Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 4, November 2008, pp. 627–635.

On existence and stability of solutions for higher order semilinear Dirichlet problems

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MS received 20 March 2007; revised 29 May 2008

Abstract. We provide existence and stability results for semilinear Dirichlet problems with nonlinearity satisfying general growth conditions. We consider the case when both the coefficients of the differential operator and the nonlinear term depend on the numerical parameter. We show applications for the fourth order semilinear Dirichlet problem.

Keywords. Dirichlet problem; dual variational method; existence of solutions; stability.

1. Introduction

The aim of the paper is to investigate by a variational approach the higher order semilinear Dirichlet problems with nonlinearities being convex in a certain interval and which are subject to some general local growth conditions. We take up the existence of the solutions and their stability in such a case when both the differential operator and the nonlinear term depend on a parameter. As far as the existence result is concerned it appears to be convenient to formulate the abstract realization of the considered problem and later prove that the abstract problem has a solution. In order to obtain stability results we use our growth conditions together with some mild additional assumption. The main point is that we do not need any global growth assumption on the nonlinearity term. It is possible since we consider the action functionals on certain sets which allows for constructing a new type of duality theory. Some earlier version of these results has been published in [3], however now the setting is more general and therefore different applications are obtained (see Theorems 1.1 and 1.2 below). The dual methods of the calculus of variations originates from [7], however in the present paper we apply a different and considerably simplified methodology in obtaining the existence result. Concerning the existence of solutions our approach is also different from the author's earlier abstract variational methods (see [2]), in its reasoning, while it gives the same existence result.

A model problem, as far as the stability and existence results are concerned, which may be comprised by our methods, is the following one for $k = 0, 1, 2, \ldots$

$$\beta_k \frac{\mathrm{d}^4}{\mathrm{d}t^4} x + \gamma_k \frac{\mathrm{d}^2}{\mathrm{d}t^2} x + \delta_k x = F_x^k(t, x),$$

$$x(0) = x(\pi) = \mathrm{sgn}(\beta_k)\dot{x}(0) = \mathrm{sgn}(\beta_k)\dot{x}(\pi) = 0, \tag{1.1}$$

where $\mathrm{sgn}(\beta_k) = 0$ in the case $\beta_k = 0$ and $\mathrm{sgn}(\beta_k) = 1$ in the case $\beta_k > 0$.

O1: For all $k = 0, 1, 2, \dots$ the constants $\beta_k, \gamma_k, \delta_k$ are such that $\beta_k - \gamma_k - |\delta_k| > 0$, $\beta_k \in [\beta', \beta'']$, $\gamma_k \in [\gamma', \gamma'']$, $\delta_k \in [\delta', \delta'']$, $\gamma'' < 0$, $\beta' \geq 0$; sequences $\{\beta_k\}_{k=1}^{\infty}, \{\gamma_k\}_{k=1}^{\infty}, \{\delta_k\}_{k=1}^{\infty}$ are convergent to $\beta_0, \gamma_0, \delta_0$.

Let $\{d_k\}_{k=1}^{\infty}$ be a sequence of decreasing positive numbers bounded away from 0 and let $d_0 > d_k$ for all $k = 1, 2, \dots$. We assume for all $k = 0, 1, 2, \dots$ that

F1: F^k, F_x^k: $[0, \pi] \times [-d_0, d_0] \to \mathbb{R}$ are Caratheodory functions, F_k is continuously differentiable and convex with respect to the second variable in $[-d_0, d_0]$ for a.e. $t \in [0, \pi]$;

$$\max_{x \in [-d_k, d_k]} |F_x^k(t, x)| \leq \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k. \tag{1.2}$$

F2: $F_x^k(t, 0) \neq 0$ for a.e. $t \in [0, \pi]$, $t \to F^k(t, 0)$ and $t \to (F^k)^*(t, 0)$ are integrable.

