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Hyperbolic unit groups and quaternion algebras

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Abstract. We classify the quadratic extensions $K = \mathbb{Q}[\sqrt{d}]$ and the finite groups G for which the group ring $\mathfrak{o}_K[G]$ of G over the ring $\mathfrak{o}_K$ of integers of K has the property that the group $\mathcal{U}_1(\mathfrak{o}_K[G])$ of units of augmentation 1 is hyperbolic. We also construct units in the $\mathbb{Z}$-order $\mathcal{H}(\mathfrak{o}_K)$ of the quaternion algebra $\mathcal{H}(K) = \left(\frac{-1,-1}{K}\right)$, when it is a division algebra.

Keywords. Hyperbolic groups; quaternion algebras; free groups; group rings; units.

1. Introduction

The finite groups G for which the unit group $\mathcal{U}(\mathbb{Z}[G])$ of the integral group ring $\mathbb{Z}[G]$ is hyperbolic, in the sense of Gromov [8], have been characterized in [13]. The main aim of this paper is to examine the hyperbolicity of the group $\mathcal{U}_1(\mathfrak{o}_K[G])$ of units of augmentation 1 in the group ring $\mathfrak{o}_K[G]$ of G over the ring $\mathfrak{o}_K$ of integers of a quadratic extension $K = \mathbb{Q}[\sqrt{d}]$ of the field $\mathbb{Q}$ of rational numbers, where d is a square-free integer $\neq 1$. Our main result (Theorem 4.7) provides a complete characterization of such group rings $\mathfrak{o}_K[G]$.

In the integral case the hyperbolic unit groups are either finite, hence have zero end, or have two or infinitely many ends (see Theorem I.8.32 of [4] and [13]); in fact, in this case, the hyperbolic boundary is either empty, or consists of two points, or is a Cantor set. In particular, the hyperbolic boundary is not a (connected) manifold. However, in the case we study here, it turns out that when the unit group is hyperbolic and non-abelian it has one end, and the hyperbolic boundary is a compact manifold of constant positive curvature (see Remark after Theorem 4.7).

Our investigation naturally leads us to study units in the order $\mathcal{H}(\mathfrak{o}_K)$ of the standard quaternion algebra $\mathcal{H}(K) = \left(\frac{-1,-1}{K}\right)$, when this algebra is a division algebra. We construct units, here called Pell and Gauss units, using solutions of certain diophantine quadratic equations. In particular, we exhibit units of norm -1 in $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$; this construction, when combined with the deep work in [5], helps to provide a set of generators for the full unit group $\mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}))$.

The work reported in this paper corresponds to the first chapter of the third author's PhD thesis [17], where analogous questions about finite semi-groups (see [10]) and RA-loops (see [14]) have also been studied.

2. Preliminaries

Let Γ be a finitely generated group with identity e and S a finite symmetric set of generators of Γ, $e \notin S$. Consider the Cayley graph $\mathcal{G} = \mathcal{G}(\Gamma, S)$ of Γ with respect to the generating set S and $d = d_S$ the corresponding metric (see chap. 1.1 of [4]). The induced metric on the vertex set Γ of $\mathcal{G}(\Gamma, S)$ is then the word metric: for $\gamma_1, \gamma_2 \in \Gamma$, $d(\gamma_1, \gamma_2)$ equals the least non-negative integer n such that $\gamma_1^{-1}\gamma_2 = s_1 s_2 \dots s_n,\ s_i \in S$. Recall that in a metric space (X, d), the Gromov product $(y.z)_x$ of elements $y, z \in X$ with respect to a given element $x \in X$ is defined to be

$$(y \cdot z)_x = \frac{1}{2}(d(y, x) + d(z, x) - d(y, z)),$$

and that the metric space X is said to be *hyperbolic* if there exists $\delta \geq 0$ such that for all $w, x, y, z \in X$,

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w,\ (y \cdot z)_w\} - \delta.$$

The group Γ is said to be hyperbolic if the Cayley graph $\mathcal{G}$ with the metric d_S is a hyperbolic metric space. This is a well-defined notion which depends only on the group Γ, and is independent of the chosen generating set S (see [8]).

A map $f\colon X \to Y$ between topological spaces is said to be *proper* if $f^{-1}(C) \subseteq X$ is compact whenever $C \subseteq Y$ is compact. For a metric space X, two proper maps (rays) $r_1, r_2\colon [0, \infty[\longrightarrow X$ are defined to be equivalent if, for each compact set $C \subset X$, there exists $n \in \mathbb{N}$ such that $r_i([n, \infty[), i = 1, 2$, are in the same path component of $X \backslash C$. Denote by $\mathrm{end}(r)$ the equivalence class of the ray r, by $\mathrm{End}(X)$ the set of the equivalence classes $\mathrm{end}(r)$, and by $|\mathrm{End}(X)|$ the cardinality of the set $\mathrm{End}(X)$. The cardinality $|\mathrm{End}(\mathcal{G}, d_S)|$ for the Cayley graph $(\mathcal{G}, d_S)$ of Γ does not depend on the generating set S; we thus have the notion of the number of ends of the finitely generated group Γ (see [4, 8]).

We next recall some standard results from the theory of hyperbolic groups:

1. Let Γ be a group. If Γ is hyperbolic, then $\mathbb{Z}^2 \not\hookrightarrow \Gamma$, where $\mathbb{Z}^2$ denotes the free Abelian group of rank 2 (Corollary III.Γ 3.10(2) of [4]).
2. An infinite hyperbolic group contains an element of infinite order (Proposition III.Γ 2.22 of [4]).
3. If Γ is hyperbolic, then there exists $n = n(\Gamma) \in \mathbb{N}$ such that $|H| \leq n$ for every torsion subgroup $H < \Gamma$ (Theorem III.Γ3.2 of [4] and Chapter 8, Corollaire 36 of [7]).

These results will be used freely in the sequel. In view of (1) above, the following observation is quite useful.

