})$ where $\mathcal{H}_0$ is any Hilbert space such that $\mathcal{A}$ is isometrically embedded into $\mathcal{B}(\mathcal{H}_0)$. The following result on the embeddability is due to Mingo and Phillips [5].

Theorem 1.2. *Let (E, β) be a Hilbert C^* $G-\mathcal{A}$ module and assume that E is countably generated as a Hilbert $\mathcal{A}$-module, that is, there is a countable set $S = \{e_1, e_2, \ldots\}$ of elements of E such that the right $\mathcal{A}$-linear span of S is dense in E. Assume furthermore that G is compact. Then (E, β) is embeddable.*

When G is the trivial singleton group, the above result was proved by Kasparov.

If the C^* algebra $\mathcal{A}$ is replaced by a von Neumann algebra $\mathcal{B}\subseteq\mathcal{B}(h)$ for some Hilbert space h and G is a locally compact group with a strongly continuous unitary representation $g\mapsto u_g\in\mathcal{B}(h)$, one can define Hilbert von Neumann $G-\mathcal{B}$ module (E, β). The only difference is that E is now a Hilbert von Neumann $\mathcal{A}$-module equipped with the natural locally convex strong operator topology, and that we replace the norm-continuity in (iii) of the above definition by the continuity of $g\mapsto\beta_g(\xi)$ (for fixed $\xi\in E$) with respect to the locally convex topology of E. In this case, we have a stronger version of Theorem 1.2 (see [2] and Theorem 4.3.5, page 99 of [7]), which is valid without the condition of E being countably generated and without the compactness of G. It should be remarked here that the trivial Hilbert von Neumann $\mathcal{B}$ module $\mathcal{B}\otimes\mathcal{H}$ is defined to be the closure of $\mathcal{B}\otimes_{\text{alg}}\mathcal{H}$ with respect to the strong operator topology inherited from $\mathcal{B}(h, h\otimes\mathcal{H})$.

In Theorem 1.2, the assumption that E is countably generated restricts the applicability of the result, since it is not always easy to check this assumption. However, under further assumptions on the group G, the C^* algebra $\mathcal{A}$ and the action of G on $\mathcal{A}$, it may be possible to prove the embedability for an arbitrary Hilbert $G-\mathcal{A}$ module. The aim of the present article is to give some such sufficient conditions.

2. Ergodic action and its implication

We say that the action α of G on a unital C^*-algebra $\mathcal{A}$ is *ergodic* if $\alpha_g(a) = a$ for all $g\in G$ if and only if a is a scalar multiplie of 1. There is a considerable amount of literature on ergodic action of compact groups, and we shall quote one interesting structure theorem which will be useful for us.

PROPOSITION 2.1

Let G be a compact group acting ergodically on a unital C^*-algebra $\mathcal{A}$. Then there is a set of elements t^π_{ij} of $\mathcal{A}$ ($\pi\in\hat{G}, i = 1,\ldots,d_\pi, j = 1,\ldots,m_\pi$), where $\hat{G}$ is the set of equivalence classes of irreducible representations of G and d_π is the dimension of the irreducible representation space denoted by π, $m_\pi(\le d_\pi)$ is a positive integer, such that the followings hold:

(i) *There is a unique faithful G-invariant state τ on $\mathcal{A}$, which is in fact a trace.*
(ii) *The linear span of $\{t^\pi_{ij}\}$ is norm-dense in $\mathcal{A}$.*
(iii) *$\{t^\pi_{ij}\}$ is an orthonormal basis of $h = L^2(\mathcal{A}, \tau)$.*
(iv) *The action of u_g coincides with the π-th irreducible representation of G on the vector space spanned by t^π_{ij}, $i = 1,\ldots,d_\pi$ for each fixed j and π.*
(v) *$\sum_{i=1,\ldots d_\pi}(t^\pi_{ij})^* t^\pi_{ik} = \delta_{jk}d_\pi 1$, where δ_{jk} denotes the Kronecker delta symbol. Thus, in particular, $\|t^\pi_{ij}\|\le\sqrt{d_\pi}$* for all *$\pi, i, j$.*

The proof can be obtained by combining the results of [6], [3] and [1].