F3: F_x^k is differentiable in x on $[-d_0, d_0]$ for a.e. $t \in [0, \pi]$ and there exists a constant $a > 0$ (independent of k) such that for a.e. $t \in [0, \pi]$,

$$\max_{x \in [-d_0, d_0]} |F_{xx}^k(t, x)| \leq a. \tag{1.3}$$

Here $(F^k)^*$ denotes the Fenchel–Young conjugate of a function F^k with respect to the second variable (see [1]). Let $L_k x = \beta_k \frac{d^4}{dt^4} x + \gamma_k \frac{d^2}{dt^2} x + \delta_k x$ with $D(L_k) = H_0^2(0, \pi) \cap H^4(0, \pi)$ in the case $\beta_k \neq 0$ and $D(L_k) = H^2(0, \pi) \cap H_0^1(0, \pi)$ otherwise for $k = 0, 1, 2, \dots$. Our main results read as follows.

Theorem 1.1. *We assume* O1, F1, F2, F3 *and that* $\{\beta_k\}_{k=1}^{\infty}$ *is a decreasing sequence bounded away from* 0. *Then, for each* $k = 1, 2, \dots$ *there exists a solution* x_k *to the problem* (1.1) *such that*

$$x_k \in X_k = \{x \in D(L_k) \colon \|\dot{x}\|_{L^2(0,\pi)} \leq (\sqrt{\pi})^{-1} d_k$$
$$x(t) \in [-d_k, d_k] \text{ on } [0, \pi]\}, \tag{1.4}$$

and $\beta_k \frac{d^4}{dt^4} x_k + \gamma_k \frac{d^2}{dt^2} x_k + \delta_k x_k \in L^{\infty}(0, \pi)$. *Let for each* $x \in X_0$ *there exists a subsequence* $\{k_i\}_{i=1}^{\infty}$ *such that*

$$F_x^{k_i}(t, x(t)) \underset{i \to \infty}{\to} F_x^0(t, x(t)) \text{ a.e.} \tag{1.5}$$

There exists

$$\bar{x} \in X_0 = \{x \in D(L_0) \colon \|\dot{x}\|_{L^2(0,\pi)} \leq (\sqrt{\pi})^{-1} d_0$$
$$x(t) \in [-d_0, d_0] \text{ on } [0, \pi]\}$$

and a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ *of the sequence* $\{x_k\}_{k=1}^{\infty}$ *such that* $\lim_{i \to \infty} x_{k_i} = \bar{x}$ *(strongly in* $H_0^1(0, \pi)$*, weakly in* $H_0^2(0, \pi)$*),* $x_{k_i} \rightrightarrows \bar{x}$ *and*

$$\beta_0 \frac{d^4}{dt^4} \bar{x} + \gamma_0 \frac{d^2}{dt^2} \bar{x} + \delta_0 \bar{x} = F_x^0(t, \bar{x}),$$

$$\bar{x}(0) = \bar{x}(\pi) = \dot{\bar{x}}(0) = \dot{\bar{x}}(\pi) = 0.$$

We also get with $\beta_0 = 0$ and $\beta_k > 0$ the following theorem.

Theorem 1.2. *We assume* O1, F1, F2, F3 *and that* $\{\beta_k\}_{k=1}^{\infty}$ *is a decreasing,* $\beta_0 = 0$*. Let for each* $x \in X_0$ *there exists a subsequence* k_i *such that* $F_x^{k_i}(t, x(t)) \underset{i\to\infty}{\rightarrow} F_x^0(t, x(t))$ *a.e. Then, for each* $k = 1, 2, \ldots$ *there exists a solution* $x_k \in H_0^2(0, \pi) \cap H^4(0, \pi)$ *to the problem*

$$\beta_k \frac{d^4}{dt^4} x + \gamma_k \frac{d^2}{dt^2} x + \delta_k x = F_x^k(t, x),$$

