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On the theorem of M Golomb

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Abstract. Let $X_1, \ldots, X_n$ be compact spaces and $X = X_1 \times \cdots \times X_n$. Consider the approximation of a function $f \in C(X)$ by sums $g_1(x_1)+\cdots+g_n(x_n)$, where $g_i \in C(X_i)$, $i = 1, \ldots, n$. In [8], Golomb obtained a formula for the error of this approximation in terms of measures constructed on special points of X, called 'projection cycles'. However, his proof had a gap, which was pointed out by Marshall and O'Farrell [15]. But the question if the formula was correct, remained open. The purpose of the paper is to prove that Golomb's formula holds in a stronger form.

Keywords. Approximation error; duality relation; projection cycle; lightning bolt; orthogonal measure; extreme measure.

1. Introduction

Let X_i, $i = 1, \ldots, n$, be compact (Hausdorff) topological spaces. Consider the approximation to a continuous function f on $X = X_1 \times \cdots \times X_n$ from the manifold

$$M = \left\{ \sum_{i=1}^{n} g_i(x_i) \colon g_i \in C(X_i), i = 1, \ldots, n \right\}.$$

The approximation error is defined as the distance from f to M:

$$E(f) \stackrel{\text{def}}{=} \operatorname{dist}(f, M) = \inf_{g \in M} \|f - g\|_{C(X)}.$$

The well-known duality relation says that

$$E(f) = \sup_{\substack{\mu \in M^{\perp} \\ \|\mu\| \le 1}} \left| \int_X f \, \mathrm{d}\mu \right|, \tag{1.1}$$

where $M^{\perp}$ is the space of regular Borel measures annihilating all functions in M and $\|\mu\|$ stands for the total variation of a measure μ. It should also be noted that the sup in (1.1) is attained by some measure μ^* with total variation $\|\mu^*\| = 1$. We are interested in the problem: is it possible to replace in (1.1) the class $M^{\perp}$ by some subclass of it consisting of measures of simple structure? For the case $n = 2$, this problem was first considered by Diliberto and Straus [4]. They showed that the measures induced by the so-called 'closed lightning bolts' are sufficient for the equality (1.1).

Let $X = X_1 \times X_2$ and π_i be the projections of X onto X_i, $i = 1, 2$. A lightning bolt (or, simply, a bolt) is a finite ordered set $\{a_1, \dots, a_k\}$ contained in X, such that $a_i \neq a_{i+1}$, for $i = 1, 2, \dots, k-1$, and either $\pi_1(a_1) = \pi_1(a_2)$, $\pi_2(a_2) = \pi_2(a_3)$, $\pi_1(a_3) = \pi_1(a_4), \dots$, or $\pi_2(a_1) = \pi_2(a_2)$, $\pi_1(a_2) = \pi_1(a_3)$, $\pi_2(a_3) = \pi_2(a_4), \dots$. A bolt $\{a_1, \dots, a_k\}$ is said to be closed if k is an even number and the set $\{a_2, \dots, a_k, a_1\}$ is also a bolt. These objects have been exploited in many works devoted to the uniform approximation of bivariate functions by univariate functions and related problems, though sometimes they appeared under different names (see, e.g., [2–7, 9–11, 13–16, 18]). In [4], they were called 'permissible lines'. The term 'lightning bolt' is due to Arnold [1].

Let $l = \{a_1, \dots, a_{2k}\}$ be a closed bolt. Consider a measure μ_l having atoms $\pm\frac{1}{2k}$ with alternating signs at the vertices of l. That is,

$$\mu_l = \frac{1}{2k}\sum_{i=1}^{2k}(-1)^{i-1}\delta_{a_i} \quad \text{or} \quad \mu_l = \frac{1}{2k}\sum_{i=1}^{2k}(-1)^{i}\delta_{a_i},$$

where δ_{a_i} is a point mass at a_i. It is clear that $\mu_l \in M^{\perp}$ and $\|\mu_l\| \leq 1$. $\|\mu_l\| = 1$ if and only if the set of vertices of the bolt l having even indices does not intersect with that having odd indices. The following duality relation was first established by Diliberto and Straus [4]

