

Planar algebra of the subgroup-subfactor

VED PRAKASH GUPTA

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India
E-mail: vpgupta@imsc.res.in

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Abstract. We give an identification between the planar algebra of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ and the G -invariant planar subalgebra of the planar algebra of the bipartite graph \star_n , where $n = [G : H]$. The crucial step in this identification is an exhibition of a model for the basic construction tower, and thereafter of the standard invariant of $R \rtimes H \subset R \rtimes G$ in terms of operator matrices.

We also obtain an identification between the planar algebra of the fixed algebra subfactor $R^G \subset R^H$ and the G -invariant planar subalgebra of the planar algebra of the ‘flip’ of \star_n .

Keywords. Planar algebras; subfactors; standard invariant.

1. Introduction

For every pair $H \subset G$ of finite groups, and an outer action α of G on the hyperfinite II_1 -factor R , we have a dual pair of subfactors $R \rtimes_{\alpha/H} H \subset R \rtimes_{\alpha} G$ (the *subgroup-subfactor*) and fixed algebra subfactor $R^G \subset R^H$.

On the other hand, Jones [5] associates a planar algebra $P(\Gamma)$ to a finite bipartite graph Γ with a spin function μ .

We show that the planar algebras of the subgroup and fixed algebra subfactors¹ are isomorphic to the invariants with respect to an action by G as planar algebra automorphisms of $P(\Gamma)$ and $P(\bar{\Gamma})$, respectively (see Theorem 5.12 and Corollary 5.16), where Γ is the bipartite graph \star_n ($n = [G : H]$), and $\bar{\Gamma}$ is its ‘flip’ (with ‘even’ and ‘odd’ vertices interchanged).

We begin §2 with a discussion of matrix algebras, matrix maps and our notational conventions for them, and go on with certain facts related to the matrix functor in the context of subfactors. Section 3 is mainly devoted to the exhibition of an explicit model for the basic construction tower of $R \rtimes H \subset R \rtimes G$ and then its relative commutants in terms of that model. In §4, we briefly recall some basic aspects of planar algebras and discuss the notion of group action on planar algebras and a related result. We then recall the notion of planar algebra of a bipartite graph as given by Jones [5], and discuss how it and its dual behave under the action of a finite group on the graph. Finally, in §5, we define the bipartite graph \star_n , where $n = [G : H]$, and analyse the planar algebra $P(\star_n)^G$. We then complete the proof of the main theorem via several lemmas; and end the paper with some corollaries.

There are references in literature giving different models for the basic construction tower and the standard invariant of the subgroup-subfactor – see, for instance, [10] and [1]. Apart

¹Our subfactors are all of type II_1 .

from that, Bhattacharyya and Landau [1] have given descriptions of planar algebras of an intermediate subfactor and a subgroup-subfactor is a particular case of it. However, the description of the planar algebra of a subgroup-subfactor that we present here is different from theirs, and is more concrete. One immediate consequence of this description is that, given any pair of finite groups $H \subset G$ with index n , the planar algebra of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ is sandwiched in between the planar algebras $P(\star_n)^{S_n}$ and $P(\star_n)$ – see Corollary 5.15.

2. Matrix functor

Given an algebra P – all our algebras will be over the field \mathbb{C} – and a (finite) index set Λ , we write $M_\Lambda(P)$ for the set of matrices with rows and columns indexed by Λ , and entries in P ; this ‘matrix algebra’ is a $*$ -algebra (resp., a II_1 factor) if P is.

An algebra map $\theta: P \rightarrow Q$ gives rise naturally to the matrix map $M_\Lambda(\theta): M_\Lambda(P) \rightarrow M_\Lambda(Q)$ defined by

$$[M_\Lambda(\theta)(A)]_{\lambda_1, \lambda_2} = \theta(A_{\lambda_1, \lambda_2}), \quad \lambda_1, \lambda_2 \in \Lambda, A \in M_\Lambda(P). \tag{1}$$

For any two index sets Λ and Γ , and an algebra P , we view $M_\Lambda(M_\Gamma(P))$ as the set of block matrices, whose blocks are determined by the index set Λ and the matrices in each block are members of $M_\Gamma(P)$. Thus we identify it with the algebra $M_{\Gamma \times \Lambda}(P)$ via the correspondence

$$M_\Lambda(M_\Gamma(P)) \ni A \mapsto \tilde{A} \in M_{\Gamma \times \Lambda}(P),$$

$$\tilde{A}_{(\gamma_1, \lambda_1), (\gamma_2, \lambda_2)} := (A_{\lambda_1, \lambda_2})_{\gamma_1, \gamma_2}. \tag{2}$$

With this convention, we shall interchangeably use A and \tilde{A} during calculations.

Thus, given algebras P and Q , finite index sets Λ and Γ , and an algebra map $\theta: P \rightarrow Q$, the matrix map

$$M_\Lambda(\theta): M_\Lambda(P) \rightarrow M_{\Gamma \times \Lambda}(Q)$$

is given by

$$[M_\Lambda(\theta)(A)]_{(\gamma_1, \lambda_1), (\gamma_2, \lambda_2)} = \theta(A_{\lambda_1, \lambda_2})_{\gamma_1, \gamma_2} =: \theta_{\gamma_1, \gamma_2}(A_{\lambda_1, \lambda_2}), \tag{3}$$

$\forall (\gamma_i, \lambda_i) \in \Gamma \times \Lambda, i = 1, 2, \forall A \in M_\Lambda(P)$.

For later reference, we record a few facts – which are mostly modifications of certain results proved in [6], and whose proofs we therefore omit.

PROPOSITION 2.1

Suppose $N \subset M \subset^{e_1} M_1$ is the basic construction for a pair of II_1 -factors $N \subset M$ (also called a subfactor) with $\tau^{-1} = [M : N] < \infty$. Then for any finite index set Λ ,

$$M_\Lambda(N) \subset M_\Lambda(M) \subset M_\Lambda(M_1) \tag{4}$$

is the basic construction for the subfactor $M_\Lambda(N) \subset M_\Lambda(M)$, where

- (i) the trace preserving conditional expectation $E_{M_\Lambda(N)}: M_\Lambda(M) \rightarrow M_\Lambda(N)$ is given by the matrix map $M_\Lambda(E_N)$; and

(ii) the Jones projection which implements the above conditional expectation $E_{M_\Lambda(N)}$ is given by the diagonal matrix $\tilde{e}_1 \in M_\Lambda(M_1)$, with diagonal entries given by

$$(\tilde{e}_1)_{\lambda,\lambda} = e_1, \forall \lambda \in \Lambda.$$

Lemma 2.2. Let $N \subset M$ be a subfactor with integer index $n = [M : N]$. Let $\{\lambda_i : i \in I\}$ be an orthonormal basis² for M/N , where $I := \{1, \dots, n\}$. Define $\theta : M \rightarrow M_I(N)$ by $\theta_{i,j}(x) = E_N(\lambda_i x \lambda_j^*), \forall x \in M, i, j \in I$. Then θ is a unital inclusion and we have an isomorphism of towers

$$(N \subset M \xrightarrow{\theta} M_I(N) \subset M_I(M)) \cong (N \subset M \subset M_1 \subset M_2). \tag{5}$$

