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On an extension of a combinatorial identity

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Abstract. Using Frobenius partitions we extend the main results of [4]. This leads to an infinite family of 4-way combinatorial identities. In some particular cases we get even 5-way combinatorial identities which give us four new combinatorial versions of Göllnitz–Gordon identities.

Keywords. n-Color partitions; lattice paths; Frobenius partitions; Göllnitz–Gordon identities; combinatorial interpretations.

1. Introduction, definitions and the main results

Agarwal in [1] proved the following theorem.

Theorem 1.1. *Given a positive integer k, let $A_k(\nu)$ denote the number of partitions of ν in which each part $\geq k$, minimal difference ≥ 2 between the parts, consecutive odd integers are not allowed if k is even and consecutive even integers are not allowed if k is odd. Then*

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2:q^2)_n} q^{n(n+k-1)}, \tag{1.1}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

This theorem for $k = 1$ and 3 reduces to the following Göllnitz–Gordon (cf. [8, 9]) identities, respectively.

Identity 1. The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear is equal to the number of partitions of n into parts which are congruent to 1, 4 or 7 (mod 8).

Identity 2. The number of partitions of n into parts differing by at least 2 among which no two consecutive even numbers appear and with each part being at least equal to 3 is equal to the number of partitions of n into parts which are congruent to 3, 4 or 5 (mod 8).

Identities 1 and 2 are the combinatorial interpretations of the following analytical identities respectively,

$$\prod_{n=1}^{\infty}(1-q^{8n-3})(1-q^{8n-5})(1-q^{8n})$$
$$=\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1+q^{2n+1})}\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2:q^2)_n}, \text{(eq. (36), p. 155 of [10])} \tag{1.2}$$

$$\prod_{n=1}^{\infty}(1-q^{8n-1})(1-q^{8n-7})(1-q^{8n})$$
$$=\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1+q^{2n+1})}\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{n(n+2)}}{(q^2:q^2)_n}, \text{(eq. (34), p. 155 of [10]).} \tag{1.3}$$

The following identity is due to Göllnitz [7].

Identity 3. The number of partitions of n into parts differing by at least 2 among which no two consecutive odd numbers appear and with each part being atleast equal to 2 is equal to the number of partitions of n into parts which are congruent to 2, 3 or 7 (mod 8).

Identity 3 is the combinatorial interpretation of the following analytical identity (Corollary 2.7, p. 21 with $q = q^2$ and $a = -q$ of [5])

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)}}{(q^2:q^2)_n} = (-q^2;q^2)_\infty(-q^3;q^4)_\infty. \tag{1.4}$$

Recently, Theorem 1.1 was extended in [4] by using n-color partitions and the lattice paths. This resulted in a 4-way extension of Göllnitz–Gordon identities 1 and 2 as well as Gollnitz identity 3. The 4-way extension provides six identities in the usual sense leading to five new combinatorial versions of the Göllnitz–Gordon identities 1 and 2 and the Göllnitz identity 3. The objective of this paper is to extend the main result of [4] using Frobenius partitions. For each value of k our extension gives an infinite family of 4-way identities. Each of the 4-way identities gives six combinatorial identities in the usual sense. In the particular cases when $k = 1$, 2 or 3, we get even 5-way identities. Our each five way identity yields 10 combinatorial identities in the usual sense including identities 1, 2 and 3 and their five more combinatorial interpretations found in [4]. Out of 10 combinatorial interpretations of the analytic identities (1.2), (1.3) and (1.4) four are entirely new. Before we reproduce the result of [4] and state the main result of this paper we recall the following definitions:

DEFINITION 1.1 [2]

An n-color partition of a positive integer is a partition in which a part of size n can come in n-different colors denoted by subscripts: $n_1, n_2, n_3, \ldots, n_n$ and the parts satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < \cdots$. Thus, for example, the n-color partitions of 3 are $3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1$.

DEFINITION 1.2 [2]

The weighted difference of two parts m_i, n_j, $m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.

Next, we recall the definition of a Frobenius partition.

DEFINITION 1.3 [6]

A two-rowed array of non-negative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}, a_1 > a_2 > \cdots > a_r \geq 0, b_1 > b_2 > \cdots > b_r \geq 0,$$

is known as a Frobenius partition of n if $n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$.

For example, $n = 28 = 4 + (6 + 5 + 2 + 0) + (5 + 3 + 2 + 1)$ and the corresponding Frobenius notation is $\begin{pmatrix} 6\,5\,2\,0 \\ 5\,3\,2\,1 \end{pmatrix}$.

DEFINITION 1.4 [3]

Lattice paths are defined as paths of finite length lying in the first quadrant. They will begin on the y-axis or on the x-axis and terminate on the x-axis. Only three moves are allowed at each step:

Northeast: from (i, j) to $(i + 1, j + 1)$,

Southeast: from (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$,

Horizontal: from $(i, 0)$ to $(i + 1, 0)$, only allowed along x-axis.

The following terminologies will be used in describing lattice paths:

Peak. Either a vertex on y-axis which is followed by a Southeast step or a vertex preceded by a Northeast step and followed by a Southeast step.