$$E(f) = \sup_{l \subset X}\left|\int_X f \mathrm{d}\mu_l\right|, \tag{1.2}$$

where $X = X_1 \times X_2$ and the sup is taken over all closed bolts of X. In fact, Diliberto and Straus obtained the formula (1.2) for the case when X is a rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axis. The same result was independently proved by Smolyak (see [18]). Yet another proof of (1.2), in the case when X is a Cartesian product of two compact Hausdorff spaces, was given by Light and Cheney [14]. For X's other than a rectangle in $\mathbb{R}^2$, the theorem under some additional assumptions appeared in the works [9, 11, 15]. But we shall not discuss these works here.

Golomb's paper [8] made a start to a systematic study of approximation of multivariate functions by various compositions, including sums of univariate functions. Golomb generalized the notion of a closed bolt to the n-dimensional case and obtained the analogue of formula (1.2) for the error of approximation from the manifold M. The objects introduced in [8] were called 'projection cycles' and they are simply sets of the form

$$p = \{b_1, \dots, b_k;\ c_1, \dots, c_k\} \subset X, \tag{1.3}$$

with the property that $b_i \neq c_j$, $i, j = 1, \dots, k$ and for all $\nu = 1, \dots, n$, the group of the ν-th coordinates of $c_1, \dots, c_k$ is a permutation of that of the ν-th coordinates of $b_1, \dots, b_k$. Some points in the b-part $(b_1, \dots, b_k)$ or c-part $(c_1, \dots, c_k)$ of p may coincide. The measure associated with p is

$$\mu_p = \frac{1}{2k}\left(\sum_{i=1}^{k}\delta_{b_i} - \sum_{i=1}^{k}\delta_{c_i}\right).$$

It is clear that $\mu_p \in M^{\perp}$ and $\|\mu_p\| = 1$. Besides, if $n = 2$, then a projection cycle is the union of closed bolts after some suitable ordering of its points. Golomb's result states that

$$E(f) = \sup_{p \subset X}\left|\int_X f \mathrm{d}\mu_p\right|, \tag{1.4}$$

where $X = X_1 \times \cdots \times X_n$ and the sup is taken over all projection cycles of X. It can be proved that in the case $n = 2$, the formulas (1.2) and (1.4) are equivalent. Unfortunately, the proof of (1.4) had a gap, which was pointed out many years later by Marshall and O'Farrell [15]. But the question if the formula (1.4) was correct, remained unsolved (see also the more recent monograph by Khavinson [11]).

In §2, we will construct families of normalized measures (that is, measures with the total variation equal to 1) on projection cycles. Each measure μ_p defined above will be a member of some family. We will also consider minimal projection cycles and measures constructed on them. By properties of these measures, we show that Golomb's formula (1.4) is valid and even in a stronger form.

2. Measures supported on projection cycles

First we are going to give an equivalent definition of a projection cycle. This will be useful in constructing of measures of simple structure and with the capability to approximate an arbitrary measure in $M^{\perp}$.

In the sequel, χ_a will denote the characteristic function of a single point set $\{a\} \subset \mathbb{R}$.

DEFINITION 2.1

Let $X = X_1 \times \cdots \times X_n$ and π_i be the projections of X onto the sets X_i, $i = 1, \ldots, n$. We say that a set $p = \{x_1, \ldots, x_m\} \subset X$ is a projection cycle if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ with nonzero real coordinates such that

$$\sum_{j=1}^{m} \lambda_j \chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n. \tag{2.1}$$

Let us give some explanatory remarks concerning Definition 2.1. Fix the subscript i. Let the set $\{\pi_i(x_j), j = 1, \ldots, m\}$ have s_i different values, which we denote by $\gamma_1^i, \gamma_2^i, \ldots, \gamma_{s_i}^i$. Then (2.1) implies that