In particular, the Jones projections $\tilde{e}_1 \in M_I(N)$ and $\tilde{e}_2 \in M_I(M)$, corresponding to e_1 and e_2 under the above tower isomorphism, are given by

$$(\tilde{e}_1)_{i,j} = E_N(\lambda_i) E_N(\lambda_j^*) \quad \text{and} \tag{6}$$

$$(\tilde{e}_2)_{i,j} = n^{-1} \lambda_i \lambda_j^*, \tag{7}$$

respectively, $\forall i, j \in I$. And the trace preserving conditional expectation $E_M : M_I(N) \rightarrow M$ is given by

$$E_M((a_{i,j})) = \sum_{i,j \in I} n^{-1} \lambda_i^* a_{i,j} \lambda_j^*, \forall (a_{i,j}) \in M_I(N). \tag{8}$$

Lemma 2.3. With $N \subset M$, $\{\lambda_i : i \in I\}$ and θ as in Lemma 2.2, for each $k \geq 1$, define $\theta^{(k)} : M \rightarrow M_{I^k}(N)$ by

$$\begin{aligned} \theta_{\underline{i}, \underline{j}}^{(k)}(x) &= \theta_{i_1, j_1}(\theta_{i_2, j_2}(\dots \theta_{i_k, j_k}(x) \dots)) \\ &= E_N(\lambda_{i_1} E_N(\lambda_{i_2} E_N(\dots E_N(\lambda_{i_k} x \lambda_{j_k}^*) \dots) \lambda_{j_2}^*) \lambda_{j_1}^*), \end{aligned}$$

for all $x \in M$, $\underline{i} = (i_1, \dots, i_k)$, $\underline{j} = (j_1, \dots, j_k) \in I^k$. Then $\theta^{(k)}$ is a unital $*$ -homomorphism for each $k \geq 1$.

3. The subgroup-subfactor

In this section, we set up the notation and the model for the subgroup-subfactor which we shall find convenient to use.

Thus, let R be the hyperfinite II_1 -factor viewed as sitting in $\mathcal{L}(L^2(R))$ as left multiplication operators. Let G be a finite group and $\alpha : G \rightarrow \text{Aut}(R)$ be an outer action of G on R , i.e., α_g is not inner for all $e \neq g \in G$.

We think of $R \rtimes_\alpha G$ as the II_1 -factor $(R \cup \{u_g : g \in G\})'' \subset \mathcal{L}(L^2(R))$, where $u_g(\hat{x}) := \widehat{\alpha_g(x)}$, $x \in R$, $g \in G$ – see §A.4 of [6].

Let H be a subgroup of the group G . Once and for all, we fix a set of representatives $\{g_1, \dots, g_n\}$ for the right H -coset decomposition of G with $g_1 = e$, i.e., $G = \sqcup_{i=1}^n H g_i$, where $n = [G : H]$. Then $\{u_{g_i} : i \in I\}$ is a (n orthonormal) basis for $R \rtimes G/R \rtimes H$, where $I := \{1, \dots, n\}$.

The following result is probably folklore.

² $\{\lambda_i\}$ satisfies $E_n(\lambda_i \lambda_j^*) = \delta_j^i, \forall i, j \in I$.

PROPOSITION 3.1

- (1) $R^G \subset R^H$ is an irreducible subfactor with index $[G : H]$ and has finite depth.
- (2) $(R^G \subset R^H) \cong (M \subset M_1)$, where M_1 is the II_1 -factor obtained by the basic construction of the subgroup-subfactor $N := R \rtimes H \subset R \rtimes G =: M$.

Taking $N = R \rtimes H \subset R \rtimes G = M$, by Lemma 2.2, with respect to the orthonormal basis $\{u_{g_i} : i \in I\}$ for M/N , we have

$$(N \subset M \subset M_1 \subset M_2) \cong (N \subset M \xrightarrow{\theta} M_I(N) \subset M_I(M)).$$

Then, by repeated applications of Proposition 2.1, we see that

$$\begin{aligned} M_{2k-1} \subset M_{2k} \subset M_{2k+1} \subset M_{2k+2} \\ \cong M_{I^k}(N) \subset M_{I^k}(M) \xrightarrow{\Theta_{k+1}} M_{I^{k+1}}(N) \subset M_{I^{k+1}}(M), \quad \forall k \geq 0, \end{aligned} \tag{9}$$

where, as in §2, for each $k \geq 1$, we have inductively identified $M_{I^k}(X)$ with $M_I(M_{I^{k-1}}(X))$ for $X \in \{N, M\}$, and $\Theta_{k+1} := M_I(\Theta_k)$ with $\Theta_1 := \theta$. Thus, we find the following.

Theorem 3.2. *With notations as in the preceding paragraph, the tower*

$$N \subset M \xrightarrow{\Theta_1} M_I(N) \subset M_I(M) \xrightarrow{\Theta_2} \dots \subset M_{I^k}(M) \xrightarrow{\Theta_{k+1}} M_{I^{k+1}}(N) \subset \dots$$

is a model for the basic construction tower of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$.

Given a k -tuple $\underline{i} \in I^k$, we write $\underline{i}_{[r,s]}$ for the $(s - r + 1)$ -tuple $(i_r, i_{r+1}, \dots, i_s)$ for all $1 \leq r < s \leq k$. In fact, if $r = 1$ (resp., $s = k$), we simply write $\underline{i}_{[s]}$ (resp., $\underline{i}_{[r]}$) for $\underline{i}_{[r,s]}$. And, for any $j \in I$, we write $(j, \underline{i}_{[r,s]})$ (resp., $(\underline{i}_{[r,s]}, j)$) for the tuple (j, i_r, \dots, i_s) (resp., (i_r, \dots, i_s, j)).

