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Vanishing of the top local cohomology modules over Noetherian rings

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Abstract. Let R be a (not necessarily local) Noetherian ring and M a finitely generated R-module of finite dimension d. Let $\mathfrak{a}$ be an ideal of R and $\mathfrak{M}$ denote the intersection of all prime ideals $\mathfrak{p} \in \operatorname{Supp}_R H_{\mathfrak{a}}^d(M)$. It is shown that

$$H_{\mathfrak{a}}^d(M) \simeq H_{\mathfrak{M}}^d(M)/\sum_{n\in\mathbb{N}} \langle\mathfrak{M}\rangle(0:_{H_{\mathfrak{M}}^d(M)} \mathfrak{a}^n),$$

where for an Artinian R-module A we put $\langle\mathfrak{M}\rangle A = \cap_{n\in\mathbb{N}} \mathfrak{M}^n A$. As a consequence, it is proved that for all ideals $\mathfrak{a}$ of R, there are only finitely many non-isomorphic top local cohomology modules $H_{\mathfrak{a}}^d(M)$ having the same support. In addition, we establish an analogue of the Lichtenbaum–Hartshorne vanishing theorem over rings that need not be local.

Keywords. Artinian modules; attached prime ideals; cohomological dimension; formally isolated; local cohomology; secondary representations.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring. Let M be a finitely generated R-module of finite dimension d and $\mathfrak{a}$ an ideal of R. The present article is concerned with the top local cohomology module $H_{\mathfrak{a}}^d(M)$. We refer the reader to [3] for more details about local cohomology. By Grothendieck's vanishing theorem (Theorem 6.1.2 of [3]), it is known that $H_{\mathfrak{a}}^i(M) = 0$ for all $i > \dim M$. So $H_{\mathfrak{a}}^d(M)$ is the last possible non-vanishing local cohomology module of M. Also, by Exercise 7.1.7 of [3] the top local cohomology module $H_{\mathfrak{a}}^d(M)$ is Artinian. There are many papers concerning the top local cohomology modules of finitely generated modules over local rings. But, as per the knowledge of the author, [2] and [4] are the only existing articles studying such local cohomology modules over general Noetherian rings. In this paper, we investigate the structure of the top local cohomology modules of finitely generated modules over rings that need not be local.

When R is local with maximal ideal $\mathfrak{m}$, it is proved that there is a natural isomorphism $H_{\mathfrak{a}}^d(M) \simeq H_{\mathfrak{m}}^d(M)/\Sigma_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0:_{H_{\mathfrak{m}}^d(M)} \mathfrak{a}^n)$, see Theorem 3.2 of [10]. As a result, in [10] a new proof is provided for the Lichtenbaum–Hartshorne vanishing theorem. In §2, we establish an analogue of the above isomorphism over rings that are not necessarily local. To be more precise, we will prove that if $\mathfrak{M}$ denotes the intersection of all prime ideals $\mathfrak{p} \in \operatorname{Supp}_R H_{\mathfrak{a}}^d(M)$, then there is a natural isomorphism

$$H_{\mathfrak{a}}^d(M) \simeq H_{\mathfrak{M}}^d(M)/\sum_{n\in\mathbb{N}}\langle\mathfrak{M}\rangle(0:_{H_{\mathfrak{M}}^d(M)}\mathfrak{a}^n).$$

This will be proved in Theorem 2.3.

Knowing more about $\operatorname{Att}_R H_{\mathfrak{a}}^d(M)$, the set of attached primes of $H_{\mathfrak{a}}^d(M)$, could lead to a better understanding of the structure of the top local cohomology module $H_{\mathfrak{a}}^d(M)$. In particular, knowing $\operatorname{Att}_R H_{\mathfrak{a}}^d(M)$ implies vanishing results for $H_{\mathfrak{a}}^d(M)$. In case R is local, the set $\operatorname{Att}_R H_{\mathfrak{a}}^d(M)$ is already determined (see e.g. [18, 10, 6]). In Theorem 2.5 below, we determine the set $\operatorname{Att}_R H_{\mathfrak{a}}^d(M)$ without the assumption that R is local, namely we show that

$$\operatorname{Att}_R H_{\mathfrak{a}}^d(M) = \{\mathfrak{p} \in \operatorname{Assh}_R M\colon\ \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

(Here for an R-module N, $\operatorname{cd}_R(\mathfrak{a}, N)$ denotes the cohomological dimension of N with respect to the ideal $\mathfrak{a}$.) Then as an application, we provide an improvement of the main result of [2]. Next, for a finitely generated R-module N so that $H_{\mathfrak{a}}^c(N)$, $c := \operatorname{cd}_R(\mathfrak{a}, N)$, is representable, we examine the set $\operatorname{Att}_R H_{\mathfrak{a}}^c(N)$.

In §3, first we show that for all ideals $\mathfrak{a}$ of R, there are only finitely many non-isomorphic top local cohomology modules $H_{\mathfrak{a}}^d(M)$ having the same support. Next, as an application of Theorems 2.3 and 2.5, we extend the Lichtenbaum–Hartshorne vanishing theorem to (not necessarily local) Noetherian rings. Namely, we prove that if $\mathfrak{M}$ is as above and T denotes the $\mathfrak{M}$-adic completion of R, then the following are equivalent:

(i) $H_{\mathfrak{a}}^d(M) = 0$.
(ii) $H_{\mathfrak{M}}^d(M) = \sum_{n\in\mathbb{N}}\langle\mathfrak{M}\rangle(0:_{H_{\mathfrak{M}}^d(M)}\mathfrak{a}^n)$.
(iii) For any integer $l \in \mathbb{N}$, there exists an $n = n(l) \in \mathbb{N}$ such that

$$0:_{H_{\mathfrak{M}}^d(M)}\mathfrak{a}^l \subseteq \langle\mathfrak{M}\rangle(0:_{H_{\mathfrak{M}}^d(M)}\mathfrak{a}^n).$$

(iv) $\dim T/\mathfrak{a}T + \mathfrak{p} > 0$ for all $\mathfrak{p} \in \operatorname{Assh}_T(M \otimes_R T)$.
(v) $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) < d$ for all $\mathfrak{p} \in \operatorname{Assh}_R M$.

Throughout the paper, for an R-module M, $\operatorname{Assh}_R M$ denotes the set of all associated prime ideals $\mathfrak{p}$ of M such that $\dim R/\mathfrak{p} = \dim M$. Also, for an Artinian R-module A, we denote $\bigcap_{n\in\mathbb{N}}\mathfrak{a}^n A$ by $\langle\mathfrak{a}\rangle A$.