$$\sum_{j} \lambda_j = 0,$$

where the sum is taken over all j such that $\pi_i(x_j) = \gamma_k^i$, $k = 1, \ldots, s_i$. Thus for fixed i, we have s_i homogeneous linear equations in $\lambda_1, \ldots, \lambda_m$. The coefficients of these equations are the integers 0 and 1. By varying i, we obtain $s = \sum_{i=1}^{n} s_i$ such equations. Hence (2.1), in its expanded form, stands for the system of these equations. One can observe that if this system has a solution $(\lambda_1, \ldots, \lambda_m)$ with nonzero real components λ_i, then it also has a solution $(n_1, \ldots, n_m)$ with nonzero integer components n_i, $i = 1, \ldots, m$. This means that in Definition 2.1, we can replace the vector λ by the vector $n = (n_1, \ldots, n_m)$, where $n_i \in \mathbb{Z}\backslash\{0\}$, $i = 1, \ldots, m$. Thus, Definition 2.1 is equivalent to the following definition.

DEFINITION 2.2

A set $p = \{x_1, \ldots, x_m\} \subset X$ is called a projection cycle if there exist nonzero integers $n_1, \ldots, n_m$ such that

$$\sum_{j=1}^{m} n_j \chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n. \tag{2.2}$$

PROPOSITION 2.3

Definition 2.2 *is equivalent to Golomb's definition of a projection cycle.*

Proof. Let $p = \{x_1, \dots, x_m\}$ be a projection cycle with respect to Definition 2.2. By b and c denote the set of all points x_i such that the integers n_i associated with them in (2.2) are positive and negative correspondingly. Write out each point x_i n_i times if $n_i > 0$ and $-n_i$ times if $n_i < 0$. Then the set $\{b; c\}$ is a projection cycle with respect to Golomb's definition (see Introduction). The inverse is also true. Let a set $p_1 = \{b_1, \dots, b_k;\ c_1, \dots, c_k\}$ be a projection cycle with respect to Golomb's definition. Here, some points b_i or c_i may be repeated. Let $p = \{x_1, \dots, x_m\}$ stand for the set p_1 , but with no repetition of its points. Let n_i show how many times x_i appear in p_1. We take n_i positive if x_i appears in the b-part of p_1 and negative if it appears in the c-part of p_1. Clearly, the set $\{x_1, \dots, x_m\}$ is a projection cycle with respect to Definition 2.2, since the integers n_i, $i = 1, \dots, m$, satisfy (2.2). □

In the sequel, we will use Definition 2.1. A pair $\langle p, \lambda\rangle$, where p is a projection cycle in X and λ is a vector associated with p by (2.1), will be called a 'projection cycle-vector pair' of X. To each such pair $\langle p, \lambda\rangle$ with $p = \{x_1, \dots, x_m\}$ and $\lambda = (\lambda_1, \dots, \lambda_m)$, we correspond the measure

$$\mu_{p,\lambda} = \frac{1}{\sum_{j=1}^{m} |\lambda_j|} \sum_{j=1}^{m} \lambda_j \delta_{x_j}. \tag{2.3}$$

Clearly, $\mu_{p,\lambda} \in M^{\perp}$ and $\|\mu_{p,\lambda}\| = 1$. We will also deal with measures supported on some certain subsets of projection cycles called minimal projection cycles. A projection cycle is said to be minimal if it does not contain any projection cycle as its proper subset. For example, the set $p = \{(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (1,1,1)\}$ is a minimal projection cycle in $\mathbb{R}^3$, since the vector $\lambda = (2, -1, -1, -1, 1)$ satisfies eq. (2.1) and there is no such vector for any other subset of p. Adding one point $(0, 1, 1)$ from the right to p, we will also have a projection cycle, but not minimal. Note that in this case, λ can be taken as $(3, -1, -1, -2, 2, -1)$.

Remark 1. A minimal projection cycle under the name of a 'loop' was introduced in the work of Klopotowski, Nadkarni and Rao [12].

To prove our main result we need some auxiliary facts. The following lemma essentially combines Lemmas 2 and 3 of Navada's paper [17].

Lemma 2.4.