COROLLARY 3.3

For each $k \geq 1$, let the Jones projections e_{2k-1} and e_{2k} be mapped to the operator matrices $\tilde{e}_{2k-1} \in M_{I^k}(N)$ and $\tilde{e}_{2k} \in M_{I^k}(M)$, respectively, under the identifications (9). Then

$$(\tilde{e}_{2k-1})_{\underline{i}, \underline{j}} = \delta_{\underline{j}}^{\underline{i}} \delta_1^{i_1}, \quad \forall \underline{i}, \underline{j} \in I^k; \text{ and} \tag{10}$$

$$(\tilde{e}_{2k})_{\underline{i}, \underline{j}} = n^{-1} \delta_{\underline{j}_{[2]}}^{i_{[2]}} g_{i_1} g_{j_1}^{-1}, \quad \forall \underline{i}, \underline{j} \in I^k. \tag{11}$$

And, for each $k \geq 0$, the trace preserving conditional expectations $E_{M_{2k}} : M_{I^{k+1}}(N) \rightarrow M_{I^k}(M)$ and $E_{M_{2k-1}} : M_{I^k}(M) \rightarrow M_{I^k}(N)$ are given by the matrix maps

$$E_{M_{2k}} = M_{I^k}(E_M) \quad \text{and} \quad E_{M_{2k-1}} = M_{I^k}(E_N), \tag{12}$$

respectively, where E_M is given as in Lemma 2.2, and as usual $M_{I^{k+1}}(T) := M_I(M_{I^k}(T))$ for $T \in \{E_M, E_N\}$.

The proof of Corollary 3.3 relies on Proposition 2.1 and Lemma 2.2, while easy induction arguments yield the following results.

Lemma 3.4. For each $k \geq 1$, we have

$$\Theta_k((a_{\underline{i}, \underline{j}}))_{\underline{u}, \underline{v}} = \theta_{u_1, v_1}(a_{u_2, v_2}), \forall (a_{\underline{i}, \underline{j}}) \in M_{I^k}(M), \underline{u}, \underline{v} \in I^{k+1}. \quad (13)$$

PROPOSITION 3.5

For each $k \geq 1$, we have

$$\Theta_k \circ \dots \circ \Theta_1 = \theta^{(k)}. \quad (14)$$

We now recall some notations and facts from [6].

Given a k -tuple $\underline{i} = (i_1, \dots, i_k) \in I^k$, $k \geq 1$, we write $\sqcap g_{\underline{i}}$ for the product $g_{i_1} g_{i_2} \dots g_{i_k}$. For each $k \geq 1$, we have an action β^k of G on the set I^k : For $\underline{i}, \underline{j} \in I^k$,

$$\beta_g^k(\underline{j}) = \underline{i} \iff H g_{j_l} g_{j_{l+1}} \dots g_{j_k} g^{-1} = H g_{i_l} g_{i_{l+1}} \dots g_{i_k}, \forall 1 \leq l \leq k.$$

For each $k \geq 1$, the relative commutants of R are given by

$$\begin{aligned} R' \cap M_{2k-1} &\cong \theta^{(k)}(R)' \cap M_{I^k}(N) = \{(x_{\underline{i}, \underline{j}}) \in M_{I^k}(N): \\ &\exists (C_{\underline{i}, \underline{j}}) \in M_{I^k}(\mathbb{C}) \text{ such that } x_{\underline{i}, \underline{j}} = C_{\underline{i}, \underline{j}} u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}} \\ &\text{and } C_{\underline{i}, \underline{j}} = 0 \text{ unless } H \sqcap g_{\underline{i}} = H \sqcap g_{\underline{j}}, \forall \underline{i}, \underline{j} \in I^k\}; \text{ and} \end{aligned} \quad (15)$$

$$\begin{aligned} R' \cap M_{2k} &\cong \theta^{(k)}(R)' \cap M_{I^k}(M) = \{(x_{\underline{i}, \underline{j}}) \in M_{I^k}(M): \\ &\exists (C_{\underline{i}, \underline{j}}) \in M_{I^k}(\mathbb{C}) \ni x_{\underline{i}, \underline{j}} = C_{\underline{i}, \underline{j}} u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}}, \underline{i}, \underline{j} \in I^k\}. \end{aligned} \quad (16)$$

From these descriptions, it immediately follows that the relative commutants in the grid of the standard invariant of $N \subset M$ are given by

$$\begin{aligned} N' \cap M_{2k-1} &\cong \theta^{(k)}(N)' \cap M_{I^k}(N) = \{(x_{\underline{i}, \underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(N): \\ &C_{\underline{i}, \underline{j}} = C_{\beta_h^k(\underline{i}), \beta_h^k(\underline{j})}, \forall h \in H, \underline{i}, \underline{j} \in I^k\}; \end{aligned} \quad (17)$$

$$\begin{aligned} N' \cap M_{2k} &\cong \theta^{(k)}(N)' \cap M_{I^k}(M) = \{(x_{\underline{i}, \underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(M): \\ &C_{\underline{i}, \underline{j}} = C_{\beta_h^k(\underline{i}), \beta_h^k(\underline{j})}, \forall h \in H, \underline{i}, \underline{j} \in I^k\}; \end{aligned} \quad (18)$$

$$\begin{aligned} M' \cap M_{2k-1} &\cong \theta^{(k)}(M)' \cap M_{I^k}(N) = \{(x_{\underline{i}, \underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(N): \\ &C_{\underline{i}, \underline{j}} = C_{\beta_g^k(\underline{i}), \beta_g^k(\underline{j})}, \forall g \in G, \underline{i}, \underline{j} \in I^k\}; \text{ and} \end{aligned} \quad (19)$$

$$\begin{aligned} M' \cap M_{2k} &\cong \theta^{(k)}(M)' \cap M_{I^k}(M) = \{(x_{\underline{i}, \underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(M): \\ &C_{\underline{i}, \underline{j}} = C_{\beta_g^k(\underline{i}), \beta_g^k(\underline{j})}, \forall g \in G, \underline{i}, \underline{j} \in I^k\}, \end{aligned} \quad (20)$$

for all $k \geq 1$, where in each of the above equations the scalar matrix $(C_{\underline{i}, \underline{j}})$ corresponds to the operator matrix $(x_{\underline{i}, \underline{j}})$ as in (15) and (16).

For each $k \geq 1$, and given k -tuples $\underline{i}, \underline{j} \in I^k$, we define operator matrices $[\underline{i}, \underline{j}]^{ev} \in M_{I^k}(M)$ and $[\underline{i}, \underline{j}]^{od} \in M_{I^k}(N)$ by

$$[\underline{i}, \underline{j}]_{u,v}^{ev} = \delta_u^i \delta_v^j u_{(\cap g_i)(\cap g_j)^{-1}} \text{ and} \tag{21}$$

$$[\underline{i}, \underline{j}]_{u,v}^{od} = \delta_u^i \delta_v^j 1_H((\cap g_i)(\cap g_j)^{-1})u_{(\cap g_i)(\cap g_j)^{-1}}, \tag{22}$$

respectively, $\forall u, v \in I^k$, where $1_H(x) := 1$ if $x \in H$ and 0 otherwise. The superscripts *ev* and *od* refer to the ambient spaces M_{2k} and M_{2k-1} , respectively.