2. Attached prime ideals

A nonzero R-module S is called *secondary* if for each $x \in R$ the multiplication map induced by x on S is either surjective or nilpotent. If S is secondary, then the ideal $\mathfrak{p} := \operatorname{Rad}(\operatorname{Ann}_R S)$ is a prime ideal and S is called $\mathfrak{p}$-secondary. For an R-module M, a secondary representation of M is an expression for M as a sum of finitely many secondary submodules of M. An R-module M is said to be *representable* if it has a secondary representation. From any secondary representation for an R-module M, one can obtain other one, say $M = S_1 + \cdots + S_n$ such that the prime ideals $\mathfrak{p}_i := \operatorname{Rad}(\operatorname{Ann}_R S_i)$, $i = 1, \ldots, n$ are all distinct and $S_j \nsubseteq \Sigma_{i\neq j} S_i$ for all $j = 1, \ldots, n$. Such a secondary representation for M is said to be minimal. It is shown that set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the chosen minimal secondary representation for M. This set is denoted by $\mathrm{Att}_R\, M$ and each element of this set is said to be an attached prime ideal of M. It is known that a representable R-module M is zero if and only if $\mathrm{Att}_R\, M = \varnothing$ and that if $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ is an exact sequence of representable R-modules and R-homomorphisms, then $\mathrm{Att}_R\, L \subseteq \mathrm{Att}_R\, M \subseteq \mathrm{Att}_R\, N \cup \mathrm{Att}_R\, L$. Also, it is known that any Artinian R-module is representable. For more information about the theory of secondary representations, see [12] or §6, Appendix of [14].

Lemma 2.1.

(i) *Let* $f\colon R \longrightarrow U$ *be a ring homomorphism and* M *a representable* U*-module. Then* M *is also representable as an* R*-module and* $\mathrm{Att}_R\, M = \{f^{-1}(\mathfrak{p})\colon \mathfrak{p} \in \mathrm{Att}_U\, M\}$.

(ii) *Let* A *be an Artinian* R*-module. Then* $\mathrm{Supp}_R\, A$ *equals* $\mathrm{Ass}_R\, A$ *and is a finite subset of* $\mathrm{Max}\, R$. *Moreover, if* $\mathrm{Supp}_R\, A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$, *then the natural* R*-homomorphism* $\psi\colon A \longrightarrow \oplus_{i=1}^t A_{\mathfrak{m}_i}$ *is an isomorphism. In particular,* $\mathrm{Att}_R\, A = \bigcup_{i=1}^t \mathrm{Att}_R\, A_{\mathfrak{m}_i}$.

(iii) *Let* $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ *be distinct maximal ideals of* R *and* $A_1, \ldots, A_t$ *Artinian* R*-modules such that* $\mathrm{Supp}_R\, A_i = \{\mathfrak{m}_i\}$ *for all* $i = 1, \ldots, t$. *Let* $A = \oplus_{i=1}^t A_i$. *Then for any ideal* $\mathfrak{a}$ *of* R *such that* $\mathfrak{a} \subseteq \mathfrak{M} := \cap_{i=1}^t \mathfrak{m}_i$, *there is a natural isomorphism*

$$\frac{A}{\sum_{n\in\mathbb{N}} \langle \mathfrak{M} \rangle (0 :_A \mathfrak{a}^n)} \simeq \bigoplus_{i=1}^t \frac{A_i}{\sum_{n\in\mathbb{N}} \langle \mathfrak{m}_i \rangle (0 :_{A_i} \mathfrak{a}^n)}.$$

Proof.

(i) holds by Proposition 4.1 of [15].

(ii) The first assertion of (ii) holds by Exercises 8.49 and 9.43 of [17]. Now, we are going to prove the second assertion of (ii). It follows by Exercise 8.49 of [17], that $A = \oplus_{i=1}^t \Gamma_{\mathfrak{m}_i}(A)$. This yields that for each i, $A_{\mathfrak{m}_i} \simeq \Gamma_{\mathfrak{m}_i}(A)$, and so $A_{\mathfrak{m}_i}$, as an R-module, is supported only at the maximal ideal $\mathfrak{m}_i$. So $\psi_{\mathfrak{m}}\colon A_{\mathfrak{m}} \longrightarrow (\oplus_{i=1}^t A_{\mathfrak{m}_i})_{\mathfrak{m}}$ is an isomorphism for any maximal ideal $\mathfrak{m}$ of R. Thus ψ is an isomorphism, as claimed. Finally, the last assertion of (ii) is immediate by (i) and the fact that for any given finitely many secondary representable R-modules $M_1, \ldots, M_t$, the direct sum $\oplus_{i=1}^t M_i$ is also representable and

$$\mathrm{Att}_R \left(\bigoplus_{i=1}^t M_i \right) = \bigcup_{i=1}^t \mathrm{Att}_R\, M_i.$$

(iii) First note that $A_{\mathfrak{m}_i} \simeq A_i$ for all $i = 1, \ldots, t$. If $\{B_i\}_{i\in\mathbb{N}}$ is an ascending chain of submodules of an R-module B, then the direct limit of $\{B/B_i\}_{i\in\mathbb{N}}$ is $B/\sum_{i\in\mathbb{N}} B_i$. Thus in view of (ii), we have the following isomorphisms:

$$\begin{aligned}\frac{A}{\sum_{n\in\mathbb{N}} \langle \mathfrak{M} \rangle (0 :_A \mathfrak{a}^n)} &\simeq \varinjlim_n \frac{A}{\langle \mathfrak{M} \rangle (0 :_A \mathfrak{a}^n)} \\ &\simeq \varinjlim_n \left[\left(\frac{A}{\langle \mathfrak{M} \rangle (0 :_A \mathfrak{a}^n)} \right)_{\mathfrak{m}_1} \oplus \cdots \oplus \left(\frac{A}{\langle \mathfrak{M} \rangle (0 :_A \mathfrak{a}^n)} \right)_{\mathfrak{m}_t} \right]\end{aligned}$$

$$\simeq \varinjlim_n \left[\frac{A_1}{\langle \mathfrak{m}_1 \rangle (0:_{A_1} \mathfrak{a}^n)} \oplus \cdots \oplus \frac{A_t}{\langle \mathfrak{m}_t \rangle (0:_{A_t} \mathfrak{a}^n)} \right]$$

$$\simeq \bigoplus_{i=1}^{t} \left[\varinjlim_n \frac{A_i}{\langle \mathfrak{m}_i \rangle (0:_{A_i} \mathfrak{a}^n)} \right]$$

$$\simeq \bigoplus_{i=1}^{t} \frac{A_i}{\sum_{n \in \mathbb{N}} \langle \mathfrak{m}_i \rangle (0:_{A_i} \mathfrak{a}^n)}.$$

■

Remark 2.2.

(i) Let $\mathfrak{a}$ be an ideal of R. For a prime ideal $\mathfrak{p}$ of R, we say that $\mathfrak{a}$ is *formally isolated at* $\mathfrak{p}$ if $\mathfrak{a} \subseteq \mathfrak{p}$ and if there is some prime ideal $\mathfrak{p}^*$ of $\hat{R}_{\mathfrak{p}}$ such that $\dim \hat{R}_{\mathfrak{p}}/\mathfrak{p}^* = ht(\mathfrak{p})$ and $\dim \hat{R}_{\mathfrak{p}}/\mathfrak{a}\hat{R}_{\mathfrak{p}} + \mathfrak{p}^* = 0$. Assume that R has finite dimension d, and let $\mathcal{P}_{\mathfrak{a}}$ denote the set of all prime ideals $\mathfrak{p}$ such that $ht(\mathfrak{p}) = d$ and such that $\mathfrak{a}$ is formally isolated at $\mathfrak{p}$. Then, by Theorem 3.3(b) of [2] for any finitely generated faithful R-module M, we have $\operatorname{Supp}_R H^d_{\mathfrak{a}}(M) = \mathcal{P}_{\mathfrak{a}}$.