Lemma 2.1. *Let A be a unital ring whose aditive group is torsion free, and let $\theta_1, \theta_2 \in A$ be two 2-nilpotent commuting elements which are $\mathbb{Z}$-linearly independent. Then $\mathcal{U}(A)$ contains a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. Set $u = 1 + \theta_1$ and $v = 1 + \theta_2$. It is clear that $u, v \in \mathcal{U}(A)$ and both have infinite order. If $1 \neq w \in \langle u\rangle \cap \langle v\rangle$, then there exists $i, j \in \mathbb{Z}\backslash\{0\}$, such that, $u^i = w = v^j$. Since $u^i = 1 + i\theta_1$ and $v^j = 1 + j\theta_2$, it follows that $i\theta_1 - j\theta_2 = 0$ and hence $\{\theta_1, \theta_2\}$ is $\mathbb{Z}$-linearly dependent, a contradiction. Hence $\mathbb{Z}^2 \simeq \langle u, v\rangle \subseteq \mathcal{U}(A)$. □

Let C_n denote the cyclic group of order n, S_3 the symmetric group of degree 3, D_4 the dihedral group of order 8, and Q_{12} the split extension $C_3 \rtimes C_4$. Let K be an algebraic number field and $\mathfrak{o}_K$ its ring of integers. The analysis of the implication for torsion subgroups G of a hyperbolic unit group $\mathcal{U}(\mathbb{Z}[\Gamma])$ leading to Theorem 3 of [13] is easily seen to remain valid for torsion subgroups of hyperbolic unit groups $\mathcal{U}(\mathfrak{o}_K[\Gamma])$. We thus have the following:

Theorem 2.2. *A torsion group G of a hyperbolic unit group $\mathcal{U}(\mathfrak{o}_K[\Gamma])$ is isomorphic to one of the following groups*:

1. C_5, C_8, C_{12}, *an Abelian group of exponent dividing* 4 *or* 6;
2. *a Hamiltonian* 2-*group*;
3. $S_3, D_4, Q_{12}, C_4 \rtimes C_4$.

We denote by $\mathcal{H}(K) = \left(\frac{a,b}{K}\right)$ the generalized quaternion algebra over K: $\mathcal{H}(K) = K[i, j : i^2 = a, j^2 = b, ji = -ij =: k]$. The set $\{1, i, j, k\}$ is a K-basis of $\mathcal{H}(K)$. Such an algebra is a totally definite quaternion algebra if the field K is totally real and a, b are totally negative. If $a, b \in \mathfrak{o}_K$, then the set $\mathcal{H}(\mathfrak{o}_K)$, consisting of the $\mathfrak{o}_K$-linear combinations of the elements $1, i, j$ and k, is an $\mathfrak{o}_K$-algebra. We denote by N the norm map $\mathcal{H}(K) \to K$, sending $x = x_1 + x_i i + x_j j + x_k k$ to $N(x) = x_1^2 - ax_i^2 - bx_j^2 + abx_k^2$.

Let $d \neq 1$ be a square-free integer, $K = \mathbb{Q}[\sqrt{d}]$. Let us recall the basic facts about the ring of integers $\mathfrak{o}_K$ (see, for example, [11] or [15]). Set

$$\vartheta = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \pmod 4 \\ (1+\sqrt{d})/2, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Then $\mathfrak{o}_K = \mathbb{Z}[\vartheta]$ and the elements $1, \vartheta$ constitute a $\mathbb{Z}$-basis of $\mathfrak{o}_K$. If $d < 0$, then

$$\mathcal{U}(\mathfrak{o}_K) = \begin{cases} \{\pm 1, \pm\vartheta\}, & \text{if } d = -1, \\ \{\pm 1, \pm\vartheta, \pm\vartheta^2\}, & \text{if } d = -3, \\ \{\pm 1\}, & \text{otherwise.} \end{cases} \tag{1}$$

If $d > 0$, then there exists a unique unit $\epsilon > 1$, called the *fundamental unit*, such that

$$\mathcal{U}(\mathfrak{o}_K) = \pm\langle\epsilon\rangle. \tag{2}$$

We need the following:

PROPOSITION 2.3

Let $K = \mathbb{Q}[\sqrt{d}]$, with $d \neq 1$ a square-free integer, be a quadratic extension of $\mathbb{Q}$, and $u \in \mathcal{U}(\mathfrak{o}_K)$. Then $u^i \equiv 1 \pmod 2$, where

$$i = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 8, \\ 2 & \text{if } d \equiv 2, 3 \pmod 4, \\ 3 & \text{if } d \equiv 5 \pmod 8. \end{cases}$$

Proof. The assertion follows immediately on considering the prime factorization of the ideal $2\mathfrak{o}_K$ (see Theorem 1, p. 236 of [3]). □

3. Abelian groups with hyperbolic unit groups

PROPOSITION 3.1

Let R be a unitary commutative ring, $C_2 = \langle g \rangle$. Then $u = a + (1-a)g$, $a \in R\backslash\{0, 1\}$ is a non-trivial unit in $\mathcal{U}_1(R[C_2])$ if, and only if, $2a - 1 \in \mathcal{U}(R)$.

Proof. Let $C_2 = \langle g \rangle$ and suppose that $u = a + (1-a)g$, $a \in R\backslash\{0, 1\}$ is a non-trivial unit in $R[C_2]$ having augmentation 1. Let $\rho\colon R[C_2] \to M_2(R)$ be the regular representation. Clearly $\rho(u) = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$. Since u is a unit, it follows that $2a - 1 = \det \rho(u) \in \mathcal{U}(R)$.

Conversely, let $a \in R\backslash\{0, 1\}$ be such that $e = 2a - 1 \in \mathcal{U}(R)$. It is then easy to see that $u = a + (1-a)g$ is a non-trivial unit in $R[C_2]$ with inverse $v = ae^{-1} + (1 - ae^{-1})g$ (Proposition I of [9]). □

PROPOSITION 3.2

The unit group $\mathcal{U}(\mathfrak{o}_K[C_2])$ is trivial if, and only if, $K = \mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$, i.e., $d < 0$.

Proof. It is clear from the description (1) of the unit group of $\mathfrak{o}_K$ that the equation

$$2a - 1 = u,\ a \in \mathfrak{o}_k\backslash\{0, 1\},\ u \in \mathcal{U}(\mathfrak{o}_K) \tag{3}$$

does not have a solution when $K = \mathbb{Q}$ or $d < 0$.

Suppose $d > 1$ and ϵ is the fundamental unit in $\mathfrak{o}_K$. In this case we have $\mathcal{U}(\mathfrak{o}_K) = \pm\langle\epsilon\rangle$. By Proposition 2.3, $\epsilon^i \in 1 + 2\mathfrak{o}_K$ for some $i \in \{1, 2, 3\}$. Consequently eq. (3) has a solution and so, by Proposition 3.1, $\mathcal{U}(\mathfrak{o}_K[C_2])$ is non-trivial. □

Theorem 3.3. *Let $\mathfrak{o}_K$ be the ring of integers of a real quadratic extension $K = \mathbb{Q}[\sqrt{d}]$, $d > 1$ a square-free integer, $\epsilon > 1$ the fundamental unit of $\mathfrak{o}_K$ and $C_2 = \langle g \rangle$. Then*

$$\mathcal{U}_1(\mathfrak{o}_K[C_2]) \cong \langle g \rangle \times \left\langle \frac{1+\epsilon^n}{2} + \frac{1-\epsilon^n}{2} g \right\rangle \cong C_2 \times \mathbb{Z},$$

where n is the order of ϵ mod $2\mathfrak{o}_K$.