Further, for each $k \geq 1$, we set $Y_k = \{(\underline{i}, \underline{j}) \in I^k \times I^k : H \cap g_{\underline{i}} = H \cap g_{\underline{j}}\}$. With these simplified notations, it is not hard to show that, for each $k \geq 1$,

- (1) G acts diagonally by β^k on $I^k \times I^k$ and Y_k is invariant under this action, i.e., $(\beta_g^k(\underline{i}), \beta_g^k(\underline{j})) \in Y_k, \forall g \in G, (\underline{i}, \underline{j}) \in Y_k$;
- (2) $\{[\underline{i}, \underline{j}]^{od} : (\underline{i}, \underline{j}) \in Y_k\}$ forms a basis for $R' \cap M_{2k-1}$; and
- (3) $\{[\underline{i}, \underline{j}]^{ev} : (\underline{i}, \underline{j}) \in I^k \times I^k\}$ forms a basis for $R' \cap M_{2k}$.

Thus, it makes sense to write $g[\underline{i}, \underline{j}]^x$ for $[\beta_g^k(\underline{i}), \beta_g^k(\underline{j})]^x, \forall g \in G, (\underline{i}, \underline{j}) \in I^k \times I^k, k \geq 1, x \in \{ev, od\}$.

Given a G -action ν on a set X , we write $G \setminus X(\nu)$ for a set of representatives of G -orbits of X under the action ν , and when the group action is clear from the context we simply denote it by $G \setminus X$. For instance, we write $G \setminus Y_k$ for a set of orbit representatives for the diagonal G -action on Y_k by β^k .

The descriptions (17)–(20) give the following list of bases for the relative commutants:

Lemma 3.6. For each $k \geq 1$, for every choice of orbit representatives $H \setminus X$ and $G \setminus X$ ($X = Y_k, I^k \times I^k$),

- (1) $\{\sum_{g \in G} g[\underline{i}, \underline{j}]^{od} : (\underline{i}, \underline{j}) \in G \setminus Y_k\}$ (resp., $\{\sum_{h \in H} h[\underline{i}, \underline{j}]^{od} : (\underline{i}, \underline{j}) \in H \setminus Y_k\}$) forms a basis for $M' \cap M_{2k-1}$ (resp., $N' \cap M_{2k-1}$).
- (2) $\{\sum_{g \in G} g[\underline{i}, \underline{j}]^{ev} : (\underline{i}, \underline{j}) \in G \setminus (I^k \times I^k)\}$ (resp., $\{\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} : (\underline{i}, \underline{j}) \in H \setminus (I^k \times I^k)\}$) forms a basis for $M' \cap M_{2k}$ (resp., $N' \cap M_{2k}$).

Since the above bases do not vary with the choice of orbit representatives, we shall no more trouble ourselves by repeating the phrase ‘given a set of representatives of K -orbits of X ’ and simply write $K \setminus X$ for such a choice.

4. Planar algebras

4.1 Introduction

We refer to [7, 8] for the basic aspects of planar algebras and thus follow the notations and terminologies therein. The goal of this section is to analyse the planar algebra associated to a bipartite graph and its dual, and to see how it responds to an appropriate action of a finite group on the bipartite graph.

Basically, a planar algebra P is a collection of vector spaces $\{P_k : k \in \text{Col}\}$ admitting an ‘action’ of the operad of coloured tangles, where $\text{Col} := \{0_{\pm}, 1, 2, \dots\}$.

For the sake of completeness, we list some important coloured tangles in figures 1–5. With respect to composition of tangles, [8] contains the following generating sets:

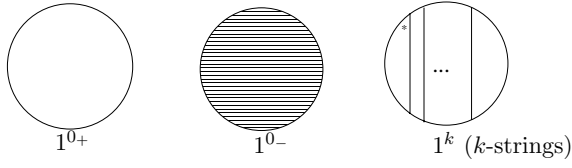


Figure 1. The unit tangles.

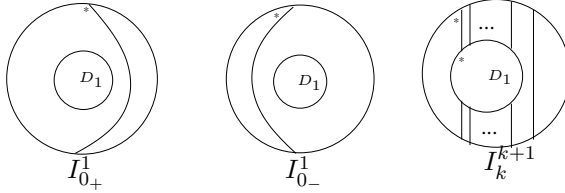


Figure 2. The inclusion tangles.

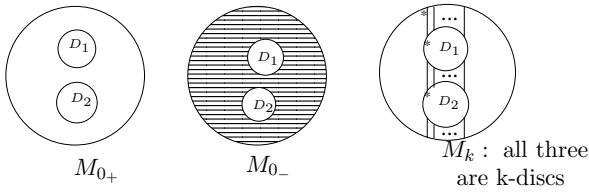


Figure 3. The multiplication tangles.

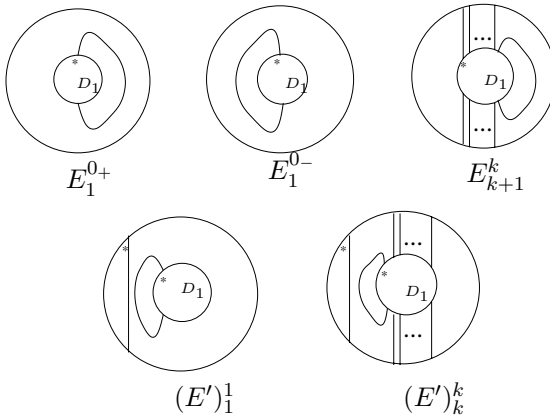


Figure 4. The conditional expectation tangles.

Theorem 4.1 [8]. Let \mathcal{T} be a collection of coloured tangles containing

- $\mathcal{G}_0 := \{1^{0\pm}\} \cup \{E_{k+1}^k, M_k, I_k^{k+1} : k \in \text{Col}\} \cup \{\mathcal{E}^{k+1}, (E')_k^k : k \geq 1\}$, or
- $\mathcal{G}_1 := \{1^{0\pm}\} \cup \{E_{k+1}^k, M_k, I_k^{k+1} : k \in \text{Col}\} \cup \{R_k : k \geq 2\}$,

and suppose \mathcal{T} is closed under composition of coloured tangles, whenever it makes sense. Then \mathcal{T} contains all coloured tangles.

DEFINITION 4.2

Let P and Q be two planar algebras. A planar algebra morphism from P to Q is a collection $\varphi = \{\varphi_k : k \in \text{Col}\}$ of linear maps $\varphi_k : P_k \rightarrow Q_k$ which commutes with all the tangle maps, i.e., if T is a k_0 -tangle with b internal boxes of colours k_1, \dots, k_b , respectively, then

$$\varphi_{k_0} \circ Z_T^P = \begin{cases} Z_T^Q \circ (\otimes_{i=1}^b \varphi_{k_i}), & \text{if } b > 0; \text{ and} \\ Z_T^Q, & \text{if } b = 0. \end{cases} \tag{23}$$

φ is said to be a planar algebra isomorphism if the maps φ_k are all linear isomorphisms.