$$x(0) = x(\pi) = \dot{x}(0) = \dot{x}(\pi) = 0$$

and there exists $\bar{x} \in H^2(0, \pi) \cap H_0^1(0, \pi)$ *a subsequence* $\{x_{k_i}\}_{i=1}^{\infty}$ *of the sequence* $\{x_k\}_{k=1}^{\infty}$ *such that* $\lim_{i\to\infty} x_{k_i} = \bar{x}$ *strongly in* $L^2(0, \pi)$*, weakly in* $H_0^1(0, \pi)$*,* $x_{k_i} \rightrightarrows \bar{x}$ *and*

$$\gamma_0 \frac{d^2}{dt^2} \bar{x} + \delta_0 \bar{x} = F_x^0(t, \bar{x}),$$

$$\bar{x}(0) = \bar{x}(\pi) = 0.$$

We believe that the above Theorem shows the advantage of our method as far as the stability results are concerned. We will also show some qualitative properties of $\bar{x}$ and x_k for $k = 1, 2, 3 \ldots$ ((see 1.4)). For the existence result we need only assume O1, F1 and F2.

Higher order problems with both Dirichlet and periodic boundary value have been investigated thoroughly lately, see for example [5], [11], [8] to mention a few works. The methods that can be applied in investigating the existence may vary due to the setting in which the problem is considered and due to a growth that is put on the nonlinear term. Among many methods used we can mention the direct method of the calculus of variations, the dual least action principle, the mountain pass geometry, the usage of topological arguments, the Krasnosielskij fixed point theorem, the strong maximum principle and many others. See [6] for review of many techniques that can be used in investigating Dirichlet problems for ODE. Our method which is variational in spirit – i.e. it relays on minimizing the suitable action functional – also uses some topological argument – i.e. it uses the idea of a linearization trick. To be precise, we consider a primal action functional for which the Dirichlet problem is an Euler–Lagrange equation, later we define a suitable dual action functional and by introducing a duality theory we investigate relations between both action functionals. Our method is based on investigating the existence of a minimum and the existence of the argument of a minimum for the dual action functional. We relate the critical value and critical point to the suitably defined point of the domain of the action functional and we get the existence of a solution to the considered problem. We investigate both functionals over suitably constructed sets of their domains. Although in the stability investigation we base ourselves on some results from [9], [10], the approach of this paper is a novel one first of all because we do not use convexity calculations and secondly because of the weaker convergence assumptions and of the fact that we allow for both the nonlinear term and the differential operator to vary with k in that respect that the 'limit' Dirichlet problem can be of a lower order. We may not obtain Theorem 1.2 using stability results from [3]. The qualitative properties of the solution that we obtain depend on the growth condition assumed and of course on the construction of the set on which the action functional is minimized.

2. The existence and stability of solutions

In order to prove the existence of solutions to a certain type of semilinear Dirichlet problems governed by ordinary differential equation of higher even order, we shall investigate the existence of solutions to the following family of abstract problems, namely

$$L_k x = G_x^k(x), \tag{2.1}$$

where for $k = 0, 1, 2, \ldots$, L_k: $D(L_k) \to Y$; $D(L_k)$, Y are separable real Hilbert spaces, $D(L_k)$ is dense in Y; the scalar product in Y is denoted by $\langle \cdot, \cdot \rangle$. L_k is a self-adjoint and positive definite linear operator. Let S_k be a densely defined self-adjoint square root operator. Moreover $S_k x \in D(S_k)$ for any $x \in D(L_k)$ and $S_k^2 = L_k$ (see [4]). On $D(S_k)$ we use a norm $\|x\|_{D(S_k)} = \|S_k x\|_Y$ which makes it into a complete space. For each $k = 0, 1, 2, \ldots$, G^k: $Y \to Y$ is a Gâteaux differentiable function; $G_x^k(0) \neq 0$.

A1: $D(L_0) \supset D(L_1) \supset D(L_2) \ldots$;
A2: For each $k = 0, 1, 2, \ldots$, $D(S_k)$ is compactly imbedded in Y;
A3: For each $k = 0, 1, 2, \ldots$, we assume that there exists a nonempty set $X_k \subset D(L_k)$ such that for each $x \in X_k$ relation

$$L_k \tilde{x} = G_x^k(x) \tag{2.2}$$

implies that $\tilde{x} \in X_k$ and sets X_k, $G_x^k(X_k)$ are weakly compact in Y.