(ii) Let M be a finitely generated R-module of finite dimension d. Let $\mathcal{P}_{\mathfrak{a},M}$ denote the set of all $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R M + \mathfrak{a})$ such that there is some prime $\mathfrak{p}^* \in \operatorname{Supp}_{\hat{R}_{\mathfrak{p}}} \hat{M}_{\mathfrak{p}}$ such that $\dim \hat{R}_{\mathfrak{p}}/\mathfrak{p}^* = d$ and $\dim \hat{R}_{\mathfrak{p}}/\mathfrak{a}\hat{R}_{\mathfrak{p}} + \mathfrak{p}^* = 0$. Then, by adapting the method of the proof of Theorem 3.3(b) of [2], one can easily deduce that $\operatorname{Supp}_R H^d_{\mathfrak{a}}(M) = \mathcal{P}_{\mathfrak{a},M}$. Also, in Corollary 4.1 below, we establish another characterization of $\mathcal{P}_{\mathfrak{a},M}$.

In the remainder of the paper, for a finitely generated R-module M of finite dimension d and an ideal $\mathfrak{a}$ of R, let $\mathcal{P}_{\mathfrak{a},M}$ be as in Remark 2.2(ii).

Theorem 2.3. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module of finite dimension d and $\mathfrak{M} = \bigcap_{\mathfrak{p} \in \mathcal{P}_{\mathfrak{a},M}} \mathfrak{p}$. There is a natural isomorphism*

$$H^d_{\mathfrak{a}}(M) \simeq H^d_{\mathfrak{M}}(M) \Big/ \sum_{n \in \mathbb{N}} \langle \mathfrak{M} \rangle (0:_{H^d_{\mathfrak{M}}(M)} \mathfrak{a}^n).$$

Proof. By Remark 2.2(ii), we have $\operatorname{Supp}_R H^d_{\mathfrak{a}}(M) = \mathcal{P}_{\mathfrak{a},M}$. Let $\operatorname{Supp}_R H^d_{\mathfrak{a}}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and for each i denote the local ring $R_{\mathfrak{m}_i}$ by R_i.

Let $\mathfrak{a}$ be an ideal of a local ring $(U, \mathfrak{n})$. By Theorem 3.2 of [10], it turns out that for any finitely generated U-module M, there is a natural isomorphism

$$H^d_{\mathfrak{a}}(M) \simeq H^d_{\mathfrak{n}}(M) \Big/ \sum_{n \in \mathbb{N}} \langle \mathfrak{n} \rangle (0:_{H^d_{\mathfrak{n}}(M)} \mathfrak{a}^n),$$

where $d = \dim M$. Observe that by the flat base change theorem (Theorem 4.3.2 of [3]) and Lemma 2.1(ii) the modules $H^d_{\mathfrak{m}_i R_i}(M_{\mathfrak{m}_i})$ and $H^d_{\mathfrak{m}_i}(M)$ are isomorphic for all $1 \leq i \leq t$. Therefore applying Lemma 2.1(ii) again, provides the following isomorphisms:

$$H^d_{\mathfrak{a}}(M) \simeq \bigoplus_{i=1}^{t} H^d_{\mathfrak{a}R_i}(M_{\mathfrak{m}_i})$$

$$\simeq \bigoplus_{i=1}^{t} \frac{H^d_{\mathfrak{m}_i R_i}(M_{\mathfrak{m}_i})}{\sum_{n\in\mathbb{N}} \langle \mathfrak{m}_i R_i\rangle (0:_{H^d_{\mathfrak{m}_i R_i}(M_{\mathfrak{m}_i})} \mathfrak{a}^n R_i)}$$

$$\simeq \bigoplus_{i=1}^{t} \frac{H^d_{\mathfrak{m}_i}(M)}{\sum_{n\in\mathbb{N}} \langle \mathfrak{m}_i\rangle (0:_{H^d_{\mathfrak{m}_i}(M)} \mathfrak{a}^n)}.$$

On the other hand, the Mayer–Vietoris sequence for local cohomology (Theorem 3.2.3 of [3]) yields the following isomorphism:

$$H^d_{\mathfrak{M}}(M) \simeq \bigoplus_{i=1}^{t} H^d_{\mathfrak{m}_i}(M).$$

This finishes the proof, by Lemma 2.1(iii). ■

Recall that for an R-module M, the cohomological dimension of M with respect to an ideal $\mathfrak{a}$ of R is defined as $\operatorname{cd}_R(\mathfrak{a}, M) := \sup\{i \in \mathbb{N}_0\colon H^i_{\mathfrak{a}}(M) \neq 0\}$. It is appropriate to list some basic properties of this notion. First of all note that, it is immediate by Grothendieck's vanishing theorem, that $\operatorname{cd}_R(\mathfrak{a}, M) \leq \dim M$. Next, note that if V is a multiplicative subset of R, then it becomes clear by the flat base change theorem, that $\operatorname{cd}_{V^{-1}R}(\mathfrak{a}V^{-1}R, V^{-1}M) \leq \operatorname{cd}_R(\mathfrak{a}, M)$. Also, if M and L are two finitely generated R-modules such that $\operatorname{Supp}_R L \subseteq \operatorname{Supp}_R M$, then Theorem 2.2 of [9] implies that $\operatorname{cd}_R(\mathfrak{a}, L) \leq \operatorname{cd}_R(\mathfrak{a}, M)$. For further details concerning this notion, we refer the reader to [11] and [9].

Lemma 2.4. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* d *a natural number. For any prime ideal* $\mathfrak{p}$ *of* R *such that* $\dim R/\mathfrak{p} \leq d$, *the following are equivalent*:

(i) $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d$.
(ii) $\mathfrak{p}$ *is the contraction to* R *of a prime ideal* $\mathfrak{p}^*$ *of* $\hat{R}$ *such that* $\dim \hat{R}/\mathfrak{p}^* = d$ *and* $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}^* = 0$.

Proof. Let M be a finitely generated R-module of dimension d. Then by the Lichtenbaum–Hartshorne vanishing theorem, it turns out that $H^d_{\mathfrak{a}}(M) \neq 0$ if and only if there exists $\mathfrak{p}^* \in \operatorname{Assh}_{\hat{R}} \hat{M}$ such that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}^* = 0$ (see e.g. Corollary 3.4 of [10]). Assume that (i) holds. Then $H^d_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$, and so there exists $\mathfrak{p}^* \in \operatorname{Assh}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$ such that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}^* = 0$. Since $H^d_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$, by Grothendieck's vanishing theorem, we have $\dim R/\mathfrak{p} = d$. Thus

$$\dim \hat{R}/\mathfrak{p}^* = \dim \hat{R}/\mathfrak{p}\hat{R} = d.$$

On the other hand, by Theorem 23.2(i) of [14], we have

$$\{\mathfrak{p}\} = \operatorname{Ass}_R(R/\mathfrak{p}) = \{Q \cap R\colon Q \in \operatorname{Ass}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})\}.$$

Hence $\mathfrak{p} = \mathfrak{p}^* \cap R$, and so (ii) follows.