If G is a Lie group, with a basis of the Lie algebra given by $\{\chi_1, \ldots, \chi_N\}$, which has a strongly continuous action θ on a Banach space F, we can consider the space of 'smooth' or C^∞-elements of F, denoted by F^∞, consisting of all $\xi \in F$ such that $G \ni g \mapsto \theta_g(\xi)$ is C^∞. It is easy to prove that (see [7] and the references therein) F^∞ is dense in F, and it is a $*$-subalgebra if F is a Banach $*$-algebra. Moreover, we equip F^∞ with a family of seminorms $\|\cdot\|_{\infty,n}$, $n = 0, 1, \ldots$ given by

$$\|\xi\|_{\infty,n} := \sum_{i_1,i_2,\ldots,i_k;k\le n,i_t\in\{1,\ldots,N\}} \|\partial_{i_1}\partial_{i_2}\ldots\partial_{i_k}\xi\|,$$

with the convention $\|\cdot\|_{\infty,0} = \|\cdot\|$ and where $\partial_j(\xi) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\theta_{\exp(t\chi_j)}(\xi)$. The space F^∞ is complete under this family of seminorms, and thus is a Fréchet space. When F is a Hilbert space or a Hilbert module, we shall also consider a map d_j given by essentially the same expression as that of ∂_j, with χ_j replaced by $i\chi_j$, and the Hilbertian seminorms $\{\|\cdot\|_{2,n}\}$ are given by

$$\|\xi\|_{2,n}^2 := \sum_{i_1,i_2,\ldots i_k;k\le n,i_t\in\{1,\ldots,N\}} \|d_{i_1}d_{i_2}\ldots d_{i_k}\xi\|_2^2,$$

with $\|\cdot\|_2$ denoting the norm of the Hilbert space (or Hilbert module) F.

More generally, if F is a complete locally convex space given by a family of seminorms $\{\|\cdot\|^{(q)}\}$, then we can consider the smooth subspace F^∞ and the maps ∂_j as above, and make it a complete locally convex space with respect to a larger family of seminorms $\{\|\cdot\|_n^{(q)}\}$ where

$$\|\xi\|_n^{(q)} := \sum_{i_1,i_2,\ldots i_k;k\le n,i_t\in\{1,\ldots,N\}} \|\partial_{i_1}\partial_{i_2}\ldots\partial_{i_k}\xi\|^{(q)}.$$

In case F is a von Neumann algebra equipped with the locally convex strong operator topology, the locally convex space F^∞ is a topological $*$-algebra, strongly dense in F.

Let us assume from now onwards (throughout the rest of the paper) that G is a compact Lie group, with a basis $\{\chi_1, \ldots, \chi_N\}$ of the Lie algebra, such that G has an ergodic action α_g on a unital C^*-algebra $\mathcal{A}$. Let $h = L^2(\mathcal{A}, \tau)$, where τ is the unique invariant faithful trace described in Proposition 2.1. Let u_g be the unitary in h induced by the action of G, that is, on the dense subspace $\mathcal{A} \subseteq h$, $u_g(a) := \alpha_g(a)$. Denote also by α the action $g \mapsto u_g \cdot u_g^*$ on $\tilde{\mathcal{A}} := \mathcal{A}'' \subseteq \mathcal{B}(h)$.

Lemma 2.2. *We have $h^\infty = \mathcal{A}^\infty$ as Fréchet spaces.*

Proof. The fact that $\mathcal{A}^\infty = h^\infty$ as sets is contained in page 200–201, Lemma 8.1.20 of [7]. We only prove that the identity map is a topological homeomorphism.