Proof. Let $u \in \mathcal{U}_1(\mathfrak{o}_K[C_2])$ be a non-trivial unit. Then, there exists $a \in \mathfrak{o}_K$ such that, $2a - 1 = \pm\epsilon^m$ for some non-zero integer m. Since n is the order of ϵ mod $2\mathfrak{o}_K$, $m = nq$ with $q \in \mathbb{Z}$. We thus have

$$\begin{aligned} u &= a + (1-a)g \\ &= \frac{1 \pm \epsilon^m}{2} + \frac{1 \mp \epsilon^m}{2} g \end{aligned}$$

$$= \frac{1 \pm \epsilon^{nq}}{2} + \frac{1 \mp \epsilon^{nq}}{2} g$$

$$= \left(\frac{1+\epsilon^n}{2} + \frac{1-\epsilon^n}{2} g \right)^q \quad \text{or} \quad g \left(\frac{1+\epsilon^n}{2} + \frac{1-\epsilon^n}{2} g \right)^q .$$

Hence $\mathcal{U}_1(\mathfrak{o}_K[C_2]) \cong \langle g \rangle \times \left\langle \frac{1+\epsilon^n}{2} + \frac{1-\epsilon^n}{2} g \right\rangle \cong C_2 \times \mathbb{Z}$. □

As an immediate consequence of the preceding analysis, we have:

COROLLARY 3.4

If K is a quadratic extension of $\mathbb{Q}$, then $\mathcal{U}_1(\mathfrak{o}_K[C_2])$ is a hyperbolic group.

COROLLARY 3.5

Let G be a non-cyclic elementary Abelian 2-group. Then $\mathcal{U}_1(\mathfrak{o}_K[G])$ is hyperbolic if, and only if, $\mathfrak{o}_K$ is imaginary.

Proof. Suppose $\mathfrak{o}_K$ is real. Since G is not cyclic, there exist $g, h \in G$, $g \neq h$, $o(g) = o(h) = 2$. By Theorem 3.3, $\mathcal{U}_1(\mathfrak{o}_K[\langle g \rangle]) \cong C_2 \times \mathbb{Z} \cong \mathcal{U}_1(\mathfrak{o}_K[\langle h \rangle])$. Since $\langle g \rangle \cap \langle h \rangle = \{1\}$, $\mathcal{U}_1(\mathfrak{o}_K[\langle g \rangle]) \cap \mathcal{U}_1(\mathfrak{o}_K[\langle h \rangle]) = \{1\}$. Therefore $\mathcal{U}_1(\mathfrak{o}_K)$ contains an Abelian group of rank 2, so it is not hyperbolic. Conversely, if $\mathfrak{o}_K$ is imaginary, then, proceeding by induction on the order $|G|$ of G, we can conclude that $\mathcal{U}_1(\mathfrak{o}_K[G])$ is trivial, and hence is hyperbolic. □

For an Abelian group G, we denote by $r(G)$ its torsion-free rank. In order to study the hyperbolicity of $\mathcal{U}_1(\mathfrak{o}_K[G])$, it is enough to determine the torsion-free rank $r(\mathcal{U}_1(\mathfrak{o}_K[G]))$. Since $\mathcal{U}(\mathfrak{o}_K[G]) \cong \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}_1(\mathfrak{o}_K[G])$, we have $r(\mathcal{U}_1(\mathfrak{o}_K[G])) = r(\mathcal{U}(\mathfrak{o}_K[G])) - r(\mathcal{U}(\mathfrak{o}_K))$. If K is an imaginary extension, then $r(\mathcal{U}(\mathfrak{o}_K[G])) = r(\mathcal{U}_1(\mathfrak{o}_K[G]))$, whereas if K is a real quadratic extension, then $r(\mathcal{U}(\mathfrak{o}_K)) = 1$, and therefore

$$r(\mathcal{U}_1(\mathfrak{o}_K[G])) = r(\mathcal{U}(\mathfrak{o}_K[G])) - 1.$$

We note that

$$\mathbb{Q}[C_n] \cong \oplus \sum_{d|n} \mathbb{Q}[\zeta_d],$$

where ζ_d is a primitive d-th root of unity, and therefore, for any algebraic number field L,

$$L[C_n] \cong \oplus \sum_{d|n} L \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_d].$$

We say that two groups are commensurable with each other when they contain finite index subgroups isomorphic to each other. Since the unit group $\mathcal{U}(\mathfrak{o}_L[C_n])$ is commensurable with $\mathcal{U}(\Lambda)$, where $\Lambda = \oplus \sum_{d|n} \mathfrak{o}_{L \otimes \mathbb{Q}[\zeta_d]}$, we essentially need to compute the torsion-free rank of $\mathfrak{o}_{K \otimes \mathbb{Q}[\zeta_d]}$ for the needed cases.

PROPOSITION 3.6

Let $K = \mathbb{Q}[\sqrt{d}]$, with d a square-free integer $\neq 1$. The table below shows the torsion-free rank of the groups $\mathcal{U}_1(\mathfrak{o}_K[C_n])$, $n \in \{2, 3, 4, 5, 6, 8\}$.

n	$r(\mathcal{U}_1(\mathfrak{o}_K[C_n]))$	n	$r(\mathcal{U}_1(\mathfrak{o}_K[C_n]))$
2	0 *if* $d < 0$ 1 *if* $d > 1$	3	1 *if* $d < 0, d \neq -3$ 0 *if* $d = -3$ 1 *if* $d > 1$
4	1 *if* $d < -1$ 0 *if* $d = -1$ 2 *if* $d > 1$	5	6 *if* $d < 0$ 2 *if* $d = 5$ 6 *if* $d \in \mathbb{Z}^+ \backslash \{1, 5\}$
6	2 *if* $d < -3$ 0 *if* $d = -3$ 3 *if* $d > 1$	8	4 *if* $d < -1$ 1 *if* $d = -1$ 4 *if* $d = 2$ 5 *if* $d > 2$

In all the cases, the computation is elementary and we omit the details.