Now, assume that (ii) holds. We have

$$d \geq \dim R/\mathfrak{p} = \dim \hat{R}/\mathfrak{p}\hat{R} \geq \dim \hat{R}/\mathfrak{p}^* = d.$$

So $\dim R/\mathfrak{p} = d$. In particular, $\mathfrak{p}^*$ is minimal over $\mathfrak{p}\hat{R}$, and so $\mathfrak{p}^* \in \operatorname{Assh}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$. Thus $H^d_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$, by the Lichtenbaum–Hartshorne vanishing theorem (its statement being commented in the beginning of the proof). Therefore $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d$, as required. ■

The following extends the main result of [6] to general Noetherian rings.

Theorem 2.5 (see Theorem 1.2 of [4]). *Let* $\mathfrak{a}$ *be an ideal of* R *and* M *a finitely generated* R*-module of finite dimension* d*. Then*

$$\operatorname{Att}_R H^d_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{Assh}_R M\colon \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d\}.$$

Proof. Assume that $\operatorname{Supp}_R H^d_{\mathfrak{a}}(M) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$. Then by the flat base change theorem and Lemma 2.1(ii), it follows that

$$\operatorname{Att}_R H^d_{\mathfrak{a}}(M) = \bigcup_{i=1}^{t} \operatorname{Att}_R H^d_{\mathfrak{a}R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}).$$

In the remainder of the proof, we will use this equality without further comment.

Let M be a finitely generated module over a local ring $(U, \mathfrak{n})$. Then by Corollary 3.3 of [10] for any ideal $\mathfrak{a}$ of U, $\operatorname{Att}_{\hat{U}} H^{\dim M}_{\mathfrak{a}}(M)$ consists of all $\mathfrak{p} \in \operatorname{Assh}_{\hat{U}} \hat{M}$ such that $\dim \hat{U}/\mathfrak{a}\hat{U} + \mathfrak{p} = 0$. Fix $1 \le i \le t$. Since

$$H^d_{\mathfrak{a}R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \simeq (H^d_{\mathfrak{a}}(M))_{\mathfrak{m}_i} \neq 0,$$

we have $\dim M_{\mathfrak{m}_i} = d$. It now follows, by Lemma 2.1(i) and Lemma 2.4 that

$$\begin{aligned} &\operatorname{Att}_{R_{\mathfrak{m}_i}} H^d_{\mathfrak{a}R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \\ &\quad = \{Q \cap R_{\mathfrak{m}_i}\colon Q \in \operatorname{Assh}_{\hat{R}_{\mathfrak{m}_i}} \hat{M}_{\mathfrak{m}_i}, \dim \hat{R}_{\mathfrak{m}_i}/\mathfrak{a}\hat{R}_{\mathfrak{m}_i} + Q = 0\} \\ &\quad = \{\mathfrak{p}R_{\mathfrak{m}_i} \in \operatorname{Assh}_{R_{\mathfrak{m}_i}} M_{\mathfrak{m}_i}\colon \operatorname{cd}_{R_{\mathfrak{m}_i}}(\mathfrak{a}R_{\mathfrak{m}_i}, R_{\mathfrak{m}_i}/\mathfrak{p}R_{\mathfrak{m}_i}) = d\}. \end{aligned}$$

Because $\dim M_{\mathfrak{m}_i} = \dim M = d$ and

$$\operatorname{Ass}_{R_{\mathfrak{m}_i}} M_{\mathfrak{m}_i} = \{\mathfrak{p}R_{\mathfrak{m}_i}\colon \mathfrak{p} \subseteq \mathfrak{m}_i \text{ and } \mathfrak{p} \in \operatorname{Ass}_R M\},$$

it follows that $\operatorname{Assh}_{R_{\mathfrak{m}_i}} M_{\mathfrak{m}_i}$ consists of all prime ideals $\mathfrak{p}R_{\mathfrak{m}_i} \in \operatorname{Ass}_{R_{\mathfrak{m}_i}} M_{\mathfrak{m}_i}$ such that $\mathfrak{p} \in \operatorname{Assh}_R M$. Hence, if $\mathfrak{p} \in \operatorname{Att}_R H^d_{\mathfrak{a}}(M)$, then $\mathfrak{p} \in \operatorname{Assh}_R M$ and $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d$.

Conversely, assume that $\mathfrak{p} \in \operatorname{Assh}_R M$ is such that $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d$. Let $\mathfrak{m} \in \operatorname{Supp}_R H^d_{\mathfrak{a}}(R/\mathfrak{p})$. Then $H^d_{\mathfrak{a}R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}) \neq 0$, and so $\dim R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}} = d$. Hence, we have $\operatorname{cd}_{R_{\mathfrak{m}}}(\mathfrak{a}R_{\mathfrak{m}}, R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}) = d$ and $\mathfrak{p}R_{\mathfrak{m}} \in \operatorname{Assh}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$. By Lemma 2.4, $\mathfrak{p}R_{\mathfrak{m}}$ is the contraction to $R_{\mathfrak{m}}$ of a prime ideal $\mathfrak{p}^*$ of $\hat{R}_{\mathfrak{m}}$ such that $\dim \hat{R}_{\mathfrak{m}}/\mathfrak{p}^* = d$ and $\dim \hat{R}_{\mathfrak{m}}/\mathfrak{a}\hat{R}_{\mathfrak{m}} + \mathfrak{p}^* = 0$. It is easy to see that $\mathfrak{p}^* \in \operatorname{Assh}_{\hat{R}_{\mathfrak{m}}} \hat{M}_{\mathfrak{m}}$, and so by Lemma 2.1(i) and the above mentioned result of [10], it turns out that $\mathfrak{p}R_{\mathfrak{m}} \in \operatorname{Att}_{R_{\mathfrak{m}}} H^d_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$. Hence $\mathfrak{p} \in \operatorname{Att}_R H^d_{\mathfrak{a}}(M)$, by using Lemma 2.1(i) again. Note that, since $\operatorname{Att}_R H^d_{\mathfrak{a}}(M)_{\mathfrak{m}}$ is not empty, it follows that $\mathfrak{m} \in \operatorname{Supp}_R H^d_{\mathfrak{a}}(M)$. ■

Example 2.6. In Corollary 3.3 of [8], the fact that the top local cohomology modules of finitely generated modules of finite dimension are Artinian is extended to a strictly larger class of modules. Namely, it is shown that if $\mathfrak{a}$ is an ideal of R and M a ZD-module of finite dimension d such that the $\mathfrak{a}$-relative Goldie dimension of any quotient of M is finite, then $H^d_{\mathfrak{a}}(M)$ is Artinian. It would be interesting to know whether the conclusion of Theorem 2.5 remains valid for this larger class of modules. Unfortunately, this is not the case, even if R is local. To this end, let $(R, \mathfrak{m})$ be a local ring with $\dim R > 0$. Take $\mathfrak{a} = \mathfrak{m}$ and $M = E(R/\mathfrak{m})$, the injective envelope of the residue field of R. Then M is a ZD-module and the $\mathfrak{a}$-relative Goldie dimension of any quotient of M is finite. We have

$$\mathrm{Att}_R\, H^0_{\mathfrak{a}}(M) = \mathrm{Att}_R\, M = \mathrm{Ass}_R\, R,$$

while the maximal ideal $\mathfrak{m}$ is the only element of the set

$$\{\mathfrak{p} \in \mathrm{Assh}_R\, M\colon\ \mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = 0\}.$$

As a corollary to Theorem 2.5, we present an improvement of the main result of [2]. In the sequel, let $\mathcal{P}_{\mathfrak{a}}$ be as in Remark 2.2(i).