Since the trace τ is finite, the Fréchet topology of $\mathcal{A}^\infty$ is stronger than that of h^∞. This implies that the identity map id, viewed as a linear map from the Fréchet space h^∞ to the Fréchet space $\mathcal{A}^\infty$ is closed, hence continuous. This completes the proof that the two Fréchet topologies on $\mathcal{A}^\infty = h^\infty$ are equivalent, i.e. $\mathcal{A}^\infty = h^\infty$ as topological spaces. □

From this lemma, it is also clear that $\tilde{\mathcal{A}}^\infty = h^\infty = \mathcal{A}^\infty$ as Fréchet spaces. Now, we shall prove a crucial technical result, which is sort of generalisation of the above lemma, with $\mathcal{A}$ replaced by $\mathcal{A} \otimes \mathcal{H}$ for an arbitrary Hilbert space $\mathcal{H}$. The main idea is to exploit the 'smooth' topology on $\mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{H}^\infty$ (which is a common subspace of both $(\mathcal{A} \otimes \mathcal{H})^\infty$ and $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$), given by the $\mathcal{A}^\infty$-valued inner product. A sequence ξ_n is Cauchy in this topology if and only if $\langle (\xi_n - \xi_m), (\xi_n - \xi_m) \rangle$ goes to 0 in the Fréchet topology of $\mathcal{A}^\infty$, which is the same as the topology of $\tilde{\mathcal{A}}^\infty$ or that of h^∞. Given a fixed element of $\tilde{\mathcal{A}} \otimes \mathcal{H}$ which is smooth, one can hope to approximate it by a sequence of elements from $\tilde{\mathcal{A}}^\infty \otimes_{\text{alg}} \mathcal{H}^\infty = \mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{H}^\infty$ in the above smooth topology, which is stronger than the norm-topology of $\mathcal{A} \otimes \mathcal{H}$, thus the limit should belong to $\mathcal{A} \otimes \mathcal{H}$. Roughly speaking, this is how the next lemma is proved; however, there are more technical ingredients, which also made it necessary to assume the norm-continuity of the map $g \mapsto \gamma_g(\xi)$ for the fixed smooth element $\xi \in (\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$.

Lemma 2.3. *Let $\mathcal{H}$ be a (not necessarily separable) Hilbert space with a unitary representation $w \equiv w_g$ of G, and let us consider the Fréchet modules $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$ and $(\mathcal{A} \otimes \mathcal{H})^\infty$ corresponding to the action $\gamma_g := \alpha_g \otimes w_g$. Let ξ be an element of $(\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$ such that $g \mapsto \gamma_g(\xi)$ is continuous in the operator-norm topology on $\tilde{\mathcal{A}} \otimes \mathcal{H}$ inherited from $\mathcal{B}(h, h \otimes \mathcal{H})$. Then ξ actually belongs to $\mathcal{A} \otimes \mathcal{H}$.*

Proof. We shall denote by $\| \cdot \|_p$ $(p \geq 1)$ the L^p-norm coming from the trace τ on $\mathcal{A}$. The identity 1 of $\mathcal{A}$ will also be viewed as a unit vector in $L^2(\tau)$. Fix an orthonormal basis $\{e_\alpha, \alpha \in T\}$ of $\mathcal{H}$ (which need not be separable), with each $e_\alpha \in \mathcal{H}^\infty$.

For a C^∞ complex-valued function f on G and an element $\eta \in \tilde{\mathcal{A}} \otimes \mathcal{H}$, we shall denote by $\gamma(f)(\eta)$ the element $\int_G f(g)\gamma_g(\eta)\mathrm{d}g \ \in \tilde{\mathcal{A}} \otimes \mathcal{H} \subseteq L^2(\tau) \otimes \mathcal{H}$, where $\mathrm{d}g$ stands for the normalised Haar measure on G and the integral is convergent in the strong-operator topology, given by $\gamma(f)(\eta)v := \int_G f(g)\gamma_g(\eta)v \ \mathrm{d}g \ \forall v \in h$. It is straightforward to see that $\gamma(f)(\eta) \in \mathcal{A} \otimes \mathcal{H}$ whenever $\eta \in \mathcal{A} \otimes_{\text{alg}} \mathcal{H}$, and in this case the integral converges in norm, since $g \mapsto \gamma_g(\eta)$ is norm-continuous for $\eta \in \mathcal{A} \otimes_{\text{alg}} \mathcal{H}$.