Theorem 3.7. *If* $K = \mathbb{Q}[\sqrt{d}]$, *with* d *a square-free integer* $\neq 1$, *then*

1. $\mathcal{U}_1(\mathfrak{o}_K[C_3])$ *is hyperbolic*;
2. $\mathcal{U}_1(\mathfrak{o}_K[C_4])$ *is hyperbolic if, and only if,* $d < 0$;
3. *for an Abelian group* G *of exponent dividing* $n > 2$, *the group* $\mathcal{U}_1(\mathfrak{o}_K[G])$ *is hyperbolic if, and only if,* $n = 4$ *and* $d = -1$, *or* $n = 6$ *and* $d = -3$;
4. $\mathcal{U}_1(\mathfrak{o}_K[C_8])$ *is hyperbolic if, and only if,* $d = -1$;
5. $\mathcal{U}_1(\mathfrak{o}_K[C_5])$ *is not hyperbolic.*

Proof. Proposition 3.6 gives us the torsion-free rank

$$r := r(\mathcal{U}_1(\mathfrak{o}_K[C_n]))$$

for $n \in \{2, 3, 4, 5, 8\}$. The group $\mathcal{U}_1(\mathfrak{o}_K[C_n])$ is hyperbolic if, and only if, $r \in \{0, 1\}$. Thus, it only remains to consider the case (3).

Suppose $n = 6$ and $\mathcal{U}_1(\mathfrak{o}_K[G])$ is hyperbolic. We, hence, have $r \in \{0, 1\}$. If G is cyclic, then, by Proposition 3.6, we have $d = -3$. If G is not cyclic, then $G \cong C_2^l \times C_3^m$, $l, m \geq 1$. Since $\mathfrak{o}_K[C_3] \hookrightarrow \mathfrak{o}_K[G]$, it follows that $d = -3$.

Conversely, if $n = 6$ and $d = -3$, then, proceeding by induction on $|G|$, it can be proved that $\mathcal{U}_1(\mathfrak{o}_K[G])$ is hyperbolic.

The case $n = 4$ can be handled similarly. □

PROPOSITION 3.8

If $K = \mathbb{Q}[\sqrt{d}]$, *with* d *square-free integer* $\neq 1$, *then* $\mathcal{U}_1(\mathfrak{o}_K[C_{12}])$ *is not hyperbolic.*

Proof. Since $K[C_{12}] \cong K \otimes_{\mathbb{Q}} [\mathbb{Q}[C_{12}]] \cong K \otimes_{\mathbb{Q}} \mathbb{Q}[C_3 \times C_4]) \cong K[C_3 \times C_4]$, we have the immersions $\mathfrak{o}_K[C_3] \hookrightarrow \mathfrak{o}_K[C_{12}]$ and $\mathfrak{o}_K[C_4] \hookrightarrow \mathfrak{o}_K[C_{12}]$. Therefore, $r(\mathcal{U}_1(\mathfrak{o}_K[C_{12}])) \geq r(\mathcal{U}_1(\mathfrak{o}_K[C_3])) + r(\mathcal{U}_1(\mathfrak{o}_K[C_4]))$.

Suppose $\mathcal{U}_1(\mathfrak{o}_K[C_{12}])$ is hyperbolic. Then, since $r(\mathcal{U}_1(\mathfrak{o}_K[C_{12}])) < 2$, we have, by Proposition 3.6, $d \in \{-3, -1\}$. We also have

$$\begin{aligned} K[C_3 \times C_4] &\cong (K[C_3])[C_4] \cong (K \oplus K[\sqrt{-3}])[C_4] \\ &\cong K[C_4] \oplus (K[\sqrt{-3}])[C_4] \\ &\cong 2K \oplus K[\sqrt{-1}] \oplus 2K[\sqrt{-3}] \oplus K[\sqrt{-3} + \sqrt{-1}]. \end{aligned}$$

Set $\mathbb{L} = \mathbb{Q}[\sqrt{-3} + \sqrt{-1}]$ and suppose $d = -3$. Then $\mathfrak{o}_K[C_{12}] \hookrightarrow 4\mathfrak{o}_K \oplus 2\mathfrak{o}_{\mathbb{L}}$ and $r(\mathcal{U}(\mathfrak{o}_{\mathbb{L}})) = 1$. Thus $r(\mathcal{U}(\mathfrak{o}_K[C_{12}])) = 2$, and we have a contradiction.

Analogously, for $d = -1$, $\mathfrak{o}_K[C_{12}] \hookrightarrow 3\mathfrak{o}_K \oplus 3\mathfrak{o}_{\mathbb{L}}$ and so $r(\mathcal{U}(\mathfrak{o}_K[C_{12}])) = 3$. Since the extensions are non-real, we have that $r(\mathcal{U}_1(\mathfrak{o}_K[C_{12}])) = r(\mathcal{U}(\mathfrak{o}_K[C_{12}])) \geq 2$, and, hence, we again have a contradiction.

We conclude that $\mathcal{U}_1(\mathfrak{o}_K[C_{12}])$ is not hyperbolic. □

4. Non-Abelian groups with hyperbolic unit groups

Theorem 2.2 classifies the finite non-Abelian groups G for which the unit group $\mathcal{U}_1(\mathbb{Z}[G])$ is hyperbolic. These groups are: S_3, D_4, Q_{12}, $C_4 \rtimes C_4$, and the Hamiltonian 2-group, where $Q_{12} = C_3 \rtimes C_4$, with C_4 acting non-trivially on C_3, and also on C_4 (see [13]).

Jespers, in [12], classified the finite groups G which have a normal non-Abelian free complement in $\mathcal{U}(\mathbb{Z}[G])$. The group algebra $\mathbb{Q}[G]$ of these groups has at most one matrix Wedderburn component which must be isomorphic to $M_2(\mathbb{Q})$.

Lemma 4.1. *Let G be a group and K a quadratic extension. If $M_2(K)$ is a Wedderburn component of $K[G]$, then $\mathbb{Z}^2 \hookrightarrow \mathcal{U}_1(\mathfrak{o}_K[G])$. In particular, $\mathcal{U}_1(\mathfrak{o}_K[G])$ is not hyperbolic.*

Proof. The ring $\Gamma = M_2(\mathfrak{o}_K)$ is a $\mathbb{Z}$-order in $M_2(K)$ and

$$X = \{e_{12}, e_{12}\sqrt{d}\} \subset \Gamma$$

is a set of commuting nilpotent elements of index 2, where e_{ij} denotes the elementary matrix. The set $\{1, \sqrt{d}\}$ is a linearly independent set over $\mathbb{Q}$, and hence so is X. Therefore, by Lemma 2.1, $\mathbb{Z}^2 \hookrightarrow \mathcal{U}_1(\Gamma) \subset \mathcal{U}_1(\mathfrak{o}_K[G])$, and so, $\mathcal{U}_1(\mathfrak{o}_K[G])$ is not hyperbolic. □

COROLLARY 4.2

If $G \in \{S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$, then $\mathcal{U}_1(\mathfrak{o}_K[G])$ is not hyperbolic.