COROLLARY 2.7

Let $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* R *and assume that* $R/\mathfrak{b}$ *has finite dimension* d. *Then*

(i) $\mathcal{P}_{\mathfrak{a},R/\mathfrak{b}} = \{\mathfrak{m} \in \mathrm{Max}\, R\colon \exists \mathfrak{p} \in \mathrm{Assh}_R(R/\mathfrak{b})$ *such that* $\mathfrak{p} \subseteq \mathfrak{m}$ *and* $\mathrm{cd}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) = d\}$. *In particular, if* R *has finite dimension* d, *then*

$$\mathcal{P}_{\mathfrak{a}} = \{\mathfrak{m} \in \mathrm{Max}\, R\colon \exists \mathfrak{p} \in \mathrm{Assh}_R\, R \text{ such that } \mathfrak{p} \subseteq \mathfrak{m} \text{ and}$$
$$\mathrm{cd}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) = d\}.$$

(ii) *For any finitely generated* R*-module* M *such that* $\mathrm{Assh}_R\, M = \mathrm{Assh}_R(R/\mathfrak{b})$, *we have* $\mathrm{Supp}_R\, H^d_{\mathfrak{a}}(M) = \mathcal{P}_{\mathfrak{a},R/\mathfrak{b}}$. *In particular,* $\mathcal{P}_{\mathfrak{a},R/\mathfrak{b}}$ *is a finite set.*

(iii) *If* $d > 0$ (*see Theorem* 1.3(*g*) *of* [4]), *then for any* M *as in* (ii), *the* $R_\mathfrak{m}$*-module* $(H^d_{\mathfrak{a}}(M))_\mathfrak{m}$ *is not finitely generated for all* $\mathfrak{m} \in \mathcal{P}_{\mathfrak{a},R/\mathfrak{b}}$.

Proof. First, it should be noted that $\mathcal{P}_{\mathfrak{a}} = \mathcal{P}_{\mathfrak{a},R}$. By Remark 2.2(ii), we have $\mathrm{Supp}_R\, H^d_{\mathfrak{a}}(R/\mathfrak{b}) = \mathcal{P}_{\mathfrak{a},R/\mathfrak{b}}$. Hence, to prove (i) and (ii), it will be enough to show that for any finitely generated R-module M with $\mathrm{Assh}_R\, M = \mathrm{Assh}_R(R/\mathfrak{b})$, $\mathrm{Supp}_R\, H^d_{\mathfrak{a}}(M)$ consists of all maximal ideals $\mathfrak{m}$ of R such that there exists a prime ideal $\mathfrak{p} \in \mathrm{Assh}_R(R/\mathfrak{b})$ with $\mathfrak{p} \subset \mathfrak{m}$ and $\mathrm{cd}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) = d$. Assume that M is a finitely generated R-module with $\mathrm{Assh}_R\, M = \mathrm{Assh}_R(R/\mathfrak{b})$, and let $\mathfrak{m} \in \mathrm{Supp}_R\, H^d_{\mathfrak{a}}(M)$. Then $H^d_{\mathfrak{a}R_\mathfrak{m}}(M_\mathfrak{m}) \neq 0$, and so by Theorem 2.5, there exists a prime ideal $Q \in \mathrm{Assh}_{R_\mathfrak{m}}\, M_\mathfrak{m}$ such that $\mathrm{cd}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, R_\mathfrak{m}/Q) = d$. But, then there exists a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ of R such that $Q = \mathfrak{p}R_\mathfrak{m}$. As we have seen in the proof of Theorem 2.5, $Q \in \mathrm{Assh}_{R_\mathfrak{m}}\, M_\mathfrak{m}$, implies that

$$\mathfrak{p} \in \mathrm{Assh}_R\, M = \mathrm{Assh}_R(R/\mathfrak{b}).$$

Conversely, let $\mathfrak{m}$ be a maximal ideal of R such that there exists a prime ideal $\mathfrak{p} \in \mathrm{Assh}_R(R/\mathfrak{b})$ with $\mathfrak{p} \subset \mathfrak{m}$ and $\mathrm{cd}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) = d$. Since $\mathrm{Var}(\mathfrak{p}R_\mathfrak{m}) \subseteq \mathrm{Supp}_{R_\mathfrak{m}}\, M_\mathfrak{m}$, by Theorem 2.2 of [9], it turns out that $\mathrm{cd}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, M_\mathfrak{m}) = d$. But, this implies that $\mathfrak{m} \in \mathrm{Supp}_R\, H^d_{\mathfrak{a}}(M)$.

To prove (iii), let $\mathfrak{m} \in \mathcal{P}_{\mathfrak{a},R/\mathfrak{b}}$. Then by part (ii), we deduce that $H^d_{\mathfrak{a}R_\mathfrak{m}}(M_\mathfrak{m}) \neq 0$. Hence, Lemma 2.1 of [2] yields that the $R_\mathfrak{m}$-module $(H^d_{\mathfrak{a}}(M))_\mathfrak{m}$ is not finitely generated. ■

Remark 2.8.

(i) Let M and N be two finitely generated R-modules of finite dimension d so that $\mathrm{Assh}_R\, N = \mathrm{Assh}_R\, M$. Having in mind Theorem 2.5, it becomes clear that $\mathrm{Att}_R\, H^d_{\mathfrak{a}}(N) = \mathrm{Att}_R\, H^d_{\mathfrak{a}}(M)$. Also, it follows by Corollary 2.7(ii) that $\mathrm{Supp}_R\, H^d_{\mathfrak{a}}(N) = \mathrm{Supp}_R\, H^d_{\mathfrak{a}}(M)$. In particular, $H^d_{\mathfrak{a}}(N) = 0$ if and only if $H^d_{\mathfrak{a}}(M) = 0$.

(ii) Let R be a ring of finite dimension d and $\mathfrak{a}$ an ideal of R. Also, let M be a finitely generated R-module. If M is faithful, then it follows by Theorem 3.3(b) of [2] that $\operatorname{Supp}_R H^d_{\mathfrak{a}}(M) = \mathcal{P}_{\mathfrak{a}}$. It is perhaps worth pointing out that by part (i), this conclusion for M remains valid under the weaker assumption that $\operatorname{Assh}_R M = \operatorname{Assh}_R R$.

The following lemma will be needed in the proof of our last result in this section.

Lemma 2.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and c a natural number. Assume that M is a finitely generated R-module such that* $\operatorname{cd}_R(\mathfrak{a}, M) \leq c$. *Then there is a natural isomorphism*

$$H^c_{\mathfrak{a}}(M/\mathfrak{b}M) \simeq H^c_{\mathfrak{a}}(M)/\mathfrak{b}H^c_{\mathfrak{a}}(M).$$

Proof. Let $U = R/\operatorname{Ann}_R M$. Since $\operatorname{Supp}_R U = \operatorname{Supp}_R M$, it follows by Theorem 2.2 of [9], that $H^i_{\mathfrak{a}U}(U) = 0$ for all $i > c$. Hence $H^c_{\mathfrak{a}U}(\cdot)$ is a right exact functor on the category of U-modules and U-homomorphisms. Thus

$$\begin{aligned} H^c_{\mathfrak{a}}(M/\mathfrak{b}M) &\simeq H^c_{\mathfrak{a}U}(U) \otimes_U M/\mathfrak{b}M \\ &\simeq (H^c_{\mathfrak{a}U}(U) \otimes_U M) \otimes_R R/\mathfrak{b} \\ &\simeq H^c_{\mathfrak{a}}(M)/\mathfrak{b}H^c_{\mathfrak{a}}(M). \end{aligned}$$

■

Theorem 2.10. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module such that $c := \operatorname{cd}_R(\mathfrak{a}, M) \neq -\infty$. Let $\mathfrak{W}$ be the set of all $\mathfrak{p} \in \operatorname{Supp}_R M$ such that* $\dim R/\mathfrak{p} = \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = c$ *and* $\mathfrak{X} := \mathfrak{W} \cap \operatorname{Ass}_R M$.