Now let us fix $\xi \in (\tilde{\mathcal{A}} \otimes \mathcal{H})^\infty$ satisfying the hypothesis of the lemma. Since $L^2(\tau)$ is separable, say with an orthonormal basis given by $\{x_1, x_2, \dots\}$, we can find, for each i, a countable subset T_i of T such that $\langle \xi 1, x_i \otimes e_\alpha \rangle = 0$ for all $\alpha \notin T_i$. Denoting by T_∞ the countable set $\bigcup_i T_i$, we have $\langle \xi 1, v \otimes e_\alpha \rangle = 0 \ \forall v \in L^2(\tau)$, for all $\alpha \notin T_\infty$. Write $T_\infty = \{e_{\alpha_1}, e_{\alpha_2}, \dots\}$. Denote by ξ_n the element in $\mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{H}^\infty$ given by $\xi_n = (\mathrm{id} \otimes P_n)\xi$, where P_n denotes the orthogonal projection onto the linear span of $\{e_{\alpha_1}, \dots, e_{\alpha_n}\}$. Let us write ξ_n as

$$\xi_n = \sum_{k=1}^{n} a_k \otimes e_{\alpha_k},$$

where $a_k = \langle (1 \otimes e_{\alpha_k}), \xi \rangle \in \tilde{\mathcal{A}}^\infty = \mathcal{A}^\infty$ for all k. It is clear that $\|\xi_n\| \leq \|\xi\|$ and $\xi_n 1 \to \xi 1$ as $n \to \infty$.

We claim that $\gamma(f)(\xi)$ indeed belongs to $\mathcal{A} \otimes \mathcal{H}$ for every $f \in C^\infty(G)$. To this end, first observe that $\eta_n := \gamma(f)(\xi_n)$ clearly belongs to $\mathcal{A} \otimes \mathcal{H}$ for all n, since $\xi_n \in \mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{H}^\infty \subseteq \mathcal{A} \otimes_{\text{alg}} \mathcal{H}$. Moreover, since $u_g 1 = 1$ for all g and $\gamma_g(\cdot) = \mathrm{ad}_{u_g} \otimes w_g$, we have

$$\lim_{n \to \infty} \eta_n 1 = \lim_{n \to \infty} \int_G f(g) \left(\sum_{k=1}^{n} u_g(a_k 1) \otimes w_g e_{\alpha_k} \right) \mathrm{d}g$$

$$= \lim_{n\to\infty} \int_G f(g)(u_g \otimes w_g)(\xi_n 1)\mathrm{d}g$$

$$= \int_G f(g)(u_g \otimes w_g)(\xi 1)\mathrm{d}g$$

(by the dominated convergence theorem)

$$= \int_G f(g)\gamma_g(\xi)1 \,\mathrm{d}g \equiv \eta 1.$$

Since for each $m, n \geq 1$, $\eta_{m,n} := \eta_m - \eta_n$ belongs to $\mathcal{A} \otimes \mathcal{H}$, for proving $\eta \in \mathcal{A} \otimes \mathcal{H}$ it is enough to prove that $\eta_{m,n} \to 0$ in the topology of $\mathcal{A} \otimes \mathcal{H}$, i.e. $x_{mn} := \langle \eta_{m,n}, \eta_{m,n} \rangle \to 0$ in the norm-topology of $\mathcal{A}$. We shall prove that $x_{mn} \to 0$ in the Fréchet topology of h^∞, which will prove that it converges to 0 also in the topology of $\mathcal{A}^\infty$ by Lemma 2.2.