For each $k = 1, 2, \ldots$, we put

$$X_k^d = S_k(X_k).$$

Of course, $X_k^d \subset D(S_k)$ and X_k^d is nonempty and we assume that it is weakly compact in $D(S_k)$.

On set X_k we will investigate J_k: $D(S_k) \to \mathbb{R}$,

$$J_k(x) = \frac{1}{2}\langle S_k x, S_k x \rangle - G^k(x)$$

for which (2.1) is the Euler–Lagrange equation and the dual functional J_{D_k}: $D(S_k) \to \mathbb{R}$ is given by the formula

$$J_{D_k}(p) = (G^k)^*(S_k p) - \frac{1}{2}\langle p, p \rangle$$

on X_k^d.

Theorem 2.1. *We assume* A1, A2, A3. *For any* $k = 0, 1, 2, \ldots$ *there exists* $x_k \in X_k$ *such that* (2.1) *is satisfied.*

Proof. By Fenchel–Young inequality and by definition of X_k^d it follows that J_{D_k} is bounded from below on X_k^d and thus we can choose a minimizing sequence $\{p_k^j\}_{j=1}^{\infty}$ which may be assumed – by A3 and by assumption of X_k^d – to be weakly convergent in $D(S_k)$ to a certain p_k. Therefore by A2 it is strongly convergent in Y, possibly up to a subsequence. Since $(G^k)^*$ is convex it follows that functional J_D is weakly lower-semicontinuous on sequence $\{p_k^j\}_{j=1}^{\infty}$. Hence $J_D(p_k) = \inf_{p \in X_k^d} J_{D_k}(p)$.

Since $p_k \in X_k^d$ there exists $x_k \in X_k$ such that relation

$$S_k x_k = p_k \tag{2.3}$$

holds. Next we observe that by the Fenchel–Young inequality

$$\begin{aligned} J_{D_k}(p_k) &= -\frac{1}{2}\langle p_k, p_k\rangle + (G^k)^*(S_k p_k) \\ &= -\langle p_k, S_k x_k\rangle + (G^k)^*(S_k p_k) + \frac{1}{2}\langle S_k x_k, S_k x_k\rangle \\ &\geq \frac{1}{2}\langle S_k x_k, S_k x_k\rangle - G^k(x_k) = J_k(x_k). \end{aligned}$$

Further we show that

$$\inf_{p\in X_k^d}\{-\langle S_k p, x_k\rangle + (G^k)^*(S_k p)\} = -G^k(x_k). \tag{2.4}$$

Indeed, by definition of X_k^d for x_k there exists $\tilde{p} \in X_k^d$ such that $S_k\tilde{x} = \tilde{p}$, where $\tilde{x} \in X_k$ is such that $L_k\tilde{x} = G_x^k(x_k)$. It follows that $S_k\tilde{p} = G_x^k(x_k)$ and by properties of the Fenchel–Young conjugate (see [1]), we get $G^k(x_k) + (G^k)^*(\tilde{p}) = \langle x_k, \tilde{p}\rangle$. In consequence

$$\langle x_k, S_k\tilde{p}\rangle - (G^k)^*(S_k\tilde{p}) = G^k(x_k)$$

and we obtain by the above and by the Fenchel–Young inequality

$$G^k(x_k) \leq \sup_{p\in X_k^d}\{\langle p, S_k x_k\rangle - (G^k)^*(S_k p)\} \leq G^k(x_k).$$

