Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 1, February 2009, pp. 37–43.

On the zeros of a polynomial

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MS received 20 June 2007; revised 10 March 2008

Abstract. For a polynomial of degree n, we have obtained an upper bound involving coefficients of the polynomial, for moduli of its p zeros of smallest moduli, and then a refinement of the well-known Eneström–Kakeya theorem (under certain conditions).

Keywords. Upper bound; p zeros of smallest moduli; refinement; Eneström–Kakeya theorem.

1. Introduction and statement of results

For a polynomial

$$q(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n, \quad a_p \neq 0,$$

of degree n, we have the following theorem due to Pellet ([3], p. 128 of [2]), giving an upper bound for the moduli of its p zeros of smallest moduli.

Theorem A. *Let*

$$q(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n, \quad a_p \neq 0, \tag{1.1}$$

be a polynomial of degree n. If the polynomial

$$Q_p(z) = |a_0| + |a_1|z + \cdots + |a_{p-1}|z^{p-1} - |a_p|z^p + |a_{p+1}|z^{p+1} + \cdots + |a_n|z^n, \tag{1.2}$$

has two positive zeros r and R, $r < R$, then $q(z)$ has exactly p zeros in the disc

$$|z| \leq r \tag{1.3}$$

and no zero in the annular ring

$$r < |z| < R. \tag{1.4}$$

In this paper we have obtained an upper bound involving coefficients of the polynomial, for moduli of its p zeros of smallest moduli and then a refinement of the well-known Eneström–Kakeya theorem (under certain conditions). More precisely we have proved the following.

Theorem 1. *Let*

$$q(z) = a_0 + a_1 z + \cdots + a_{p-1} z^{p-1} + a_p z^p + \cdots + a_n z^n \tag{1.5}$$

be a polynomial of degree n such that $a_p \neq a_{p-1}$ for some $p \in \{1, \ldots, n\}$. Set

$$M = M_p := \sum_{j=p+1}^{n} |a_j - a_{j-1}| + |a_n| (1 \leq p \leq n-1),\, M_n := |a_n| \tag{1.6}$$

and

$$m = m_p := \sum_{j=1}^{p-1} |a_j - a_{j-1}|\, (2 \leq p \leq n),\, m_1 := 0. \tag{1.7}$$

Suppose that

$$\frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1} < 1 \tag{1.8}$$

and that

$$|a_0| + m \frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1} < \left(\frac{p}{M}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}. \tag{1.9}$$

Then $q(z)$ has at least p zeros in

$$|z| < \frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1}. \tag{1.10}$$

And then we have considered a class of polynomials for which a condition similar to (1.8) is definitely true ($p \neq n$), and the corresponding result is as follows.

Theorem 2. *Let*

$$q(z) = a_0 + a_1 z + \cdots + a_{p-1} z^{p-1} + a_p z^p + \cdots + a_n z^n \tag{1.11}$$

be a polynomial of degree n such that

$$a_p \neq a_{p-1} \text{ for some } p \in \{1, 2, \ldots, n-1\},$$

$$|\arg a_k - \beta| \leq \alpha \leq \pi/2,\, k = 0, 1, \ldots, n, \text{ for some real } \beta \text{ and } \alpha \tag{1.12}$$

and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|. \tag{1.13}$$

Set

$$L = L_p := |a_n| + (|a_n| - |a_p|)\cos\alpha + \sum_{j=p+1}^{n} (|a_j| + |a_{j-1}|)\sin\alpha, \tag{1.14}$$

and

$$l = l_p := (|a_{p-1}| - |a_0|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha \; (2 \le p \le n-1), l_1 := 0. \tag{1.15}$$

Suppose that

$$|a_0| + l\frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1} < \left(\frac{p}{L}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}. \tag{1.16}$$

Then $q(z)$ has at least p zeros in

$$|z| < \frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1}. \tag{1.17}$$

Remark 1. The condition similar to (1.8) is

$$\frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1} < 1, \tag{1.18}$$

which is definitely true, as

$$\begin{aligned} |a_p - a_{p-1}| &\le (|a_p| - |a_{p-1}|)\cos\alpha + (|a_p| + |a_{p-1}|)\sin\alpha, \\ &\qquad \text{(by Lemma 2, (1.12) and (1.13)),} \\ &\le L, \text{(by (1.14), (1.13) and (1.12)).} \end{aligned}$$

By taking

$$\alpha = \beta = 0,$$

in Theorem 2, we get the following.