For this, first note that for $\beta_1, \beta_2 \in (\tilde{\mathcal{A}} \otimes \mathcal{H})$, we have

$$\|\langle \beta_1, \beta_2 \rangle\|_2^2 = \tau(\beta_2^* \beta_1 \beta_1^* \beta_2) \leq \|\beta_1\|^2 \tau(\beta_2^* \beta_2),$$

hence $\|\langle \beta_1, \beta_2 \rangle\|_2 \leq \|\beta_1\| \|\beta_2\|_2$. Moreover, since $\tau(x^* x) = \tau(x x^*)$, we have $\|x\|_2 = \|x^*\|_2 \; \forall x \in \tilde{\mathcal{A}}$. Thus, $\|\langle \beta_2, \beta_1 \rangle\|_2 = \|\langle \beta_1, \beta_2 \rangle^*\|_2 = \|\langle \beta_1, \beta_2 \rangle\|_2$. It follows that

$$\|\langle \gamma(f)(\beta), \gamma(f)(\beta) \rangle\|_2$$

$$\leq \int_G \int_G |f(g)|\,|f(h)|\,\|\langle \gamma_g(\beta), \gamma_h(\beta) \rangle\|_2 \,\mathrm{d}g\mathrm{d}h$$

$$\leq \int_G \int_G |f(g)|\,|f(h)|\,\|\gamma_g(\beta)\|\,\|\gamma_h(\beta)\|_2 \,\mathrm{d}g\mathrm{d}h$$

$$\leq \int_G \int_G |f(g)|\,|f(h)|\,\|\beta\|\,\|\beta\|_2 \,\mathrm{d}g\mathrm{d}h$$

$$= C(f)^2 \|\beta\| \|\beta\|_2,$$

where $C(f) := \int_G |f| \mathrm{d}g$. Let us now fix an ordered k-tuple $I = (i_1, \ldots, i_k)$ (k nonnegative integer), and for any ordered subset $J = (j_1, \ldots, j_p)$ of I, $\beta \in (\mathcal{A} \otimes \mathcal{H})^\infty$, we shall abbreviate $\partial_{j_1} \ldots \partial_{j_p} \beta$ and $\chi_{j_1} \ldots \chi_{j_p} f$ by $\partial_J \beta$ and f_J respectively. Note that

$$\partial_J \gamma(f)(\beta) = \gamma(f_J)(\beta).$$

Using this as well as the Leibniz formula $\partial_I \langle \beta, \beta \rangle = \sum_J \langle \partial_J \beta, \partial_{I-J} \beta \rangle$ (with J varying over all ordered subsets of I), we have the following:

$$\|\partial_I x_{mn}\|_2$$

$$\leq \sum_{J:\, J \text{ ordered subset of } I} \|\langle \partial_J(\eta_m - \eta_n), \partial_{I-J}(\eta_m - \eta_n) \rangle\|_2$$

$$\leq \sum_J C(f_J) C(f_{I-J}) \|\xi_m - \xi_n\| \|\xi_m - \xi_n\|_2$$

$$\leq 2^k C^2 \|\xi_m - \xi_n\| \|\xi_m - \xi_n\|_2 \quad (\text{where } C := \max\{C(f_J) \colon J \subset I, \text{ ordered subset}\})$$

$$\leq 2^{k+1} C^2 \|\xi\| \|\xi_m - \xi_n\|_2,$$

since the number of ordered subsets of I is 2^k and $\|\xi_n\| \leq \|\xi\|$ for all n. We also have $\|\xi_m - \xi_n\|_2^2 = \tau(\langle \xi_m - \xi_n, \xi_m - \xi_n \rangle) = \langle (\xi_m - \xi_n)1, (\xi_m - \xi_n)1 \rangle \to 0$ as $m, n \to \infty$. This proves $x_{m,n} \to 0$ in the topology of h^∞, hence in the topology of $\mathcal{A}^\infty$ as well, so $\eta = \gamma(f)(\xi) \in \mathcal{A} \otimes \mathcal{H}$.