As in the above definition, we shall follow the convention that the empty tensor product of vector spaces is taken to be the scalars \mathbb{C} .

Repeated applications of the compatibility condition for tangle maps with respect to composition of tangles give the following useful result.

Lemma 4.3. Let P and Q be planar algebras, and $\varphi_k : P_k \rightarrow Q_k, k \in \text{Col}$ be linear maps. If \mathcal{T} is the collection of those tangles T for which the equation in Definition 4.2 holds, then \mathcal{T} is closed under composition of tangles.

We shall primarily deal with subfactor planar algebras. Basically, these planar algebras are connected (i.e., $\dim P_{0\pm} = 1$), have positive modulus (i.e., loops come out as constants), are spherical (i.e., tangle maps for 0_{\pm} -tangles are isotopy invariant on the 2-sphere), their constituent vector spaces are finite dimensional C^* -algebras, and the tangle maps satisfy the $*$ -compatibility condition

$$Z_T(x_1 \otimes \dots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \dots \otimes x_b^*), \tag{24}$$

where T is a coloured tangle as in Definition 4.2 and $x_i \in P_{k_i}, 1 \leq i \leq b$, and T^* is the adjoint of T – see [8]. Along with these, if the modulus is δ , then for each $k \geq 0$, the pictorial trace $\text{tr}_{k+1} : P_{k+1} \rightarrow \mathbb{C}$ defined by

$$\text{tr}_{k+1}(x)1_{0+} = \delta^{-k-1} Z_{E_{k+1}^{0+}}^P(x), \forall x \in P_{k+1}, \tag{25}$$

where $E_{k+1}^{0+} := E_1^{0+} \circ E_2^1 \circ \dots \circ E_{k+1}^k$, is a faithful positive trace on P_{k+1} .

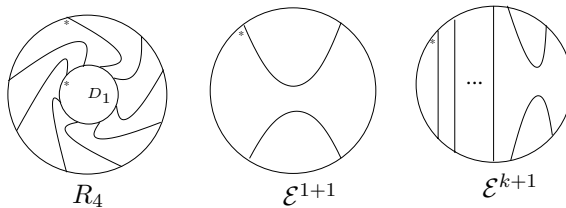


Figure 5. The rotation and Jones’ projections tangles.

Lemma 4.4 [9]. If P is a connected and irreducible³ planar algebra with modulus $\delta > 0$, then P is spherical.

Remark 4.5. For a connected and spherical planar algebra P with modulus $\delta > 0$, and each P_k being a finite-dimensional C^* -algebra,

- (1) the pictorial traces $\text{tr}_m: m \in \text{Col}$ are consistent with respect to inclusions and thus define a global trace on P , where for 0_{\pm} the trace for $P_{0_{\pm}} \cong \mathbb{C}$ is the obvious identity map; and
- (2) if tr_m is faithful for all $m \geq 1$, then for each $k \in \text{Col}$, $\frac{1}{\delta} Z_{E_{k+1}^k}^P$ (resp., $\frac{1}{\delta} Z_{(E')_{k+1}^{k+1}}^P$) is the unique tr_{k+1} preserving conditional expectation from P_{k+1} onto P_k (resp., $P_{1,k+1}$), where $P_{1,k+1} := \text{Image}(Z_{(E')_{k+1}^{k+1}}^P)$.

The importance of planar algebras in subfactor theory lies in the following theorem of Jones:

Theorem 4.6 [3]. For every extremal II_1 -subfactor $N \subset M$ of finite index δ^2 , taking $P_{0_{\pm}} = \mathbb{C}$ and $P_k = N' \cap M_{k-1}, k \geq 1$, there exists a unique subfactor planar algebra structure on the collection $P^{N \subset M} := \{P_k: k \in \text{Col}\} =: P$ satisfying:

- (1) $Z_{E_{k+1}^k}^P(1) = \delta e_k, \forall k \geq 1$;
- (2) $Z_{(E')_k^k}^P(x) = \delta E_{M' \cap M_{k-1}}(x), \forall x \in N' \cap M_{k-1}, \forall k \geq 1$;
- (3) $Z_{E_{k+1}^k}^P(x) = \delta E_{N' \cap M_{k-1}}(x), \forall x \in N' \cap M_k, \forall k \in \text{Col}$, where for $k = 0_{\pm}$, the equation is read as

$$Z_{E_{1}^{0_{\pm}}}^P(x) = \delta \text{tr}_M(x), \forall x \in N' \cap M.$$

We now discuss group actions on planar algebras.

DEFINITION 4.7

Let G be a finite group and $P = \{P_k: k \in \text{Col}\}$ be a planar algebra. We say that G acts on P if for each $k \in \text{Col}$, we have group homomorphisms $\alpha_k: G \rightarrow GL(P_k)$ such that, for each $g \in G, \alpha(g) := \{\alpha_k(g): k \in \text{Col}\}$ is a planar algebra automorphism of P .

For convenience, we write gx for the element $\alpha_k(g)(x)$, for $g \in G, x \in P_k$ and $k \in \text{Col}$. Under such an action, taking

$$P_k^G = \{x \in P_k: gx = x, \forall g \in G\}, k \in \text{Col},$$

we note that the collection $P^G := \{P_k^G: k \in \text{Col}\}$ is a planar sub-algebra of P . Furthermore, the above action induces a G -action on the dual planar algebra ${}^-P$ – see [8] as well. Indeed, for each $k > 0$, since ${}^-P_k = P_k, G$ acts on ${}^-P_k$ as it does on P_k ; and as ${}^-P_{0_{\pm}} = P_{0_{\mp}}$, again G acts on these as it did on $P_{0_{\mp}}$. The constituent vector spaces of ${}^-(P^G)$ and $({}^-P)^G$ being the same, the following is a tautology:

PROPOSITION 4.8

If a finite group G acts on a planar algebra P , then with the induced G -action on ${}^-P$, the planar algebras ${}^-(P^G)$ and $({}^-P)^G$ are isomorphic.

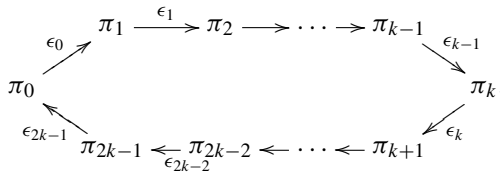
³A planar algebra P is said to be irreducible if $\dim P_1 = 1$.