Proof. We have that $K[G] \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[G])$. For each of the groups under consideration, $M_2(\mathbb{Q})$ is a Wedderburn component of $\mathbb{Q}[G]$; it therefore follows that $M_2(K)$ is a Wedderburn component of $K[G]$. The preceding lemma implies that $\mathcal{U}_1(\mathfrak{o}_K[G])$ is not hyperbolic. □

If H is a non-Abelian Hamiltonian 2-group, then $H = E \times Q_8$, where E is an elementary Abelian 2-group and Q_8 is the quaternion group of order 8. Since Q_8 contains a cyclic subgroup of order 4, it follows, by Theorem 3.7, that *if $\mathcal{U}_1(\mathfrak{o}_K[Q_8])$ is hyperbolic, then $\mathfrak{o}_K$ is not real.*

PROPOSITION 4.3

If G is a Hamiltonian 2-group of order greater than 8, then $\mathcal{U}_1(\mathfrak{o}_K[G])$ is not hyperbolic.

Proof. Let $G = E \times Q_8$ with E elementary Abelian of order $2^n > 1$. We then have $K[G] = K[E \times Q_8] \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[E \times Q_8]) \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[E])[Q_8] \cong K \otimes_{\mathbb{Q}} (2^n\mathbb{Q})[Q_8] \cong (2^n K)[Q_8]$. If $d = -1$, it is well-known that $K Q_8$ has a Wedderburn component isomorphic to $M_2(K)$ and hence, by Lemma 4.1, $\mathcal{U}_1(\mathfrak{o}_K Q_8)$ is not hyperbolic. If $d < -1$, then, by Proposition 3.6, $r(\mathcal{U}_1(\mathfrak{o}_K[C_4])) = 1$. Since C_4 is a subgroup of Q_8, it follows that $\mathcal{U}_1((2^n\mathfrak{o}_K)[C_4])$ embeds into $\mathcal{U}_1(\mathfrak{o}_K[G])$. Thus, since $\mathcal{U}_1(\prod_{2^n} \mathfrak{o}_K[C_4])$ has rank $2^n \geq 2$, $\mathcal{U}_1(\mathfrak{o}_K[G])$ is not hyperbolic. □

In view of the above Proposition, it follows that Q_8 is the only Hamiltonian 2-group for which $\mathcal{U}_1(\mathfrak{o}_K[G])$ can possibly be hyperbolic, and in this case $\mathfrak{o}_K$ is the ring of integers of an imaginary extension. By Lemma 4.1, $K[Q_8]$ can not have a matrix ring as a Wedderburn component. Since $\mathbb{Q}[Q_8] \cong 4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})$, we have $K[Q_8] \cong K \otimes_{\mathbb{Q}} (4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})) \cong 4K \oplus \mathcal{H}(K)$; hence $K[Q_8]$ must be a direct sum of division rings, or equivalently, has no non-zero nilpotent elements. In particular, $\mathcal{H}(K)$ is a division ring.

Theorem 4.4. *Let $K = \mathbb{Q}[\sqrt{d}]$, with d square-free integer $\neq 1$. Then $K[Q_8]$ is a direct sum of division rings if, and only if, one of the following holds*:

(i) $d \equiv 1 \pmod 8$;
(ii) $d \equiv 2$, *or* $3 \pmod 4$, *or* $d \equiv 5 \pmod 8$, *and* $d > 0$.

Proof. The assertion follows from Theorem 2.3 of [1]; Theorem 1, p. 236 of [3] and Theorem 3.2 of [16]. □

COROLLARY 4.5

If $K = \mathbb{Q}[\sqrt{d}]$, where d is a negative square-free integer, then the group $\mathcal{U}_1(\mathfrak{o}_K[Q_8])$ is not hyperbolic if $d \not\equiv 1 \pmod 8$.

Let $\mathbb{H}$: $\mathbb{C}\times]0, \infty[$ be the upper half-space model of three-dimensional hyperbolic space and Iso($\mathbb{H}$) its group of isometries. In the quaternion algebra $\mathcal{H} := \mathcal{H}(-1, -1)$ over $\mathbb{R}$, with its usual basis, we may identify $\mathbb{H}$ with the subset $\{z + rj : z \in \mathbb{C}, r \in \mathbb{R}^+\}$. The group $PSL(2, \mathbb{C})$ acts on $\mathbb{H}$ in the following way:

$$\begin{aligned} \varphi:\quad & PSL(2,\mathbb{C}) \times \mathbb{H} \longrightarrow \mathbb{H} \\ & (M, P) \qquad\qquad \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} P := (aP + b)(cP + d)^{-1}, \end{aligned}$$

where $(cP + d)^{-1}$ is calculated in $\mathcal{H}$. Explicitly, $MP = M(z + rj) = z^* + r^* j$, with

$$z^* = \frac{(az + b)(\overline{cz} + \overline{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2 r^2} \quad \text{and} \quad r^* = \frac{r}{|cz + d|^2 + |c|^2 r^2}.$$

Let K be an algebraic number field and $\mathfrak{o}_K$ its ring of integers. Let

$$SL_1(\mathcal{H}(\mathfrak{o}_K)) := \{x \in \mathcal{H}(\mathfrak{o}_K) : N(x) = 1\},$$

where N is the norm in $\mathcal{H}(K)$. Clearly the groups $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ and $\mathcal{U}(\mathfrak{o}_K) \times SL_1(\mathcal{H}(\mathfrak{o}_K))$ are commensurable. Consider the subfield $F = K[i] \subset \mathcal{H}(K)$ which is a maximal subfield in $\mathcal{H}(K)$. The inner automorphism σ,

$$\begin{aligned} \sigma:\quad & \mathcal{H}(K) \longrightarrow \mathcal{H}(K) \\ & x \qquad \mapsto \quad jxj^{-1}, \end{aligned}$$

fixes F. The algebra $\mathcal{H}(K) = F \oplus Fj$ is a crossed product and embeds into $M_2(\mathbb{C})$ as follows:

$$\begin{aligned} \Psi:\quad & \mathcal{H}(K) \hookrightarrow M_2(\mathbb{C}) \\ & x + yj \mapsto \begin{pmatrix} x & y \\ -\sigma(y) & \sigma(x) \end{pmatrix}. \end{aligned} \tag{4}$$

This embedding enables us to view $SL_1(\mathcal{H}(\mathfrak{o}_K))$ and $SL_1(\mathcal{H}(K))$ as subgroups of $SL(2, \mathbb{C})$ and hence $SL_1(\mathcal{H}(K))$ acts on $\mathbb{H}$.