Hence (2.4) follows. By inequality

$$-\frac{1}{2}\langle p, p\rangle \leq -\langle p, S_k x_k\rangle + \frac{1}{2}\langle S_k x_k, S_k x_k\rangle,$$

where x_k is defined by (2.3), we see that

$$\begin{aligned} J_{D_k}(p_k) = \inf_{p\in X_k^d} J_{D_k}(p) &= \inf_{p\in X_k^d}\{-\frac{1}{2}\langle p, p\rangle + (G^k)^*(S_k p)\} \\ &\leq \inf_{p\in X_k^d}\{-\langle p, S_k x_k\rangle + \frac{1}{2}\langle S_k x_k, S_k x_k\rangle + (G^k)^*(S_k p)\} \\ &= \frac{1}{2}\langle S_k x_k, S_k x_k\rangle + \inf_{p\in X_k^d}\{-\langle p, S_k x_k\rangle + (G^k)^*(S_k p)\} \leq J_k(x_k). \end{aligned}$$

Thus $J_{D_k}(p_k) = J_k(x_k)$. In consequence, we have the equality

$$G^k(x_k) + (G^k)^*(S_k p_k) - \langle S_k p_k, x_k\rangle = 0.$$

Hence we have the relation

$$S_k p_k = G_x^k(x_k). \tag{2.5}$$

From (2.3) and (2.5) the assertion of the Theorem follows. ■

3. Existence and stability of solutions for Dirichlet problem (1.1)

Lemma 3.1. *We assume* O1, F1, F2. *For each $x \in X_k$ there exists a solution $u \in X_k$ to the problem*

$$\beta_k \frac{d^4}{dt^4}u + \gamma_k \frac{d^2}{dt^2}u + \delta_k u = F_x^k(t, x(t)),$$
$$x(0) = x(\pi) = \operatorname{sgn}(\beta_k)\dot{x}(0) = \operatorname{sgn}(\beta_k)\dot{x}(\pi) = 0. \tag{3.1}$$

Moreover $\|\ddot{u}\|^2_{L^2(0,\pi)} \le \frac{1}{\beta_k \pi}(\beta_k - 2\gamma_k)d_k$ in the case $\beta_k \neq 0$.

Proof. We see that by standard arguments there exists $u \in H_0^2(0,\pi) \cap H^4(0,\pi)$ in the case $\beta_k > 0$ and $u \in H_0^1(0,\pi) \cap H^2(0,\pi)$ in the case $\beta_k = 0$ such that (3.1) is satisfied. Due to Poincare inequality, Schwartz inequality, O1 and (1.2) we obtain

$$(\beta_k - \gamma_k - |\delta_k|)\|\dot{u}\|^2_{L^2(0,\pi)} \le \frac{(\beta_k - \gamma_k - |\delta_k|)}{\sqrt{\pi}} d_k \|\dot{u}\|_{L^2(0,\pi)}.$$

Hence $\|\dot{u}\|_{L^2(0,\pi)} \le (\sqrt{\pi})^{-1} d_k$ and by Sobolev's inequality we now get $\max_{t\in[0,\pi]} |u(t)| \le d_k$. Thus $u \in X_k$. In the case $\beta_k \neq 0$ from relation

$$\int_0^\pi \beta_k \left(\frac{d^4}{dt^4}u(t)\right) u(t)dt + \int_0^\pi \gamma_k \left(\frac{d^2}{dt^2}u(t)\right) u(t)dt + \int_0^\pi \delta_k u^2(t)dt$$
$$= \int_0^\pi F_x^k(t, x(t))u(t)dt$$

we see that

$$\beta_k \|\ddot{u}\|^2_{L^2(0,\pi)} - \gamma_k \|\dot{u}\|^2_{L^2(0,\pi)} + \delta_k \|u\|^2_{L^2(0,\pi)} \le \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k^2$$

and therefore

$$\beta_k \|\ddot{u}\|^2_{L^2(0,\pi)} \le \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k^2 - \gamma_k \|\dot{u}\|^2_{L^2(0,\pi)} + |\delta_k| \|u\|^2_{L^2(0,\pi)}.$$

So

$$\|\ddot{u}\|^2_{L^2(0,\pi)} \le \frac{1}{\beta_k}\left(\frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k^2 - \gamma_k \frac{d_k^2}{\pi} + |\delta_k| \frac{d_k^2}{\pi}\right) = \frac{1}{\beta_k \pi}(\beta_k - 2\gamma_k)d_k.$$