4.2 Planar algebra of the bipartite graph

Jones [5] associated a planar algebra to a given finite (possibly with multiple edges), connected bipartite graph with a ‘spin function’. However, in [5], there is a slight conflict between the notion of ‘state’ and its ‘compatibility’ with ‘loops’. Thus the planar algebra that we associate here to a data as above is an appropriate modification of the one given there; and the results obtained there hold verbatim.

Let $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ be a finite, connected bipartite graph with even and odd vertices \mathcal{U}^+ and \mathcal{U}^- , respectively, edge set \mathcal{E} and a spin function $\mu: \mathcal{U} := \mathcal{U}^+ \sqcup \mathcal{U}^- \rightarrow (0, \infty)$.

For each $k > 0$, we denote a loop on Γ of length $2k$, based at a vertex $\pi_0 \in \mathcal{U}^+$ by a pair (π, ϵ) of maps $\pi: \{0, 1, \dots, 2k - 1\} \rightarrow \mathcal{U}$ and $\epsilon: \{0, 1, \dots, 2k - 1\} \rightarrow \mathcal{E}$ such that $\pi_{2i} \in \mathcal{U}^+$ and $\pi_{2i+1} \in \mathcal{U}^-$ for all $0 \leq i \leq k - 1$; and for $0 \leq i \leq 2k - 1$, $\epsilon(i)$ is the edge joining the vertices π_i and π_{i+1} , where we follow, here and elsewhere, the convention of labelling the vertices of $2k$ -loops modulo $2k$, thus $\pi_{2k} = \pi_0$. The following pictorial form of the loop (π, ϵ) is quite useful:



For each $k > 0$ and a $2k$ -loop (π, ϵ) , for every $1 \leq i < j \leq 2k$, we define

$$\pi_{[i, j]} = \pi_i \xrightarrow{\epsilon_i} \pi_{i+1} \rightarrow \dots \xrightarrow{\epsilon_{j-1}} \pi_j,$$

and its reverse $\widetilde{\pi_{[i, j]}}$ to be the path

$$\widetilde{\pi_{[i, j]}} = \pi_j \xrightarrow{\epsilon_{j-1}} \pi_{j-1} \rightarrow \dots \xrightarrow{\epsilon_i} \pi_i;$$

and if $0 \leq r < s \leq 2l$ is such that $\lambda_r = \pi_j$ for some $2l$ -loop (λ, η) , then we define the concatenation $\pi_{[i, j]} \circ \lambda_{[r, s]}$ to be the path

$$\pi_{[i, j]} \circ \lambda_{[r, s]} = \pi_i \xrightarrow{\epsilon_i} \pi_{i+1} \rightarrow \dots \xrightarrow{\epsilon_{j-1}} \pi_j (= \lambda_r) \xrightarrow{\eta_r} \dots \xrightarrow{\eta_{s-1}} \lambda_s.$$

Further, for any two vertices $u^+ \in \mathcal{U}^+$ and $u^- \in \mathcal{U}^-$, we set

$$\mathcal{E}(u^+, u^-) = \{\epsilon \in \mathcal{E}: \epsilon \text{ joins } u^+ \text{ and } u^-\}.$$

For each $k > 0$, let $P_k(\Gamma)$ be the \mathbb{C} -vector space whose basis consists of loops on Γ of length $2k$ based at vertices in \mathcal{U}^+ ; and as for $k = 0_{\pm}$, a loop of length 0 is just a vertex on Γ , we set $P_{0_{\pm}}(\Gamma)$ to be the \mathbb{C} -vector space with \mathcal{U}^{\pm} as basis. We wish to give a planar algebra structure on the collection

$$P(\Gamma) := \{P_k(\Gamma): k \in \text{Col}\}.$$

DEFINITION 4.9

A state σ of a coloured tangle T is a function

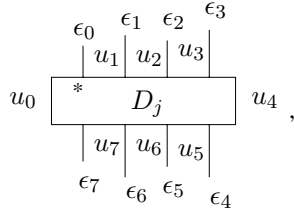
$$\sigma: \{\text{regions of } T\} \sqcup \{\text{strings of } T\} \rightarrow \mathcal{U} \sqcup \mathcal{E},$$

such that

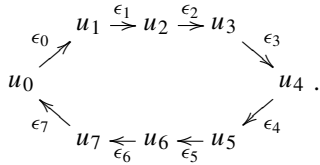
- (1) $\sigma(\{\text{unshaded regions}\}) \subset \mathcal{U}^+, \sigma(\{\text{shaded regions}\}) \subset \mathcal{U}^-;$
- (2) $\sigma(\{\text{strings}\}) \subset \mathcal{E};$ and
- (3) if a string s lies in the closure of two regions r_1 and r_2 , then $\sigma(s)$ is the edge joining the vertices $\sigma(r_1)$ and $\sigma(r_2)$.

Note that a state σ of a k_0 -tangle T with b internal boxes of colours k_1, \dots, k_b , respectively, induces unique loops at the internal as well as the external boxes of T .

Suppose around the j -th internal box D_j of T , say of colour $k_j = 4$, the state σ looks like



then, by condition (3) of Definition 4.9, we get a loop



And if $k_j \in \{0_{\pm}\}$, then the state σ simply induces a vertex in \mathcal{U}^+ or \mathcal{U}^- according as D_j is a 0_+ - or 0_- -box. We take this vertex as the $2k_j$ -loop induced by σ at D_j . Similarly, σ also induces a $2k_0$ -loop at the external box D_0 . We denote this unique $2k_j$ -loop by the pair $(\pi_{D_j}^\sigma, \epsilon_{D_j}^\sigma)$, $0 \leq j \leq b$. For each $0 \leq j \leq b$, we say that the state σ is compatible with a $2k_j$ -loop (π, ϵ) at its j -th box D_j if $(\pi_{D_j}^\sigma, \epsilon_{D_j}^\sigma) = (\pi, \epsilon)$. We can now define the tangle maps.

Given a k_0 -tangle T as above, we first isotope it to a ‘standard form’, i.e.,

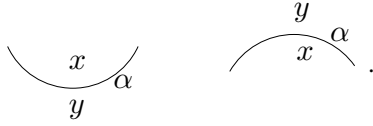
- first rotate all the internal boxes of T so that their $*$ -vertices are on top left-corner; and
- then isotope all the strings, if necessary, so that any singularity of the y -coordinate function for strings is either a local maximum or a local minimum.