COROLLARY 1

Let

$$q(z) = a_0 + a_1 z + \cdots + a_{p-1}z^{p-1} + a_p z^p + \cdots + a_n z^n \tag{1.19}$$

be a polynomial of degree n, with

$$a_n \ge a_{n-1} \ge \cdots \ge a_p > a_{p-1} \ge \cdots \ge a_1 \ge a_0 > 0, \quad 1 \le p \le n. \tag{1.20}$$

Suppose that

$$\begin{aligned} &(p+1)^p(2a_n - a_p)^{p-1}\{(p+1)a_0(2a_n - a_p) \\ &\quad + p(a_p - a_{p-1})(a_{p-1} - a_0)\} < p^p(a_p - a_{p-1})^{p+1}. \end{aligned} \tag{1.21}$$

Then $q(z)$ has at least p zeros in

$$|z| < \frac{p}{p+1}\frac{a_p - a_{p-1}}{2a_n - a_p}. \tag{1.22}$$

Remark 2. As

$$\frac{a_p - a_{p-1}}{2a_n - a_p} < 1, \quad \text{(by (1.20))},$$

we have

$$\frac{p}{p+1}\frac{a_p - a_{p-1}}{2a_n - a_p} < 1.$$

Remark 3. We have included the value

$$p = n$$

also, in Corollary 1, as under the hypotheses of Corollary 1, for

$$p = n,$$

the condition (1.18) is definitely satisfied, L and l remaining the same, as in Theorem 2, with the sum in (1.14) being assumed to be zero for

$$p = n.$$

Further, for

$$p = n,$$

Corollary 1 is a refinement of the following.

Theorem B (Eneström–Kakeya theorem) (p. 136 of [2]). *The polynomial* $q(z) = \sum_{j=0}^{n} a_j z^j$, *of degree n, with*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

has all its zeros in

$$|z| \leq 1,$$

under the conditions

$$a_n > a_{n-1} \geq a_{n-2} \geq \cdots \geq a_2 \geq a_1 \geq a_0 > 0, \tag{1.23}$$

$$(n+1)^n a_n^{n-1}\{(n+1)a_0 a_n + n(a_n - a_{n-1})(a_{n-1} - a_0)\}$$
$$< n^n (a_n - a_{n-1})^{n+1}. \tag{1.24}$$

The polynomial

$$q(z) = 6.1z^5 + 0.1z^4 + 0.05z^3 + 0.03z^2 + 0.02z + 0.01,$$

satisfies conditions (1.23) *and* (1.24), *and has all the zeros in*

$$|z| < 50/61 (\approx 0.82), \quad (\textit{by } (1.22) \textit{ with } p = n = 5),$$

a result, better than the one, obtainable by Eneström–Kakeya theorem, according to which all the zeros of $q(z)$ *will lie in*

$$|z| \leq 1.$$

Remark 4. The polynomial

$$q(z) = 6.2z^5 + 6.2z^4 + 6.19z^3 + 6.1z^2 + 0.1z + 0.01$$

satisfies the conditions of Corollary 1, with

$$p = 2,$$

and has at least 2 zeros in

$$|z| < 40/63(\approx 0.635), \text{ (by (1.22) with } p = 2 \text{ and } n = 5).$$

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

Lemma 1. *If a_j and a_{j-1} are two complex numbers with*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2,$$
$$|\arg a_{j-1} - \beta| \leq \alpha \leq \pi/2,$$

for certain real β and α, then

$$|a_j - a_{j-1}|^2 \leq (|a_j| - |a_{j-1}|)^2 \cos^2\alpha + (|a_j| + |a_{j-1}|)^2 \sin^2\alpha.$$

This lemma is due to Govil and Rahman (proof of Theorem 2 of [1]).

Lemma 2. *Under the same hypothesis, as in Lemma* 1,

$$|a_j - a_{j-1}| \leq ||a_j| - |a_{j-1}|| \cos\alpha + (|a_j| + |a_{j-1}|) \sin\alpha.$$

Proof of Lemma 2. It follows easily from Lemma 1.