(1) *The vector* $\lambda = (\lambda_1, \dots, \lambda_m)$ *associated with a minimal projection cycle* $p = (x_1, \dots, x_m)$ *is unique up to multiplication by a constant.*
(2) *If in* (1), $\sum_{j=1}^{m} |\lambda_j| = 1$, *then all the numbers* λ_j, $j = 1, \dots, m$, *are rational.*

Proof. Let $\lambda^1 = (\lambda_1^1, \dots, \lambda_m^1)$ and $\lambda^2 = (\lambda_1^2, \dots, \lambda_m^2)$ be any two vectors associated with p. That is,

$$\sum_{j=1}^{m} \lambda_j^1 \chi_{\pi_i(x_j)} = 0 \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j^2 \chi_{\pi_i(x_j)} = 0, \quad i = 1, \dots, n.$$

After multiplying the second equality by $c = \frac{\lambda_1^1}{\lambda_1^2}$ and subtracting from the first, we obtain that

$$\sum_{j=2}^{m}(\lambda_j^1 - c\lambda_j^2)\chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n.$$

Now since the cycle p is minimal, $\lambda_j^1 = c\lambda_j^2$, for all $j = 1, \ldots, m$.

The second part of the proposition is a consequence of the first part. Indeed, let $n = (n_1, \ldots, n_m)$ be a vector with the nonzero integer coordinates associated with p. Then the vector $\lambda' = (\lambda_1', \ldots, \lambda_m')$, where $\lambda_j' = \frac{n_j}{\sum_{j=1}^{m}|n_j|}$, $j = 1, \ldots, m$, is also associated with p. All the coordinates of λ' are rational and therefore by the first part of the proposition, it is the unique vector satisfying $\sum_{j=1}^{m}|\lambda_j'| = 1$. □

By this proposition, a minimal projection cycle p uniquely (up to a sign) defines the measure

$$\mu_p = \sum_{j=1}^{m}\lambda_j\delta_{x_j}, \quad \sum_{j=1}^{m}|\lambda_j| = 1.$$

Lemma 2.5 (*see* [17]). *Let μ be a normalized orthogonal measure on a projection cycle $l \subset X$. Then it is a convex combination of normalized orthogonal measures on minimal projection cycles of l. That is*,

$$\mu = \sum_{i=1}^{s} t_i\mu_{l_i}, \quad \sum_{i=1}^{s} t_i = 1, \quad t_i > 0,$$

where l_i, $i = 1, \ldots, s$, are minimal projection cycles in l.

This lemma follows from the result of Navada (see Theorem 2 of [17]): Let $S \subset X_1 \times \cdots \times X_n$ be a finite set. Then any extreme point of the convex set of measures μ on S, $\mu \in M^\perp$, $\|\mu\| \le 1$, has its support on a minimal projection cycle contained in S.

Remark 2. In the case $n = 2$, Lemma 2.5 was proved by Medvedev (see p. 77 of [11]).

Lemma 2.6 (*see p.* 73 *of* [11]). *Let $X = X_1 \times \cdots \times X_n$ and π_i be the projections of X onto the sets X_i, $i = 1, \ldots, n$. In order that a measure $\mu \in C(X)^*$ be orthogonal to the subspace M, it is necessary and sufficient that*

$$\mu \circ \pi_i^{-1} = 0, \ i = 1, \ldots, n.$$

Lemma 2.7 (*see p.* 75 *of* [11]). *Let $\mu \in M^\perp$ and $\|\mu\| = 1$. Then there exist a net of measures $\{\mu_\alpha\} \subset M^\perp$ weak* converging in $C(X)^*$ to μ and satisfying the following properties*:

(1) $\|\mu_\alpha\| = 1$;
(2) *The closed support of each μ_α is a finite set.*

Theorem 2.8. *The error of approximation from the manifold M obeys the equality*

$$E(f) = \sup_{l \subset X}\left\|\int_X f\,\mathrm{d}\mu_l\right\|,$$

where the sup *is taken over all minimal projection cycles of X.*

Proof. Let $\tilde{\mu}$ be a measure with finite support $\{x_1, \dots, x_m\}$ and orthogonal to the space M. Put $\lambda_j = \tilde{\mu}(x_j)$, $j = 1, \dots, m$. By Lemma 2.6, $\tilde{\mu}(\pi_i^{-1}(\pi_i(x_j))) = 0$, for all $i = 1, \dots, n$, $j = 1, \dots, m$. Fix the indices i and j. Then we have the equation $\sum_k \lambda_k = 0$, where the sum is taken over all indices k such that $\pi_i(x_k) = \pi_i(x_j)$. Varying i and j, we obtain a system of such equations, which concisely can be written as

$$\sum_{k=1}^{m} \lambda_k \chi_{\pi_i(x_k)} = 0, \quad i = 1, \dots, n.$$

This means that the finite support of $\tilde{\mu}$ forms a projection cycle. Therefore, a net of measures approximating the given measure μ in Lemma 2.7 are all of the form (2.3).