Mountain. A section of the path, which starts on either the x-axis or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

Plain. A section of path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a Southeast step and ends at a vertex followed by a Northeast step.

The *height* of a vertex is its y-coordinate. The *weight* of a vertex is x-coordinate. The *weight of a path* is the sum of the weights of its peaks.

In [4], Agarwal and Rana proved the following theorem:

Theorem 1.2. *Let $B_k(\nu)$ denote the number of n-color partitions of ν such that the parts $\geq k$, parts used are of the type $(2l-1)_1$ and $(2l)_2$ if k is odd, $(2l-1)_2$ and $(2l)_1$ if k is even. The weighted difference between any two parts is nonnegative and even. Let $C_k(\nu)$ denote the number of lattice paths of weight ν which start at* $(0, 0)$, *have no valley above height* 0, *there is a plain of length* $(k-1)+2s$, $s \geq 0$, *in the beginning of the path, other plains, if any, are of even lengths, the height of each peak of odd* (*resp.*, *even*) *weight is* 1 (*resp.*, 2) *if k is odd and* 2 (*resp.*, 1) *if k is even. Then*

$$A_k(\nu) = B_k(\nu) = C_k(\nu), \tag{1.5}$$

for all ν, where $A_k(\nu)$ is as defined in Theorem 1.1.

In this paper we propose to prove the following theorems which extend Theorem 1.2.

Theorem 1.3. *For k an odd positive integer, let $D_k(\nu)$ denote the number of Frobenius partitions of ν such that*

(1.a) $a_i \geq b_i$; *more specifically,*

(1.b) $a_i = \begin{cases} b_i & \text{if } a_i \text{ and } b_i \text{ are of same parity} \\ b_i + 1 & \text{if } a_i \text{ and } b_i \text{ are of opposite parity} \end{cases}$

(1.c) $b_i > a_{i+1}$, $1 \leq i \leq r-1$, *and*

(1.d) $a_i \geq (k - b_i - 1)$.

Let $E_k(\nu)$ denote the number of n-color partitions of ν such that

(1.e) *only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the form $(2l-1)_1$ or $(2l)_2$,*

(1.f) *the size of each part is $\geq k$, and*

(1.g) *the weighted difference of any two parts is nonnegative and even. Then $D_k(\nu) = E_k(\nu)$, for all ν.*

Remark 1. Here $E_k(\nu)$ is the function $B_k(\nu)$ of Theorem 1.2 when k is odd.

Theorem 1.4. *For a non zero even integer k, let $F_k(\nu)$ denote the number of Frobenius partitions of ν such that*

(1.a) $a_i > b_i$; *more specifically,*

(1.b) $a_i = \begin{cases} b_i + 2 & \text{if } a_i \text{ and } b_i \text{ are of same parity} \\ b_i + 1 & \text{if } a_i \text{ and } b_i \text{ are of opposite parity} \end{cases}$

(1.c) $b_i > a_{i+1}$, $1 \leq i \leq r-1$, *and*

(1.d) $a_i \geq (k - b_i - 1)$.

Let $G_k(\nu)$ denote the number of n-color partitions of ν such that

(1.e) *only the second copy of the odd parts and the first copy of the even parts are used, that is, the parts are of the form $(2l-1)_2$ or $(2l)_1$,*

(1.f) *the size of each part is $\geq k$, and*

(1.g) *the weighted difference of any two parts is nonnegative and even. Then*

$$F_k(\nu) = G_k(\nu), \text{ for all } \nu.$$

Remark 2. Here $G_k(\nu)$ is the function $B_k(\nu)$ of Theorem 1.2 when k is even.

In our next section we give the detailed proof of Theorem 1.3 and sketch the proof of Theorem 1.4.

In §3 we shall discuss some particular cases which provide four new combinatorial versions of the identities 1, 2 and 3 in addition to the six versions given in [4].

2. Proof of Theorem 1.3

We establish a 1–1 correspondence between the Frobenius partitions enumerated by $A_k(\nu)$ and the n-color partitions enumerated by $B_k(\nu)$. We do this by mapping each column $\binom{a}{b}$

of the Frobenius partitions to a single part m_i of n-color partitions enumerated by $B_k(\nu)$. The mapping ϕ is

$$\phi\colon \binom{a}{b} \to (a+b+1)_{a-b+1}, \tag{2.1}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1}\colon m_i = \begin{cases} \begin{pmatrix} m/2 \\ (m-2)/2 \end{pmatrix} & \text{if } m \equiv 0 \pmod 2, \\ \begin{pmatrix} (m-1)/2 \\ (m-1)/2 \end{pmatrix} & \text{if } m \equiv 1 \pmod 2. \end{cases} \tag{2.2}$$

Clearly (2.1) and (1.b) imply (1.e). (2.1) and (1.d) imply (1.f).