(i) *If* $\mathfrak{b} := \bigcap_{\mathfrak{p}\in\mathfrak{X}}\mathfrak{p}$, *then* $\mathcal{P}_{\mathfrak{a},R/\mathfrak{b}} \subseteq \operatorname{Supp}_R H^c_{\mathfrak{a}}(M)$.
(ii) *If $H^c_{\mathfrak{a}}(M)$ is representable, then* $\mathfrak{X} \subseteq \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$.
(iii) *Assume that $H^c_{\mathfrak{a}}(M)$ is representable. If* $\mathfrak{p} \in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ *is such that* $\dim R/\mathfrak{p} = c$, *then* $\mathfrak{p} \in \mathfrak{W}$.

Proof. By p. 263, Proposition 4 of [1], there is a submodule N of M such that $\operatorname{Ass}_R(M/N) = \mathfrak{X}$. In particular, $\dim M/N = c$. Since $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$, by Theorem 2.2 of [9], we have $H^i_{\mathfrak{a}}(N) = 0$ for all $i > c$. Thus, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

provides the following exact sequence of local cohomology modules

$$\cdots \longrightarrow H^c_{\mathfrak{a}}(N) \longrightarrow H^c_{\mathfrak{a}}(M) \longrightarrow H^c_{\mathfrak{a}}(M/N) \longrightarrow 0.$$

Thus $\operatorname{Supp}_R H^c_{\mathfrak{a}}(M/N) \subseteq \operatorname{Supp}_R H^c_{\mathfrak{a}}(M)$, and so (i) follows by Corollary 2.7(ii). If $H^c_{\mathfrak{a}}(M)$ is representable, then the above exact sequence implies that $\operatorname{Att}_R H^c_{\mathfrak{a}}(M/N) \subseteq \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$, and so (ii) follows by Theorem 2.5.

Next, we prove (iii). Let $\mathfrak{p} \in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$ be such that $\dim R/\mathfrak{p} = c$. By 2.5 of [12], there is a submodule N of $H^c_{\mathfrak{a}}(M)$ such that $\mathfrak{p} = N :_R H^c_{\mathfrak{a}}(M)$. Hence $\mathfrak{p}H^c_{\mathfrak{a}}(M) \subseteq N$, and so by Lemma 2.9, it turns out that $H^c_{\mathfrak{a}}(M)/N$ is isomorphic to a quotient of $H^c_{\mathfrak{a}}(M/\mathfrak{p}M)$. Now, by the independence theorem (Theorem 4.2.1 of [3]), we have the following isomorphisms:

$$\begin{aligned} H^c_{\mathfrak{a}}(M/\mathfrak{p}M) &\simeq H^c_{\mathfrak{a}R/\mathfrak{p}}(M/\mathfrak{p}M) \\ &\simeq H^c_{\mathfrak{a}R/\mathfrak{p}}(R/\mathfrak{p}) \otimes_{R/\mathfrak{p}} M/\mathfrak{p}M \\ &\simeq H^c_{\mathfrak{a}}(R/\mathfrak{p}) \otimes_R M. \end{aligned}$$

Thus $H^c_{\mathfrak{a}}(M/\mathfrak{p}M)$ is Artinian and $\mathfrak{p} \in \mathrm{Att}_R\, H^c_{\mathfrak{a}}(M/\mathfrak{p}M)$. Because, by Corollary 3.3 of [7] for an Artinian R-module A and a finitely generated R-module N, we have

$$\mathrm{Att}_R(A \otimes_R N) = \mathrm{Att}_R\, A \cap \mathrm{Supp}_R\, N,$$

the conclusion follows by Theorem 2.5. ■

3. Lichtenbaum–Hartshorne vanishing theorem

Let the situation be as in Theorem 2.5. In the case that the ideal $\mathfrak{a}$ is the intersection of finitely many maximal ideals of R, we can find a better description of the set $\mathrm{Att}_R\, H^d_{\mathfrak{a}}(M)$. We do this in the next result. The last assertion of this result might be considered as the generalization of Grothendieck's non-vanishing theorem to semi-local rings.

PROPOSITION 3.1

Assume that $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ *are maximal ideals of* R *and* M *a finitely generated nonzero* R*-module of finite dimension* d. *Let* $\mathfrak{a} = \bigcap_{i=1}^t \mathfrak{m}_i$. *Then*

$$\mathrm{Att}_R\, H^d_{\mathfrak{a}}(M) = \left\{ \mathfrak{p} \in \mathrm{Assh}_R\, M \colon \exists\, 1 \le i \le t \text{ such that } \mathfrak{p} \subseteq \mathfrak{m}_i \text{ and } \mathrm{ht}\, \frac{\mathfrak{m}_i}{\mathfrak{p}} = d \right\}.$$

In particular, if R *is semi-local with the only maximal ideals* $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$, *then* $\mathrm{Att}_R\, H^d_{\mathfrak{a}}(M) = \mathrm{Assh}_R\, M$, *and so* $H^d_{\mathfrak{a}}(M) \neq 0$.

Proof. Let $1 \le i \le t$. Since $\mathrm{Supp}_R\, H^d_{\mathfrak{m}_i}(M) \subseteq \{\mathfrak{m}_i\}$, by Lemma 2.1(ii) and the flat base change theorem, it turns out that $H^d_{\mathfrak{m}_i}(M) \simeq H^d_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i})$. Hence, applying the Mayer–Vietoris sequence for local cohomology provides the following natural isomorphisms

$$H^d_{\mathfrak{a}}(M) \simeq \bigoplus_{i=1}^t H^d_{\mathfrak{m}_i}(M) \simeq \bigoplus_{i=1}^t H^d_{\mathfrak{m}_i R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}).$$

By Theorem 2.2 of [13], for a finitely generated module M over a local ring $(U, \mathfrak{n})$, we have $\mathrm{Att}_U\, H^d_{\mathfrak{n}}(M) = \mathrm{Assh}_U\, M$, where $d = \dim M$. Thus by Lemma 2.1(i), we conclude that

$$\mathrm{Att}_R\, H^d_{\mathfrak{a}}(M) = \bigcup_{i=1}^t \{\mathfrak{p} \in \mathrm{Assh}_R\, M \colon \mathfrak{p}R_{\mathfrak{m}_i} \in \mathrm{Assh}_{R_{\mathfrak{m}_i}} M_{\mathfrak{m}_i} \text{ and } \dim R_{\mathfrak{m}_i}/\mathfrak{p}R_{\mathfrak{m}_i} = d\}$$

$$= \left\{ \mathfrak{p} \in \mathrm{Assh}_R\, M \colon \exists\, 1 \le i \le t \text{ such that } \mathfrak{p} \subseteq \mathfrak{m}_i \text{ and } \mathrm{ht}\, \frac{\mathfrak{m}_i}{\mathfrak{p}} = d \right\}.$$

The last assertion is immediate by the first one. ■

Remark 3.2. Let A be an Artinian R-module. Suppose that $\mathrm{Supp}_R\, A = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ and put $\mathfrak{M} = \cap_{i=1}^t \mathfrak{m}_i$. Let T denote the $\mathfrak{M}$-adic completion of R.