To complete the proof of the lemma, using the norm-continuity of the map $g \mapsto \gamma_g(\xi)$, we choose for each $n \geq 1$ a nonempty open subset U_n of G such that $\|\gamma_g(\xi) - \xi\| \leq \frac{1}{n}$ for all $g \in U_n$, and then choose $f_n \in C^\infty(G)$ with $\mathrm{supp}(f_n) \subseteq U_n$ satisfying $f_n \geq 0$ and $\int_G f_n \mathrm{d}g = 1$. It is easy to see that

$$\begin{aligned}
&\|\gamma(f_n)(\xi) - \xi\| \\
&= \left\| \int_G f_n(g)\gamma_g(\xi)\mathrm{d}g - \xi \int_G f_n(g)\mathrm{d}g \right\| \\
&\leq \int_G \|f_n(g)(\gamma_g(\xi) - \xi)\|\mathrm{d}g \\
&\leq \frac{1}{n} \int_G f_n(g)\mathrm{d}g = \frac{1}{n}.
\end{aligned}$$

Thus, ξ is the operator-norm limit of the sequence $\gamma(f_n)(\xi) \in \mathcal{A} \otimes \mathcal{H}$, so $\xi \in \mathcal{A} \otimes \mathcal{H}$. □

3. Main results on equivariant embedding of Hilbert modules

Let (E, β) be a $G - \mathcal{A}$ module, where $\mathcal{A}$ and G are as in the previous section. In this final section, we shall prove that any such (E, β) is embeddable.

Lemma 3.1. *We can find a Hilbert space $\mathcal{K}$, a strongly continuous unitary representation $g \mapsto V_g \in \mathcal{B}(\mathcal{K})$ and an $\mathcal{A}$-linear isometry $\Gamma_0\colon E \to \mathcal{B}(h, \mathcal{K})$, such that $\Gamma_0\beta_g(\xi) = V_g(\Gamma_0\xi)u_g^{-1}$, and moreover, the complex linear span of elements of the form $\Gamma_0(\xi)w$, where $\xi \in E$ and $w \in h$, is dense in $\mathcal{K}$.*

Proof. The proof of this result is adapted from [2] and page 99–101, Theorem 4.3.5 of [7]. We shall give only a brief sketch of the arguments involved, omitting the details. We consider first the formal vector space (say $\mathcal{V}$) spanned by symbols (ξ, w), with $\xi \in E$ and $w \in h$, and define a semi-inner product on this formal vector space by setting

$$\langle (\xi, w), (\xi', w') \rangle = \langle w, \langle \xi, \xi' \rangle w' \rangle,$$

where $\langle \xi, \xi' \rangle$ denotes the $\mathcal{A}$-valued inner product on E. By extending this semi-inner product by linearity and then taking quotient by the subspace (say $\mathcal{V}_0$) consisting of elements of zero norm we get a pre-Hilbert space, and its completion under the pre-inner product is denoted by $\mathcal{K}$. We also define $\Gamma_0\colon E \to \mathcal{B}(h, \mathcal{K})$ by setting

$$(\Gamma_0(\xi))w := [\xi, w],$$

where $[\xi, w]$ represents the equivalence class of (ξ, w) in $\mathcal{S} \equiv \mathcal{V}/\mathcal{V}_0 \subseteq \mathcal{K}$. That it is an isometry is verified by straightforward calculations. Next, we define V_g on $\mathcal{S}$ by