3. Proofs of the theorems

Proof of Theorem 1. We wish to prove that $q(z)$ has at least p zeros in

$$|z| < \rho := \frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1}.$$

Note that $\rho < 1$ by (1.8), which implies that the polynomials $q(z)$ and $g(z) := (1-z)q(z)$ have the same zeros in $|z| < \rho$. So, we may show that $g(z)$ has at least p zeros in $|z| < \rho$.

Setting

$$\phi(z) := a_0 + \sum_{j=1}^{p-1} (a_j - a_{j-1})z^j, \; \psi(z) = (a_p - a_{p-1})z^p$$
$$+ \sum_{j=p+1}^{n} (a_j - a_{j-1})z^j - a_n z^{n+1},$$

we write $g(z) = \phi(z) + \psi(z)$.

With M and m as in (1.6) and (1.7), respectively, we easily see that for $|z| = \rho$, which is less than 1 by hypothesis, we have

$$|\psi(z)| \geq |a_p - a_{p-1}|\rho^p - \rho^{p+1}\left\{\sum_{j=p+1}^{n} |a_j - a_{j-1}|\rho^{j-(p+1)} + |a_n|\rho^{n-p}\right\},$$

$$\geq |a_p - a_{p-1}|\rho^p - \rho^{p+1}M,$$

$$= \left(\frac{p}{M}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1},$$

$$> |a_0| + m\frac{p}{M}\frac{|a_p - a_{p-1}|}{p+1}$$

by (1.9). Thus

$$|\psi(z)| > |a_0| + \rho m, \quad (|z| = \rho). \tag{3.1}$$

On the other hand, we have

$$|\phi(z)| \leq |a_0| + \sum_{j=1}^{p-1} |a_j - a_{j-1}|\rho^j \leq |a_0| + \rho m, \quad (|z| = \rho). \tag{3.2}$$

From (3.1) and (3.2) it follows that $|\psi(z)| > |\phi(z)|$ for $|z| = \rho$. Since $\psi(z)$ has at least p zeros in the disc $|z| < \rho$, so has the function $g(z) := \psi(z) + \phi(z)$, by Rouché's theorem. This is what we wanted to prove.

Proof of Theorem 2. As (1.18) is true, proof of Theorem 2 is similar to the proof of Theorem 1, with certain changes:

L instead of M,

l instead of m,

$\frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1}$ instead of $\frac{p}{M}\frac{|a_p - a_{p-1}|}{p+1}$,

where, on $|z| = \rho$,

$$|\psi(z)| \geq |a_p - a_{p-1}|\rho^p - \rho^{p+1}\left\{\sum_{j=p+1}^{n} |a_j - a_{j-1}|\rho^{j-(p+1)} + |a_n|\rho^{n-p}\right\},$$

$$\geq |a_p - a_{p-1}|\rho^p - \rho^{p+1}\left(\sum_{j=p+1}^{n} |a_j - a_{j-1}| + |a_n|\right),$$

$$\geq |a_p - a_{p-1}|\rho^p - \rho^{p+1}\left\{\sum_{j=p+1}^{n}(|a_j| - |a_{j-1}|)\cos\alpha\right.$$

$$\left.+\sum_{j=p+1}^{n}(|a_j|+|a_{j-1}|)\sin\alpha+|a_n|\right\}, \quad \text{(by Lemma 2 and eq. (1.13))},$$

$$= |a_p - a_{p-1}|\rho^p - \rho^{p+1}L, \quad \text{(by eq. (1.14))},$$

and

$$|\phi(z)| \leq |a_0| + \sum_{j=1}^{p-1}|a_j - a_{j-1}|\rho^j,$$

$$\leq |a_0| + \rho\left(\sum_{j=1}^{p-1}|a_j - a_{j-1}|\right),$$

$$= |a_0| + \rho\left\{\sum_{j=1}^{p-1}(|a_j| - |a_{j-1}|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha\right\},$$

(by Lemma 2 and eq. (1.13)),

$$= |a_0| + \rho l, \quad \text{(by eq. (1.15))}.$$

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