■

Of course, X_k is weakly compact in $H_0^1(0,\pi)$ and thus in $Y = L^2(0,\pi)$. Thus by (1.2) we have conditions A1–A3 satisfied. Therefore by Theorem 2.1, we get

Theorem 3.2. *Let us assume* O1, F1, F2. *For any $k = 0, 1, 2, \ldots$ there exists a solution $x_k \in X_k$ to the Dirichlet problem* (1.1) *satisfying relation $\|\ddot{x}_k\|_{L^2(0,\pi)} \le \frac{1}{\beta_k \pi}(\beta_k - 2\gamma_k)d_k$ in case $\beta_k > 0$.*

Now we may prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.2 for each $k = 1, 2, \dots$ there exists a solution to (1.1) satisfying relation $\|\ddot{x}_k\|_{L^2(0,\pi)} \le \frac{1}{\beta_k \pi}(\beta_k - 2\gamma_k)d_k$. Since $d_k \le d_0$ for $k = 1, 2, \dots$ it follows that $X_k \subset X_0$ for all $k = 1, 2, \dots$. Indeed, for all $x_k \in X_k$ we get $x_k(t) \in [-d_k, d_k] \subset [-d_0, d_0]$, $\|\dot{x}_k\|_{L^2(0,\pi)} \le (\sqrt{\pi})^{-1} d_k \le (\sqrt{\pi})^{-1} d_0$. Since the sequence $\{x_k\}_{k=1}^{\infty}$ is bounded in $H_0^2(0, \pi)$, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ weakly convergent $H_0^2(0, \pi)$ – and which up to subsequence may be assumed to be strongly convergent in $H_0^1(0, \pi)$ – to $\bar{x} \in X_0$. By Sobolev's inequality we now get that $x_k \rightrightarrows \bar{x}$, possibly up to a subsequence. By (1.2) and O1 we get that for any $x \in X_0$ the following estimation holds: $|F_x^k(t, x(t))| \le \frac{(\beta'' - \gamma')}{\pi} d_0$ for a.e. $t \in [0, \pi]$. Therefore $\beta_k \frac{d^4}{dt^4} x_k + \gamma_k \frac{d^2}{dt^2} x_k + \delta_k x_k$ is bounded in $L^{\infty}(0, \pi)$. Hence it is weakly convergent in $L^2(0, \pi)$, up to a subsequence. Moreover by construction of X_0 and by the above remarks it follows that $\beta_k \frac{d^4}{dt^4} x_k + \gamma_k \frac{d^2}{dt^2} x_k + \delta_k x_k \rightharpoonup \beta_0 \frac{d^4}{dt^4} \bar{x} + \gamma_0 \frac{d^2}{dt^2} \bar{x} + \delta_0 \bar{x}$. Indeed, for any $f \in C_0^{\infty}(0, \pi)$ we get

$$\begin{aligned} &\int_0^{\pi} \left(\beta_k \frac{d^4}{dt^4} x_k(t) + \gamma_k \frac{d^2}{dt^2} x_k(t) + \delta_k x_k(t) \right) f(t) dt \\ &= \beta_k \int_0^{\pi} x_k(t) \frac{d^4}{dt^4} f(t) dt + \gamma_k \int_0^{\pi} x_k(t) \frac{d^2}{dt^2} f(t) dt + \delta_k \int_0^{\pi} x_k(t) f(t) dt \\ &\to \beta_0 \int_0^{\pi} \bar{x}(t) \frac{d^4}{dt^4} f(t) dt + \gamma_0 \int_0^{\pi} \bar{x}(t) \frac{d^2}{dt^2} f(t) dt + \delta_0 \int_0^{\pi} \bar{x}(t) f(t) dt \\ &= \int_0^{\pi} \left(\beta_0 \frac{d^4}{dt^4} \bar{x}(t) + \gamma_0 \frac{d^2}{dt^2} \bar{x}(t) + \delta_0 \bar{x}(t) \right) f(t) dt. \end{aligned}$$

Since $C_0^{\infty}(0, \pi)$ is dense in $L^2(0, \pi)$ we get the assertion.