(i) Sharp [16] showed that A has a natural structure as a module over T. Let $\theta \colon R \longrightarrow T$ denote the natural ring homomorphism. The T-module structure of A is such that for any element $r \in R$ the multiplication by r on A has the same effect as the multiplication of $\theta(r) \in T$. Furthermore, a subset of A is an R-submodule of A if and only if it is a T-submodule of A.

(ii) Let $\mathfrak{a} \subseteq \mathfrak{b}$ denote two ideals of R and $B := \Sigma_{n\in\mathbb{N}}\langle\mathfrak{b}\rangle(0:_A \mathfrak{a}^n)$. By Theorem 2.4 of [10], the following are equivalent:

(a) For any $l \in \mathbb{N}$, there is an integer $n = n(l)$ such that $0:_A \mathfrak{M}^l \subseteq \langle\mathfrak{b}\rangle(0:_A \mathfrak{a}^n)$.

(b) $B = A$.

(c) $\mathrm{Rad}(\mathfrak{p} + \mathfrak{a}T) \subsetneq \mathrm{Rad}(\mathfrak{p} + \mathfrak{b}T)$ for all $\mathfrak{p} \in \mathrm{Att}_T\, A$.

(iii) Let $\mathfrak{a}$, $\mathfrak{b}$ and B be as in (ii), and let $A = S_1 + \cdots + S_n$ be a minimal secondary representation of A as a T-module. We can order the elements of $\mathrm{Att}_T\, A = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ such that for an integer $0 \le l \le n$, $\mathrm{Rad}(\mathfrak{p}_i + \mathfrak{a}T) \subsetneq \mathrm{Rad}(\mathfrak{p}_i + \mathfrak{b}T)$ for all $1 \le i \le l$, while $\mathrm{Rad}(\mathfrak{p}_i + \mathfrak{a}T) = \mathrm{Rad}(\mathfrak{p}_i + \mathfrak{b}T)$ for all $l + 1 \le i \le n$. Then $S_1 + \cdots + S_l$ is a minimal secondary representation of B. This follows by Theorem 2.8 of [10]. Also, it is a routine check to see that $\Sigma_{i=l+1}^n (S_i + B)/B$ is a minimal secondary representation of A/B as a T-module.

Theorem 3.3. *Let M be a finitely generated R-module of finite dimension d and $\mathcal{P}$ a finite subset of* Max R. *Let $\mathfrak{M} = \cap_{\mathfrak{m}\in\mathcal{P}}\mathfrak{m}$ and T denote the $\mathfrak{M}$-adic completion of R.*

(i) *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathcal{P}_{\mathfrak{a},M} = \mathcal{P}_{\mathfrak{b},M} = \mathcal{P}$. If either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathrm{Att}_T\, H^d_{\mathfrak{a}}(M) \subseteq \mathrm{Att}_T\, H^d_{\mathfrak{b}}(M)$, then $H^d_{\mathfrak{a}}(M)$ is isomorphic to a quotient of $H^d_{\mathfrak{b}}(M)$.*

(ii) *Let $\mathfrak{a}$ and $\mathfrak{b}$ be as in* (i). *If $\mathrm{Att}_T\, H^d_{\mathfrak{a}}(M) = \mathrm{Att}_T\, H^d_{\mathfrak{b}}(M)$, then $H^d_{\mathfrak{a}}(M) \simeq H^d_{\mathfrak{b}}(M)$.*

(iii) *For all ideals $\mathfrak{c}$ of R, there are at most $2^{|\mathrm{Assh}_T(M\otimes_R T)|}$ non-isomorphic top local cohomology modules $H^d_{\mathfrak{c}}(M)$ such that $\mathrm{Supp}_R\, H^d_{\mathfrak{c}}(M) = \mathcal{P}$.*

Proof. Let $A = H^d_{\mathfrak{M}}(M)$,

$$B_1 = \sum_{n\in\mathbb{N}}\langle\mathfrak{M}\rangle(0:_A \mathfrak{a}^n)$$

and

$$B_2 = \sum_{n\in\mathbb{N}}\langle\mathfrak{M}\rangle(0:_A \mathfrak{b}^n).$$

Then, Theorem 2.3 yields the natural isomorphisms $H^d_{\mathfrak{a}}(M) \simeq A/B_1$ and $H^d_{\mathfrak{b}}(M) \simeq A/B_2$. Let $A = S_1 + \cdots + S_n$ be a minimal secondary representation of A as a T-module and set

$$\mathfrak{Z}_j := \mathrm{Att}_T\, A \setminus \mathrm{Att}_T(A/B_j)$$

for $j = 1, 2$. Then by Remark 3.2(iii), $B_j = \Sigma_{\mathfrak{p}_i\in\mathfrak{Z}_j} S_i$ for $j = 1, 2$. Thus, if either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathrm{Att}_T\, H^d_{\mathfrak{a}}(M) \subseteq \mathrm{Att}_T\, H^d_{\mathfrak{b}}(M)$, then $B_2 \subseteq B_1$, and so $H^d_{\mathfrak{a}}(M)$ is isomorphic to a quotient of $H^d_{\mathfrak{b}}(M)$. Also, if $\mathrm{Att}_T\, H^d_{\mathfrak{a}}(M) = \mathrm{Att}_T\, H^d_{\mathfrak{b}}(M)$, then $B_1 = B_2$, and so $H^d_{\mathfrak{a}}(M) \simeq H^d_{\mathfrak{b}}(M)$.

Next, we are going to prove (iii). Since T and $\Pi_{\mathfrak{m}\in\mathcal{P}}\hat{R}_{\mathfrak{m}}$ are isomorphic as R-modules, by the flat base change theorem and Remark 3.2(i) we have the following isomorphisms of T-modules:

$$H^d_{\mathfrak{M}T}(M \otimes_R T) \simeq H^d_{\mathfrak{M}}(M) \otimes_R T \simeq H^d_{\mathfrak{M}}(M).$$

Next, as $\mathfrak{M}T$ is the intersection of all maximal ideals of the semi-local ring T, it follows by Proposition 3.1 that

$$\mathrm{Att}_T\, H^d_{\mathfrak{M}}(M) = \mathrm{Att}_T\, H^d_{\mathfrak{M}T}(M \otimes_R T) = \mathrm{Assh}_T(M \otimes_R T).$$

Now, the claim follows by part (ii). ■

As an immediate application of Theorem 3.3, we deduce Theorem 1.6 and Proposition 1.5 of [5].

COROLLARY 3.4

Let $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module. Let* $d = \dim M$.