PROPOSITION 4.6

Let $K = \mathbb{Q}[\sqrt{d}]$, $d \equiv 1 \pmod 8$ *a square-free negative integer, and* $\mathfrak{o}_K$ *its ring of integers. Then* $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ *and* $\mathcal{U}(\mathfrak{o}_K[Q_8])$ *are hyperbolic groups.*

Proof. Observe that $SL_1(\mathcal{H}(\mathfrak{o}_K))$ acts on the space $\mathbb{H}$ and, hence, is a discrete subgroup of $SL_2(\mathbb{C})$ (see Theorem 10.1.2, p. 446 of [6]). The quotient space $Y := \mathbb{H}/SL_1(\mathcal{H}(\mathfrak{o}_K))$ is a Riemannian manifold of constant curvature -1 and, since $\mathbb{H}$ is simply connected, we have that $SL_1(\mathcal{H}(\mathfrak{o}_K)) \cong \pi_1(Y)$. Since $d \equiv 1 \pmod 8$, $\mathcal{H}(K)$ is a division ring and, therefore, co-compact and Y is compact (see Theorem 10.1.2, item (3) of [6]). Hence $SL_1(\mathcal{H}(\mathfrak{o}_K))$ is hyperbolic (see Example 2.25.5 of [2]). Since $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ and $\mathcal{U}(\mathfrak{o}_K) \times SL_1(\mathcal{H}(\mathfrak{o}_K))$ are commensurable and $\mathcal{U}(\mathfrak{o}_K) = \{-1, 1\}$, it follows that $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ is hyperbolic. Since $\mathcal{U}(\mathfrak{o}_K[Q_8]) \cong \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ and $\mathcal{U}(\mathfrak{o}_K) \cong C_2$, we conclude that $\mathcal{U}(\mathfrak{o}_K[Q_8])$ is hyperbolic. □

Combining the results in the present and the preceding section, we have the following main result.

Theorem 4.7. *Let* $K = \mathbb{Q}[\sqrt{d}]$, *with* d *square-free integer* $\neq 1$, *and* G *a finite group. Then* $\mathcal{U}_1(\mathfrak{o}_K[G])$ *is hyperbolic if, and only if,* G *is one of the groups listed below and* $\mathfrak{o}_K$ *(or* K*) is determined by the corresponding value of* d:

1. $G \in \{C_2, C_3\}$ *and* d *arbitrary*;
2. G *is an Abelian group of exponent dividing* n *for*: $n = 2$ *and* $d < 0$, *or* $n = 4$ *and* $d = -1$, *or* $n = 6$ *and* $d = -3$.
3. $G = C_4$ *and* $d < 0$.
4. $G = C_8$ *and* $d = -1$.
5. $G = Q_8$ *and* $d < 0$ *and* $d \equiv 1 \pmod 8$.

Remark. If the group $\mathcal{U}(\mathfrak{o}_K[Q_8])$ is hyperbolic, then the hyperbolic boundary $\partial(\mathcal{U}(\mathfrak{o}_K[Q_8])) \cong \mathbb{S}^2$, the Euclidean sphere of dimension 2, and End$(\mathcal{U}(\mathfrak{o}_K[Q_8]))$ has one element (see Example 2.25.5 of [2]). Note that if $\mathcal{U}(\mathbb{Z}[G])$ is an infinite non-Abelian hyperbolic group, then $\partial(\mathcal{U}(\mathbb{Z}[G]))$ is totally disconnected and is a Cantor set. So, in this case, $\mathcal{U}(\mathbb{Z}[G])$ has infinitely many ends and also is a virtually free group, (Theorem 2 of [13] and §3 of [8]). However, if $\mathcal{U}(\mathfrak{o}_K[G])$ is a non-Abelian hyperbolic group, then $\mathcal{U}(\mathfrak{o}_K[G])$ is an infinite group which is not virtually free, it has one end and $\partial(\mathcal{U}(\mathbb{Z}[G]))$ is a smooth manifold.

5. Pell and Gauss units

When the algebra $\mathcal{H}(K)$ is isomorphic to $M_2(K)$, it is known how to construct the unit group of a $\mathbb{Z}$-order up to a finite index. Nevertheless, if $\mathcal{H}(K)$ is a division ring, this is a highly non-trivial task; see [5], for example. In this section we study a construction of units of $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ in the case when the quaternion algebra $\mathcal{H}(K)$ is a division ring.

In the sequel, $K = \mathbb{Q}[\sqrt{-d}]$ is an imaginary quadratic extension with d a square-free integer congruent to $7 \pmod 8$, and $\mathfrak{o}_K$ the ring of integers of the field K. Note that $s(K)$, the *stufe* of K, is 4, the quaternion algebra $\mathcal{H}(K)$ is a division ring and $\mathcal{U}(\mathfrak{o}_K) = \{\pm 1\}$. Thus, if $u = u_1 + u_i i + u_j j + u_k k \in \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$, then its norm $N(u) = u_1^2 + u_i^2 + u_j^2 + u_k^2 = \pm 1$; furthermore, if any of the coefficients u_1, u_i, u_j, u_k is zero, then $N(u) = 1$, $s(K)$ being 4.

The representation of u, given by (4), is

$$[u] := \Psi(u) = \begin{pmatrix} u_1 + u_i i & u_j + u_k i \\ -u_j + u_k i & u_1 - u_i i \end{pmatrix} \in M_2(\mathbb{C}).$$

Denote by χ_u the characteristic polynomial of $[u]$, and by m_u its minimal polynomial. The degree $\partial(\chi_u)$ of χ_u is 2 and therefore $\partial(m_u) \leq 2$. If $\partial(m_u) = 1$, then $m_u(X) = X - z_0$, $z_0 \in \mathbb{C}$, and therefore $u = z_0$. Note that the characteristic polynomial is $\chi_u(X) = X^2 - \mathrm{trace}([u])X + \det([u])$, where $\mathrm{trace}([u]) = u_1 + u_i i + \sigma(u_1 + u_i i) = 2u_1$ and $\det([u]) = \pm 1$:

$$\chi_u(X) = X^2 - 2u_1 X \pm 1.$$

PROPOSITION 5.1

Let $u = u_1 + u_i i + u_j j + u_k k \in \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$. Then the following statements hold:

1. $u^2 = 2u_1 u - N(u)$.
2. *If $N(u) = 1$, then u is a torsion unit if, and only if, $u_1 \in \{-1, 0, 1\}$ and the order of u is* 1, 2, *or* 4.
3. *If $N(u) = -1$, then order of u is infinite.*