Let now $\mu_{p,\lambda}$ be any measure of the form (2.3). Since $\mu_{p,\lambda} \in M^{\perp}$ and $\|\mu_{p,\lambda}\| = 1$, we can write

$$\left| \int_X f \mathrm{d}\mu_{p,\lambda} \right| = \left| \int_X (f - g) \mathrm{d}\mu_{p,\lambda} \right| \leq \|f - g\|, \tag{2.4}$$

where g is an arbitrary function in M. It follows from (2.4) that

$$\sup_{\langle p,\lambda \rangle} \left| \int_X f \mathrm{d}\mu_{p,\lambda} \right| \leq E(f), \tag{2.5}$$

where the sup is taken over all projection cycle-vector pairs of X.

Consider the general duality relation (1.1). Let μ_0 be a measure attaining the supremum in (1.1) and $\{\mu_{p,\lambda}\}$ be a net of measures of the form (2.3) approximating μ_0 in the weak* topology of $C(X)^*$. We already know that this is possible. For any $\varepsilon > 0$, there exists a measure μ_{p_0,λ_0} in $\{\mu_{p,\lambda}\}$ such that

$$\left| \int_X f \mathrm{d}\mu_0 - \int_X f \mathrm{d}\mu_{p_0,\lambda_0} \right| < \varepsilon.$$

From the last inequality we obtain that

$$\left| \int_X f \mathrm{d}\mu_{p_0,\lambda_0} \right| > \left| \int_X f \mathrm{d}\mu_0 \right| - \varepsilon = E(f) - \varepsilon.$$

Hence,

$$\sup_{\langle p,\lambda \rangle} \left| \int_X f \mathrm{d}\mu_{p,\lambda} \right| \geq E(f). \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$\sup_{\langle p,\lambda \rangle} \left| \int_X f \mathrm{d}\mu_{p,\lambda} \right| = E(f). \tag{2.7}$$

By Lemma 2.5,

$$\mu_{p,\lambda} = \sum_{i=1}^{s} t_i \mu_{l_i},$$

where l_i, $i = 1, \ldots, s$, are minimal projection cycles in p and $\sum_{i=1}^{s} t_i = 1,\ t_i > 0$. Let k be an index in the set $\{1, \ldots, s\}$ such that

$$\left|\int_X f \mathrm{d}\mu_{l_k}\right| = \max\left\{\left|\int_X f \mathrm{d}\mu_{l_i}\right|, \quad i = 1, \ldots, s\right\}.$$

Then

$$\left|\int_X f \mathrm{d}\mu_{p,\lambda}\right| \leq \left|\int_X f \mathrm{d}\mu_{l_k}\right|. \tag{2.8}$$

Now since

$$\left|\int_X f \mathrm{d}\mu_l\right| \leq E(f),$$

for any minimal cycle l, from (2.7) and (2.8) we obtain the assertion of the theorem. □

Remark 3. Theorem 2.8 not only proves Golomb's formula, but also improves it. Indeed, based on Proposition 2.3, one can easily observe that the formula (1.4) is equivalent to the formula

$$E(f) = \sup_{\langle p,\lambda\rangle}\left|\int_X f \mathrm{d}\mu_{p,\lambda}\right|,$$

where the sup is taken over all projection cycle-vector pairs $\langle p, \lambda\rangle$ of X provided that all the numbers $\lambda_i \big/ \sum_{j=1}^{m} |\lambda_j|$, $i = 1, \ldots, m$, are rational. But by Lemma 2.4, minimal projection cycles enjoy this property.

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