(i) *If either* $\mathfrak{a} \subseteq \mathfrak{b}$ *or* $\mathrm{Att}_{\hat{R}}\, H^d_{\mathfrak{a}}(M) \subseteq \mathrm{Att}_{\hat{R}}\, H^d_{\mathfrak{b}}(M)$, *then* $H^d_{\mathfrak{a}}(M)$ *is isomorphic to a quotient of* $H^d_{\mathfrak{b}}(M)$.
(ii) *If* $\mathrm{Att}_{\hat{R}}\, H^d_{\mathfrak{a}}(M) = \mathrm{Att}_{\hat{R}}\, H^d_{\mathfrak{b}}(M)$, *then* $H^d_{\mathfrak{a}}(M) \simeq H^d_{\mathfrak{b}}(M)$.
(iii) *The number of non-isomorphic top local cohomology modules* $H^d_{\mathfrak{c}}(M)$ *is at most* $2^{|\mathrm{Assh}_{\hat{R}}\, \hat{M}|}$ *for all ideals* $\mathfrak{c}$ *of* R.

Example 3.5. It might be of interest to ask whether one can replace $\hat{R}$ by R in Corollary 3.4(ii). But, as we show in the sequel, this is not the case. To this end, we use an example of Brodmann and Sharp (see Exercise 8.2.9 of [3]). Let K be a field of characteristic 0. Let $R' := K[X, Y, Z]$, $\mathfrak{m}' := (X, Y, Z)$ and $\mathfrak{b} = (Y^2 - X^2 - X^3)$. Set $R := (R'/\mathfrak{b})_{\mathfrak{m}'/\mathfrak{b}}$ and let $\mathfrak{p}$ denote the extension of the ideal

$$(X + Y - YZ, (Z - 1)^2(X + 1) - 1)$$

of R' to R. As it is mentioned in Exercise 8.2.9 of [3], it follows that R is a 2-dimensional local domain and that $\mathfrak{p}\hat{R}$ is a prime ideal of $\hat{R}$ with $\dim \hat{R}/\mathfrak{p}\hat{R} = 1$. Also, it follows that $H^2_{\mathfrak{p}}(R) \neq 0$ (see again Exercise 8.2.9 of [3]). So $\mathrm{Att}_{\hat{R}}\, H^2_{\mathfrak{p}}(R)$ is not empty. Now, let $\mathfrak{p}^*$ be a minimal associated prime ideal of $\hat{R}$ such that $\mathfrak{p}^* \subseteq \mathfrak{p}\hat{R}$. Then the inclusion must be strict, because otherwise we would have

$$\mathfrak{p} = \mathfrak{p}\hat{R} \cap R = \mathfrak{p}^* \cap R \in \mathrm{Ass}_R\, \hat{R} = \{(0)\},$$

a contradiction. This yields that $\dim \hat{R}/\mathfrak{p}^* = 2$, and so $\mathfrak{p}^* \in \mathrm{Assh}_{\hat{R}}\, \hat{R}$. On the other hand, we have

$$\dim \hat{R}/\mathfrak{p}\hat{R} + \mathfrak{p}^* = \dim \hat{R}/\mathfrak{p}\hat{R} = 1.$$

Hence $\mathfrak{p}^*$ does not belong to $\mathrm{Att}_{\hat{R}}\, H^2_{\mathfrak{p}}(R)$. Thus, if $\mathfrak{m}$ denotes the maximal ideal of the local ring R, then

$$\emptyset \neq \mathrm{Att}_{\hat{R}}\, H^2_{\mathfrak{p}}(R) \subsetneqq \mathrm{Att}_{\hat{R}}\, H^2_{\mathfrak{m}}(R) = \mathrm{Assh}_{\hat{R}}\, \hat{R}.$$

In particular, it becomes clear that $H^2_{\mathfrak{p}}(R)$ and $H^2_{\mathfrak{m}}(R)$ are not isomorphic. On the other hand, we have

$$\mathrm{Att}_R\, H^2_{\mathfrak{p}}(R) = \mathrm{Att}_R\, H^2_{\mathfrak{m}}(R) = \{(0)\}.$$

We therefore conclude that it is not possible to replace $\hat{R}$ by R in Corollary 3.4(ii).

The following is an analogue of the Lichtenbaum–Hartshorne vanishing theorem for general Noetherian rings.

Theorem 3.6. *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module of finite dimension d. Let $\mathfrak{M} = \bigcap_{\mathfrak{m}\in\mathcal{P}_{\mathfrak{a},M}} \mathfrak{m}$ and T denote the $\mathfrak{M}$-adic completion of R. Then the following are equivalent*:

(i) $H^d_{\mathfrak{a}}(M) = 0$.
(ii) $H^d_{\mathfrak{M}}(M) = \sum_{n\in\mathbb{N}} \langle\mathfrak{M}\rangle(0:_{H^d_{\mathfrak{M}}(M)} \mathfrak{a}^n)$.
(iii) *For any integer $l \in \mathbb{N}$, there exists an $n = n(l) \in \mathbb{N}$ such that*

$$0:_{H^d_{\mathfrak{M}}(M)} \mathfrak{a}^l \subseteq \langle\mathfrak{M}\rangle(0:_{H^d_{\mathfrak{M}}(M)} \mathfrak{a}^n).$$

(iv) $\dim T/\mathfrak{a}T + \mathfrak{p} > 0$ *for all* $\mathfrak{p} \in \operatorname{Assh}_T(M \otimes_R T)$.
(v) $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) < d$ *for all* $\mathfrak{p} \in \operatorname{Assh}_R M$.

Proof. Let $\mathfrak{p} \in \operatorname{Assh}_T(M \otimes_R T)$. Then, it is easy to see that $\dim T/\mathfrak{a}T + \mathfrak{p} > 0$ if and only if $\operatorname{Rad}(\mathfrak{p} + \mathfrak{a}T) \subsetneq \operatorname{Rad}(\mathfrak{p} + \mathfrak{M}T)$. Therefore, the equivalence of the conditions (i), (ii) and (iv) is clear by Theorem 2.3 and Remark 3.2(ii). Note that in the proof of Theorem 3.3, we have seen that $\operatorname{Att}_T H^d_{\mathfrak{M}}(M) = \operatorname{Assh}_T(M \otimes_R T)$.

Since $\mathfrak{a} \subseteq \mathfrak{M}$, any element of $H^d_{\mathfrak{M}}(M)$ is annihilated by some power of $\mathfrak{a}$. Thus (iii) $\Rightarrow$ (ii) becomes clear.

(ii) $\Rightarrow$ (iii). Let $A = H^d_{\mathfrak{M}}(M)$ and l a fixed natural number. Then $(0:_A \mathfrak{a}^l)/\langle\mathfrak{M}\rangle(0:_A \mathfrak{a}^l)$ is a Noetherian R-module and so the sequence $\{(0:_A \mathfrak{a}^l) \cap \langle\mathfrak{M}\rangle(0:_A \mathfrak{a}^n)\}_{n\in\mathbb{N}}$ satisfies the ascending chain condition. Thus, it follows by Lemma 2.1 of [10] that (ii) implies (iii).

By Grothendieck's vanishing theorem, it turns out that $\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) \leq d$ for all $\mathfrak{p} \in \operatorname{Supp}_R M$. Therefore, the equivalence (i) and (v) is immediate by Theorem 2.5. ■

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