$$V_g[\xi, w] := [\beta_g(\xi), u_g w],$$

and verify that it is indeed an isometry, and since its range clearly contains a total subset, V_g extends to a unitary on $\mathcal{K}$. Furthermore, $V_g V_h = V_{gh}$ and $V_e = I$ (where e is the identity of G) on $\mathcal{S}$, and hence on the whole of $\mathcal{K}$. The strong continuity of $g \mapsto V_g$ is also easy to see. Indeed, it is enough to prove that $g \mapsto V_g X$ is continuous for any X of the form $[\xi, v]$, $\xi \in E$, $v \in h$. But

$$\|V_g([\xi, v]) - [\xi, v]\|^2 = 2\langle[\xi, v], [\xi, v]\rangle - \langle V_g([\xi, v]), [\xi, v]\rangle \\ - \langle[\xi, v], V_g([\xi, v])\rangle,$$

and we have

$$\begin{aligned}&\langle V_g([\xi, v]), [\xi, v]\rangle - \langle[\xi, v], [\xi, v]\rangle \\ &\quad = \langle(u_g v - v)\langle\beta_g(\xi), \xi\rangle v\rangle + \langle v, \langle(\beta_g(\xi) - \xi), \xi\rangle v\rangle \\ &\quad \to 0\end{aligned}$$

as $g \to e$, since by assumption $\lim_{g\to e}(\beta_g(\xi) - \xi) = 0$ in the norm topology of E, and $\lim_{g\to e} u_g v = v$. □

In view of the above result, we assume without loss of generality that $E \subset \mathcal{B}(h, \mathcal{K})$ (with the natural Hilbert module structure inherited from that of $\mathcal{B}(h, \mathcal{K})$), and $\beta_g(\cdot) = V_g \cdot u_g^{-1}$. Consider the strong operator closure $\tilde{E}$ of E in $\mathcal{B}(h, \mathcal{K})$. It is a Hilbert von Neumann $\tilde{\mathcal{A}}$ module (where $\tilde{\mathcal{A}}$ is the weak closure of $\mathcal{A}$ in h). Moreover, the G-action $\beta_g = V_g \cdot u_g^{-1}$ can be extended to the whole of $\mathcal{B}(h, \mathcal{K})$, and denoted again by β_g. Clearly, this action leaves $\tilde{E}$ invariant, hence $(\tilde{E}, \beta)$ is a Hilbert von Neumann $G - \tilde{\mathcal{A}}$ module. Let us recall that by $\tilde{E}^\infty$ we denote the locally convex space of elements ξ in $\tilde{E}$ such that $g \mapsto \beta_g(\xi)$ is C^∞ in the strong operator topology of $\tilde{E}$.

Theorem 3.2. *There exist a Hilbert space k_0, a unitary representation w_g of G in k_0 and an isometry Σ from $\mathcal{K}$ to $h \otimes k_0$ such that*

(i) Σ *is equivariant in the sense that* $\Sigma V_g = (u_g \otimes w_g)\Sigma$ *for all* g;
(ii) $\Sigma\xi \in \mathcal{A} \otimes k_0$ *for all* $\xi \in E$.

Proof. The statement (i) is contained in page 99, Theorem 4.3.5 of [7]. For proving (ii), we note that E^∞ (with respect to the action β) is mapped by Σ into $(\tilde{\mathcal{A}} \otimes k_0)^\infty$ (with respect to the action $\gamma_g := \mathrm{ad}_{u_g} \otimes w_g$), and moreover, for $\xi \in E$, $g \mapsto \gamma_g(\Sigma(\xi)) = \Sigma\beta_g(\xi)$ is norm-continuous since $g \mapsto \beta_g(\xi)$ is so and Σ is isometry. Thus, (ii) follows from Lemma 2.3. □

It follows from the above theorem that E can be equivariantly embedded in the trivial $G - \mathcal{A}$ module $(\mathcal{A} \otimes k_0, \alpha \otimes w)$. In particular, we have

Theorem 3.3. *If a compact Lie group G has an ergodic action on a unital C^*-algebra $\mathcal{A}$, then every Hilbert C^* $G - \mathcal{A}$ module is embeddable.*

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