Given $2k_j$ -loops $\{(\pi^{(j)}, \epsilon^{(j)}); 0 \leq j \leq b\}$ (with the convention that $2k_j = 0_{\pm}$, if $k_j = 0_{\pm}$, and $(\pi^{(j)}, \epsilon^{(j)})$ is an appropriate element of \mathcal{U}), we define the coefficient of $(\pi^{(0)}, \epsilon^{(0)})$ in the vector⁴ $Z_T^{P(\Gamma)}((\pi^{(1)}, \epsilon^{(1)}) \otimes \dots \otimes (\pi^{(b)}, \epsilon^{(b)}))$ by

$$\begin{aligned}
 & Z_T^{P(\Gamma)}((\pi^{(1)}, \epsilon^{(1)}) \otimes \dots \otimes (\pi^{(b)}, \epsilon^{(b)}))_{(\pi^{(0)}, \epsilon^{(0)})} \\
 &= \sum_{\left. \begin{array}{l} \text{states of } T \\ \text{compatible with} \\ (\pi^{(j)}, \epsilon^{(j)}) \text{ at } D_j, \\ \forall 0 \leq j \leq b \end{array} \right\}} \prod \mu_\alpha, \tag{26} \\
 & \qquad \qquad \qquad \left. \begin{array}{l} \text{singularities } \alpha \\ \text{of } y\text{-coordinate} \\ \text{on strings} \end{array} \right\}
 \end{aligned}$$

⁴We follow the convention that empty tensor product denotes the scalars.

where $\mu_\alpha := \mu_x/\mu_y$ taking $x = \sigma$ (inner region at α), $y = \sigma$ (outer region at α).

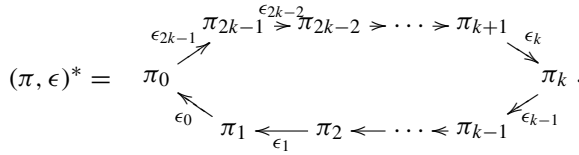


This gives the required tangle map $Z_T^{P(\Gamma)}: \otimes_{i=1}^b P_{k_i}(\Gamma) \rightarrow P_{k_0}(\Gamma)$. The fact that this definition does not depend upon the way we isotope the tangle T to a standard form and that these tangle maps satisfy the compatibility conditions for composition of tangles is precisely the crux of Theorem 3.1 in [5].

**-structure on $P(\Gamma)$*

There is a natural $*$ -algebra structure on each $P_k(\Gamma)$, such that the $*$ -compatibility condition (24) holds:

On $P_{0\pm}(\Gamma)$, we define $(u^\pm)^* = u^\pm$. And for each $k > 0$, given a $2k$ -loop (π, ϵ) , we define



This gives an involution on $P_k(\Gamma)$, with respect to the multiplication induced by $Z_{M_k}^{P(\Gamma)}$, and the $*$ -compatibility condition (24) holds.

4.3 Some calculations

For a coloured tangle T , it is quite illustrative to write the coefficient $Z_T^{P(\Gamma)}((\pi^{(1)}, \epsilon^{(1)}) \otimes \dots \otimes (\pi^{(b)}, \epsilon^{(b)}))_{(\pi^{(0)}, \epsilon^{(0)})}$ as a picture, as shown by the self-explanatory example in eq. (27), where T is the coloured tangle obtained by removing the labels from the picture.

$$\begin{aligned}
 Z_T^{P(\Gamma)}((\pi, \epsilon) \otimes (\lambda, \eta))_{(\theta, \nu)} &= \theta_0 \begin{array}{c} \nu_0 \quad \theta_1 \quad \nu_1 \quad \theta_2 \quad \nu_2 \\ \begin{array}{|c|} \hline * \\ \hline \end{array} \\ \begin{array}{|c|} \hline * \epsilon_0 \quad \pi_1 \quad \epsilon_1 \pi_2 \quad \epsilon_2 \\ \pi_0 \quad \epsilon_5 \quad \pi_5 \quad \epsilon_4 \quad \pi_4 \quad \epsilon_3 \\ \hline \end{array} \\ \begin{array}{|c|} \hline * \eta_0 \quad \lambda_1 \quad \eta_1 \\ \lambda_0 \quad \eta_3 \quad \lambda_3 \quad \eta_2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline \theta_3 \\ \hline \end{array} \\ \nu_5 \theta_5 \quad \nu_4 \quad \theta_4 \quad \nu_3 \end{array} \\
 &= \delta_{\pi^{[0,2]}}^{\theta^{[0,2]}} \delta_{\pi^{[4,6]}}^{\theta^{[4,6]}} \delta_{\lambda^{[1,3]}}^{\theta^{[2,4]}} \delta_{\lambda^{[3,4] \circ \lambda^{[0,1]}}^{\pi^{[2,4]}}. \tag{27}
 \end{aligned}$$

With the help of this diagrammatic approach, we list the expressions of some tangle maps, which we shall need in the sequel.

$$Z_{I_k^{k+1}}^P(\Gamma)((\pi, \epsilon)) = \sum \left\{ \begin{array}{l} \text{possible } v \in \mathcal{U}, \\ \epsilon \in \mathcal{E}(\pi_k, v) \end{array} \right\} \begin{array}{c} \begin{array}{c} \epsilon_0 \nearrow \pi_1 \xrightarrow{\epsilon_1} \dots \xrightarrow{\epsilon_{k-1}} \pi_k \searrow \epsilon \\ \pi_0 \nwarrow \epsilon_{2k-1} \pi_{2k-1} \ll \dots \ll \pi_k \swarrow \epsilon \end{array} \\ \forall 2k\text{-loops } (\pi, \epsilon). \end{array} \tag{28}$$

$$Z_{M_k}^P(\Gamma)((\pi, \epsilon) \otimes (\lambda, \eta)) = \delta_{\lambda_{[0,k]}^{\widetilde{\pi_{[k,2k]}}}^{\pi_{[k,2k]}}} \begin{array}{c} \begin{array}{c} \epsilon_0 \nearrow \pi_1 \xrightarrow{\epsilon_1} \dots \xrightarrow{\epsilon_{k-1}} \pi_{k-1} \searrow \epsilon_{k-1} \\ \pi_0 \nwarrow \eta_{2k-1} \lambda_{2k-1} \eta_{2k-2} \dots \ll \lambda_{k+1} \swarrow \eta_k \end{array} \\ \forall 2k\text{-loops } (\pi, \epsilon), (\lambda, \eta). \end{array} \tag{29}$$

$$Z_{(E')_k^k}^P(\Gamma)((\pi, \epsilon)) = \delta_{\pi_{[0,1]}^{\widetilde{\pi_{[2k-1,2k]}}}^{\pi_{[2k-1,2k]}}} \frac{\mu_{\pi_0}^2}{\mu_{\pi_1}^2} \sum \left\{ \begin{array}{l} \text{possible } u \in \mathcal{U}^+, \\ \epsilon \in \mathcal{E}(u, \pi_1) \end{array} \right\} \begin{array}{c} \begin{array}{c} \epsilon \nearrow \pi_1 \xrightarrow{\epsilon_1} \dots \xrightarrow{\epsilon_{k-1}} \pi_{k-1} \searrow \epsilon_{k-1} \\ u \nwarrow \epsilon \pi_{2k-1} \ll \dots \ll \pi_{k+1} \swarrow \epsilon_k \end{array} \\ \forall 2k\text{-loops } (\pi, \epsilon). \end{array} \tag{30}$$

$$Z_{\mathcal{E}^{k+1}}^P(\Gamma)(1) = \sum \left\{ \begin{array}{l} 2(k+1)\text{-loops } (\pi, \epsilon): \\ \pi_{[0,k-1]} = \widetilde{\pi_{[k+3,2k]}}, \\ \pi_{[k-1,k]} = \widetilde{\pi_{[k,k+1]}}, \\ \pi_{[k+1,k+2]} = \widetilde{\pi_{[k+2,k+3]}} \end{array} \right\} \frac{\mu_{\pi_k} \mu_{\pi_{k+2}}}{\mu_{\pi_{k-1}}^2} (\pi, \epsilon). \tag{31}$$