Proof.
(1) is obvious.
(2) Suppose $N(u) = 1$ and u is a torsion unit of order n, say. If $X^2 - 2u_1 X + \eta(u) = (X - \zeta_1)(X - \zeta_2)$, then ζ_i, $i = 1, 2$ are roots of unity and $\zeta_1\zeta_2 = 1$. It follows that $2u_1 = \zeta_1 + \zeta_2$ is a real number. Since $u \in \mathcal{H}(\mathfrak{o}_K)$ and $\{1, \vartheta\}$ is an integral basis of $\mathfrak{o}_K$, it follows that $u_1 \in \mathbb{Z}$. From the equality $2u_1 = \zeta_1 + \zeta_2$, we have $2|u_1| = |\zeta_l + \zeta_2| \leq 2$, and therefore $u_1 \in \{-1, 0, 1\}$. If $u_1 = 0$, then $u^2 = -1$ and therefore $o(u) = 4$. If $u_1 = \pm 1$, then $\chi_u(X) = X^2 \mp 2X + 1 = (X \mp 1)^2$, and therefore $0 = \chi_u(u) = (u \mp 1)^2 \in \mathcal{H}(K)$; hence $u = \pm 1$.
(3) If $N(u) = -1$, then $u^2 = 2u_1 u + 1$, $(u^2)_1 = 2u_1^2 + 1$, $\eta(u^2) = 1$. If u were a torsion unit, then, by (2) above, $(u^2)_1 \in \{-1, 0.1\}$. If $(u^2)_1 = 0$, then $1/2 = -u_1^2 \in \mathfrak{o}_K$, which is not possible. If $(u^2)_1 = 1$, then $u_1 = 0$, and therefore $u^2 = 1$ yielding $u = \pm 1$ which is not the case, because $N(u) = -1$. Finally, if $(u^2)_1 = -1$, then $u_1^2 = -1$ which implies that $\sqrt{-1} \in K$ which is also not the case, because $\mathcal{H}(K)$ is a division ring. Hence $u \in \mathcal{U}(\mathfrak{o}_K)$ is an element of infinite order. □

Let $\xi \neq \psi$ be elements of $\{1, i, j, k\}$. Suppose

$$u := m\sqrt{-d}\xi + p\psi, \quad p, m \in \mathbb{Z}, \tag{5}$$

is an element in $\mathcal{H}(\mathfrak{o}_K)$ having norm 1. Then

$$p^2 - m^2 d = 1, \tag{6}$$

i.e., (p, m) is a solution of the Pell's equation $X^2 - dY^2 = 1$. Let $\mathbb{L} := \mathbb{Q}[\sqrt{d}]$. Equation (6) implies that $\epsilon = p + m\sqrt{d}$ is a unit in $\mathfrak{o}_{\mathbb{L}}$. Conversely, if $\epsilon = p + m\sqrt{d}$ is a unit of norm 1 in $\mathfrak{o}_{\mathbb{L}}$, then, necessarily, $p^2 - m^2 d = 1$, and, therefore, for any choice of ξ, ψ in $\{1, i, j, k\}$, $\xi \neq \psi$,

$$m\sqrt{-d}\xi + p\psi \tag{7}$$

is a unit in $\mathcal{H}(\mathfrak{o}_K)$. In particular,

$$u_{(\epsilon,\psi)} := p + m\sqrt{-d}\psi,\ \ \psi \in \{i, j, k\} \tag{8}$$

is a unit in $\mathcal{H}(\mathfrak{o}_K)$.

With the notations as above, we have:

PROPOSITION 5.2

1. *If* $1 \notin \mathrm{supp}(u)$, *the support of* u, *then* u *is a torsion unit.*
2. *If* $\epsilon = p + m\sqrt{d}$ *is a unit in* $\mathfrak{o}_{\mathbb{L}}$, *then*

$$u^n_{(\epsilon,\psi)} = u_{(\epsilon^n,\psi)}$$

for all $\psi \in \{i, j, k\}$ *and* $n \in \mathbb{Z}$.

Proof. If $1 \notin \mathrm{supp}(u)$, then $u_1 = 0$; therefore, by Proposition 5.1, u is torsion unit.

Let $\mu = A + B\sqrt{d}$ and $\nu = C + D\sqrt{d}$, be units in $\mathfrak{o}_{\mathbb{L}}$. Then $u_{(\mu,\psi)} = A + B\sqrt{-d}\psi$ and $u_{(\nu,\psi)} = C + D\sqrt{-d}\psi$ are units in $\mathcal{H}(\mathfrak{o}_K)$. We have

$$\mu\nu = AC + dBD + (AD + BC)\sqrt{d}.$$

Also $u_{(\mu,\psi)}u_{(\nu,\psi)} = (AC + dBD) + (AD + BC)\sqrt{-d}\psi = u_{(\mu\nu,\psi)}$. It follows that we have $u^n_{(\epsilon,\psi)} = u_{(\epsilon^n,\psi)}$ for all $\psi \in \{i, j, k\}$ and $n \in \mathbb{Z}$. □

The units (7) constructed above are called 2-*Pell units*.

PROPOSITION 5.3

Let $L = \mathbb{Q}[\sqrt{2d}]$, $2d$ *square-free*, ξ, ψ, ϕ *pairwise distinct elements in* $\{1, i, j, k\}$ *and* $p, m \in \mathbb{Z}$. *Then the following are equivalent*:

(i) $u := m\sqrt{-d}\xi + p\psi + (1 - p)\phi \in \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$.
(ii) $\epsilon := (2p - 1) + m\sqrt{2d} \in \mathcal{U}(\mathfrak{o}_L)$.

Proof. If u is a unit in $\mathcal{H}(\mathfrak{o}_K)$, then $N(u) = -m^2d + p^2 + (1 - p)^2 = 1$, i.e., $2p^2 - 2p - m^2d = 0$, and thus $(2p - 1)^2 - m^2 2d = 1$. Consequently, $\epsilon = (2p - 1) + m\sqrt{2d}$ is invertible in $\mathfrak{o}_L$. The steps being reversible, the equivalence of (i) and (ii) follows. □

The units constructed above are called 3-*Pell units*. We shall next determine units of the form $u = m\sqrt{-d} + (m\sqrt{-d})i + pj + qk$, with $m, p, q \in \mathbb{Z}$ and $N(u) = -2m^2d + p^2 + q^2 = 1$. Set $p + q =: r$ and consider the equation

$$2p^2 - 2pr - 2m^2d + r^2 - 1 = 0. \tag{9}$$

Theorem 5.4. *If* $r = 1$, *then equation* (9) *has a solution in* $\mathbb{Z}$, *and for each such solution*, $u = m\sqrt{-d} + (m\sqrt{-d})i + pj + qk$ *is a unit in* $\mathcal{H}(\mathfrak{o}_K)$ *of norm* 1.