Now suppose we have two adjacent columns $\binom{a}{b}$ and $\binom{c}{d}$ in a Frobenius partition with $\phi\colon \binom{a}{b} = m_i$ and $\phi\colon \binom{c}{d} = n_j$. Then since $\binom{a}{b} \to (a+b+1)_{a-b+1} = m_i$ and $\binom{c}{d} \to (c+d+1)_{c-d+1} = n_j$, we have

$$\begin{aligned} ((m_i - n_j)) &= m - n - i - j \\ &= (a+b+1) - (c+d+1) - (a-b+1) - (c-d+1) \\ &= 2b - 2c - 2 = 2(b-c) - 2 \geq 0, \end{aligned}$$

in view of (1.c), which implies (1.g).

To see the reverse implication we note that (2.2) implies (1.a), (2.2) and (1.e) imply (1.b) and (2.2) and (1.f) imply (1.d).

Further we see that under ϕ^{-1}:

$$b - c = \begin{cases} \frac{((m_i-n_j))}{2} + 1 & \text{if } m \equiv 1, n \equiv 1 \pmod 2, \\ \frac{((m_i-n_j))}{2} + 1 & \text{if } m \equiv 0,\ n \equiv 1 \pmod 2, \\ \frac{((m_i-n_j))}{2} + 1 & \text{if } m \equiv 1,\ n \equiv 0 \pmod 2, \\ \frac{((m_i-n_j))}{2} + 1 & \text{if } m \equiv 0,\ n \equiv 0 \pmod 2. \end{cases} \tag{2.3}$$

From above it is obvious that $b - c$ is a positive integer as $((m_i - n_j))$ is non negative and even. Thus we have established the bijection between Frobenius partitions on one hand and the n-color partitions on the other hand. To illustrate the bijection we have constructed, we give an example for $\nu = 8$, $k = 1$ shown in table 1.

Proof of Theorem 1.4. The proof of Theorem 1.4 is similar to the proof of Theorem 1.3 with mapping ϕ as

$$\phi\colon \binom{a}{b} \to (a+b+1)_{a-b},$$

Table 1.

Frobenius partitions enumerated by $A_1(8)$	Images under ϕ
$\begin{pmatrix}4\\3\end{pmatrix}$	8_2
$\begin{pmatrix}3 & 0\\3 & 0\end{pmatrix}$	$7_1 + 1_1$
$\begin{pmatrix}3 & 1\\2 & 0\end{pmatrix}$	$6_2 + 2_2$
$\begin{pmatrix}2 & 1\\2 & 1\end{pmatrix}$	$5_1 + 3_1$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1}\colon m_i = \begin{cases} \begin{pmatrix} m/2 \\ (m-2)/2 \end{pmatrix} & \text{if } m \equiv 0 \pmod 2, \\ \begin{pmatrix} (m+1)/2 \\ (m-3)/2 \end{pmatrix} & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

3. Particular cases of Theorems 1.3 and 1.4

Case 1. When $k = 1$, Theorem 1.3 in view of the Identity (1.2) reduces to the following 5-way identity.

$$A_1(\nu) = B_1(\nu) = C_1(\nu) = D_1(\nu) = X_1(\nu), \tag{3.1}$$

where $X_1(\nu)$ is the number of partitions of ν into parts which are congruent to 1, 4 or 7 (mod 8).

The case $A_1(\nu) = X_1(\nu)$ of (3.1) is the Göllnitz–Gordon first identity. In fact out of the 10 two-way identities induced by (3.1), 6 induced by $A_1(\nu) = B_1(\nu) = C_1(\nu) = X_1(\nu)$ are given in [4] and the remaining four which we get because of the function $D_1(\nu)$ are new.

Case 2. When $k = 3$, Theorem 1.3 in view of identity (1.3) reduces to the following 5-way identity.

$$A_3(\nu) = B_3(\nu) = C_3(\nu) = D_3(\nu) = X_3(\nu), \tag{3.2}$$

where $X_3(\nu)$ is the number of partitions of ν into parts which are congruent to 3, 4 or 5 (mod 8).

The case $A_3(\nu) = X_3(\nu)$ of (3.2) is the Göllnitz–Gordon second identity. As in Case 1, here also out of the 10 two-way identities induced by (3.2), 6 induced by $A_3(\nu) = B_3(\nu) = C_3(\nu) = X_3(\nu)$ are given in [4] and the remaining four which we get because of the function $D_3(\nu)$ are new.

Case 3. When $k = 2$, Theorem 1.4 in view of identity (1.4) reduces to the following 5-way identity:

$$A_2(\nu) = B_2(\nu) = C_2(\nu) = F_2(\nu) = X_2(\nu), \tag{3.3}$$

where $X_3(\nu)$ is the number of partitions of ν into parts which are congruent to 2, 3 or 7 (mod 8).

The case $A_2(\nu) = X_2(\nu)$ of (3.3) is the Göllnitz identity. And out of the 10 two-way identities induced by (3.3), 6 induced by $A_2(\nu) = B_2(\nu) = C_2(\nu) = X_2(\nu)$ are given in [4] and the remaining four which we get because of the function $D_2(\nu)$ are new.

Remark. We have provided new combinatorial interpretations of identities 1, 2 and 3 by using n-color partitions, lattice paths and Frobenius partitions. The question which arises here is: are there other combinatorial objects which can be used to interpret (1.2)–(1.4) combinatorially?

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