Now we employ assumption (1.5) for $x = \bar{x}$ taking a suitable subsequence. We prove that

$$\lim_{n \to \infty} F_x^{k_n}(t, x_{k_n}(t)) = F_x^0(t, \bar{x}(t)) \text{ a.e.} \tag{3.2}$$

Indeed, we get

$$\begin{aligned} &F_x^{k_n}(t, x_{k_n}(t)) - F_x^0(t, \bar{x}(t)) \\ &\quad = F_x^{k_n}(t, x_{k_n}(t)) - F_x^{k_n}(t, \bar{x}(t)) + F_x^{k_n}(t, \bar{x}(t)) - F_x^0(t, \bar{x}(t)). \end{aligned} \tag{3.3}$$

By (1.3) and by the mean value theorem we observe that

$$|F_x^{k_n}(t, x_{k_n}(t)) - F_x^{k_n}(t, \bar{x}(t))| \le a|x_{k_n}(t) - \bar{x}(t)|.$$

Since $\{x_{k_n}\}_{n=1}^{\infty}$ is uniformly convergent it follows that

$$\lim_{n \to \infty} (F_x^{k_n}(t, x_{k_n}(t)) - F_x^{k_n}(t, \bar{x}(t))) = 0.$$

Thus from (3.3) using the above and (1.5) we obtain (3.2). Since a weak limit is equal to an almost everywhere limit we get

$$\beta_0 \frac{d^4}{dt^4} \bar{x} + \gamma_0 \frac{d^2}{dt^2} \bar{x} + \delta_0 \bar{x} = F_x^0(t, \bar{x}(t)).$$

■

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. Reasoning similarly to the proof of Theorem 1.1 we get that the sequence $\{x_k\}_{k=1}^{\infty}$ is bounded in $H_0^1(0,\pi)$ and thus there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$weakly convergent $H_0^1(0,\pi)$ – which, up to a subsequence, may be assumed to be strongly convergent in $L^2(0,\pi)$ – to $\bar{x} \in X_0$. Therefore the sequence $\{x_{k_n}\}_{n=1}^{\infty}$ is convergent a.e. Now we get relation

$$|F_x^{k_n}(t, x_{k_n}(t)) - F_x^{k_n}(t, \bar{x}(t))| \leq a|x^{k_n}(t) - \bar{x}(t)| \to 0$$

almost everywhere. Hence the assertion follows. □

We now provide an example of a function for which our growth conditions F1–F3 are satisfied.

Example 3.3. Let $d_k > 0$ and $F^k(t,x) = f(t)x + G^k(x)$, where $f \in L^\infty(0,\pi)$, $G^k \in C^2(R)$ is an arbitrary function convex in $\mathbb{R}$ and where

$$|f(t)| \leq \frac{(\beta_k - \gamma_k - |\delta_k|)}{\pi} d_k - \max\{|G_x^k(-d_k)|, |G_x^k(d_k)|\}. \tag{3.4}$$

Indeed, by (3.4) we obtain that (1.2) now reads

$$\max_{x\in[-d_k,d_k]} |f(t) + G_x^k(x)| \leq |f(t)| + \max_{x\in[-d_k,d_k]} |G_x^k(x)|$$
$$= |f(t)| + \max\{|G_x^k(-d_k)|, |G_x^k(d_k)|\} \leq \frac{(\beta_k - \gamma_k - |\delta_k|)}{\sqrt{\pi}} d_k.$$

The other assumptions are satisfied at hand.

Remark 1. The method we describe applies for any Dirichlet problem of even order greater or equal to 4 in case the differential operator is positive definite and nonlinearity satisfies some growth conditions that allow for getting a type of assumption A3 satisfied.

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