Proof. Viewed as a quadratic equation in p, (9) has real roots

$$p = \frac{1 \pm \sqrt{1+4m^2d}}{2}.$$

To obtain a solution in $\mathbb{Z}$, we need the argument under the radical to be a square; we thus need to solve the diophantine equation

$$X^2 - 4dY^2 = 1. \tag{10}$$

Let $\epsilon = x + y\sqrt{d}$, with $x,\ y \in \mathbb{Z}$, be a unit in $\mathfrak{o}_{\mathbb{L}}$ having infinite order. Replacing ϵ by ϵ^2, if necessary, we can assume that y is even. We then have $x^2 - y^2d = 1$, and so x must be odd. Taking $m = y/2$ and $p = \frac{1\pm x}{2}$, we obtain a solution of (10) in $\mathbb{Z}$. Clearly, for such a solution, the element u lies in $\mathcal{H}(\mathfrak{o}_K)$ and has norm 1. □

Using Gauss' result which states that a positive integer n is a sum of three squares if, and only if, n is not of the form $4^a(8b-1)$, where $a \geq 0$ and $b \in \mathbb{Z}$, it is easy to see that, for every integer $m \equiv 2 \pmod 4$, the integers $m^2d - 1$ and $m^2d + 1$ can be expressed as sums of three squares. We can thus construct units $u = m\sqrt{-d} + pi + qj + rk \in \mathcal{H}(\mathfrak{o}_K)$ having prescribed norm 1 or -1; we call such units *Gauss units*.

Example. In [5], all units exhibited in $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$ are of norm 1. We present some units of norm -1 in this ring. The previous theorem guarantees the existence of integers p, q, r, such that

$$u = 6\sqrt{-7} + pi + qj + rk$$

is a unit of norm -1. Indeed,

$$(p, q, r) \in \{(\pm 15, \pm 5, \pm 1),\ (\pm 13, \pm 9, \pm 1),\ (\pm 11, \pm 11, \pm 3)\},$$

and the triples obtained by permutation of coordinates, are all possible integral solutions. In [5], the authors have constructed a set S of generators of the group $SL_1(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}))$. If v_0 is a unit of $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})$ having norm -1, then clearly $\langle v_0, S\rangle = \mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}))$. Thus, for example, taking $v_0 = 6\sqrt{-7} + 15i + 5j + k$, we have

$$\mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]})) = \langle v_0, S\rangle. \tag{11}$$

The set $\left\{1, \frac{1+\sqrt{-7}}{2}\right\}$ is an integral basis of $R = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$. Consider units of the form

$$\frac{m+\sqrt{-d}}{2} \pm \left(\frac{m-\sqrt{-d}}{2}\right) i + pj.$$

These are neither Pell nor Gauss units. Those of norm ± 1, are solutions of the equation

$$m^2 + 2p^2 = \pm 2 + d \tag{12}$$

in $\mathbb{Z}$. The main result of [5] states that if $d = 7$, then the units of norm 1 of the above type, together with the trivial units i and j, generate the group $SL_1(\mathcal{H}(R))$.

For $d \equiv 7 \pmod 8$, there are no units of norm -1 of the above type, since, in this case, the equation $m^2 + 2p^2 = -2 + d$ has no solution in $\mathbb{Z}$, as can be easily seen working module 8.

In case $d \neq 7$, we give some more examples of negative norm units of the form $\frac{m+\sqrt{-d}}{2} \pm \left(\frac{m-\sqrt{-d}}{2}\right)i + pj$.

If $d = 15$, then eq. (12) becomes $m^2 + 2p^2 = 17$; the pairs $(m, p) \in \{(3, 2), (3, -2), (-3, 2), (-3, -2)\}$ are its integral solutions. For $m = 3$ either $p = 2$ or $p = -2$ and so there are 8 units. Each coefficient of u is distinct, hence for each solution (m, p) there are 3! units with the same support, thus there are 36 different units for a given fixed support. By Proposition 5.1, all these units have infinite order if $u_1 \notin \{-1, 0, 1\}$. If $1 \in \operatorname{supp}(u)$, then either $\{i, j\} \subset \operatorname{supp}(u)$ or $\{i, k\} \subset \operatorname{supp}(u)$, or $\{j, k\} \subset \operatorname{supp}(u)$. Therefore there are 108 of these units and, for example,

$$\frac{3+\sqrt{-15}}{2} + \left(\frac{3-\sqrt{-15}}{2}\right) j - 2k$$

is one of them.

If $1 \notin \operatorname{supp}(u)$, then u is a torsion unit, so there are 36 torsion units of this type. One of them is the unit

$$\left(\frac{-3-\sqrt{-15}}{2}\right) i + \left(\frac{-3+\sqrt{-15}}{2}\right) j + 2k,$$

of order 4.

For $d = 31$, we obtain $m^2 + 2p^2 = 33$ whose solutions in $\mathbb{Z}$ are: $(m, p) \in \{(1, 4), (1, -4), (-1, 4), (-1, -4)\}$.

As another example of a unit of norm -1 in a quaternion algebra, we may mention that, in $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-23}]})$, $u = 5\sqrt{-23} + 23i + 6j + 3k$ is a unit of norm -1.

We next exhibit some Gauss units of norm 1. For $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-15}]})$, there exist p, q, r, such that $u = 10\sqrt{-15} + pi + qj + rk$ is a unit of norm 1. In fact, (36, 14, 3), (36, 13, 6), (32, 21, 6), (30, 24, 5) are some of the possible choices for (p, q, r). For $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-23}]})$, $u = 2\sqrt{-23} + 8i + 5j + 2k$ is a unit of norm 1. It is interesting to note that $u = 3588\sqrt{-23} + 12168i + 12167j$ is a Gauss unit, although 4 divides 3588.

We conclude with the following result:

Theorem 5.5. *Let $K = \mathbb{Q}[\sqrt{-d}]$, $0 < d \equiv 7 \pmod 8$ and $\mathfrak{o}_K$ the ring of integers of K. If $\epsilon = p + m\sqrt{d}$ is a unit in $\mathbb{Z}[\sqrt{d}]$, and $x := u_{(\epsilon,\psi)}$, $y := u_{(\epsilon,\psi')}$ are two 2-Pell units in $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$, where ψ and $\psi' \in \{i, j, k\}$ and $\psi \neq \psi'$, then there exists a natural number m such that $\langle x^m, y^m\rangle$ is a free group of rank 2.*

Proof. By Proposition 4.6, $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ is a hyperbolic group. In view of Proposition III.Γ. 3.20 of [4], there exists a natural m, such that, $\langle x^m, y^m\rangle$ is a free group of rank at most 2. However, Proposition 5.2, item (2) ensures that $\langle x\rangle \cap \langle y\rangle = \{1\}$. Therefore, $\langle x^m, y^m\rangle$ has rank at least 2, and hence $\langle x^m, y^m\rangle$ is a free group of rank 2. □

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