4.4 Flip and dual

Given a bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$, we define another bipartite graph $\overline{\Gamma} = (\mathcal{U}^-, \mathcal{U}^+, \mathcal{E})$ such that if ϵ is an edge in Γ joining an even vertex u_+ with an odd vertex u_- , then in $\overline{\Gamma}$, ϵ is an edge joining the even vertex u_- with the odd vertex u_+ . We call the bipartite graph $\overline{\Gamma}$ to be the flip of the graph Γ .

Given a finite, connected bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E}, \mu)$, we note that its flip $\overline{\Gamma}$ is also finite and connected, and μ is a spin function for $\overline{\Gamma}$ as well.

Theorem 4.10. *The dual of the planar algebra of a finite, connected bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E}, \mu)$ is isomorphic to the planar algebra of its flip $\overline{\Gamma} = (\mathcal{U}^-, \mathcal{U}^+, \mathcal{E}, \mu)$, i.e.,*

$$-P(\Gamma) \cong P(\overline{\Gamma}).$$

Proof. We have ${}^{-}P(\Gamma)_{0\pm} = P_{\mp 0}(\Gamma) = \mathbb{C}[\mathcal{U}^{\mp}] = P_{\pm 0}(\bar{\Gamma})$; and for each $k > 0$,

$${}^{-}P(\Gamma)_k = P_k(\Gamma) = \mathbb{C}[2k\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^+], \text{ and}$$

$$P(\bar{\Gamma})_k = \mathbb{C}[2k\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^-].$$

Let $\varphi_{0\pm}: {}^{-}P(\Gamma)_{0\pm} \rightarrow P_{0\pm}(\bar{\Gamma})$ be the identity morphism of the underlying vector space; and for each $k > 0$, define $\varphi_k: {}^{-}P(\Gamma)_k \rightarrow P_k(\bar{\Gamma})$ to be the map whose action on basis vectors is given by

$$\varphi_k((\pi, \epsilon)) = \frac{\mu_{\pi_0} \mu_{\pi_k}}{\mu_{\pi_1} \mu_{\pi_{k+1}}} \pi_1 \begin{array}{c} \nearrow^{\epsilon_1} \pi_2 \xrightarrow{\epsilon_2} \dots \rightarrow \pi_k \searrow^{\epsilon_k} \pi_{k+1} \\ \nwarrow_{\epsilon_0} \pi_0 \xleftarrow{\epsilon_{2k-1}} \dots \leftarrow \pi_{k+2} \swarrow_{\epsilon_{k+1}} \end{array}, \tag{32}$$

for all $2k$ -loops (π, ϵ) on Γ based at vertices in \mathcal{U}^+ . Clearly, $\varphi_k((\pi, \epsilon))$ is a $2k$ -loop on Γ based at a vertex in \mathcal{U}^- .

And, as the above correspondence is a bijection between $2k$ -loops on Γ and those on its flip, each φ_k is a linear isomorphism. We claim that $\varphi := \{\varphi_k; k \in \text{Col}\}$ is a planar algebra morphism from ${}^{-}P(\Gamma)$ onto $P(\bar{\Gamma})$.

Let \mathcal{T} be the collection of tangles T for which the equation in Definition 4.2 holds. By Lemma 4.3, \mathcal{T} is closed under composition of tangles. Thus, in order to show that \mathcal{T} contains all the coloured tangles, it is enough to show, by Theorem 4.1, that it contains the collection

$$\mathcal{G}_1 = \{1^{0\pm}\} \cup \{I_k^{k+1}, M_k, E_{k+1}^k; k \in \text{Col}\} \cup \{R_k; k \geq 2\}.$$

The verification of this assertion is a routine, if laborious, exercise using the pictorial approach to tangle maps as described above. □

4.5 Group action on a bipartite graph with spin function

DEFINITION 4.11

Let $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ be a bipartite graph with a spin function μ . We say that a finite group G acts on Γ if

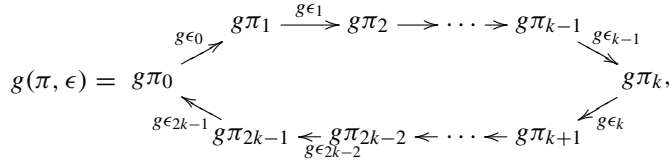
- G acts on each of the sets $\mathcal{U}^+, \mathcal{U}^-$ and \mathcal{E} ; such that
- if ϵ is an edge joining the vertices u_+ and u_- , then $g\epsilon$ is an edge between the vertices gu_+ and gu_- .

Further, we say that G acts on (Γ, μ) if G acts on Γ and the spin function μ is constant on G -orbits of \mathcal{U}^{\pm} .

Theorem 4.12. *Let $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ be a finite, connected bipartite graph with a spin function μ , and suppose that a finite group G acts on (Γ, μ) . Then there is a canonical G -action on the planar algebra $P(\bar{\Gamma})$, and its G -invariant planar sub-algebra $P(\bar{\Gamma})^G$ is isomorphic to the dual planar algebra ${}^{-}(P(\Gamma)^G)$, i.e.,*

$$P(\bar{\Gamma})^G \cong {}^{-}(P(\Gamma)^G).$$

Proof. For each $k > 0$, the action of G on (Γ, μ) induces an action on the set $\{2k\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^+\}$ given by



for all $2k$ -loops (π, ϵ) and $g \in G$.

Already, G acts on \mathcal{U}^\pm , thus we get a group action on $P_k(\Gamma)$, for all $k \in \text{Col}$. Thanks to the invariance of the spin function on the G -orbits of \mathcal{U}^\pm , Lemma 4.3 and Theorem 4.1, a bit of straight forward checking readily shows that it is in fact a G -action on the planar algebra $P(\Gamma)$.

And the second assertion that $P(\bar{\Gamma})^G \cong^- (P(\Gamma)^G)$ is a mere consequence of Proposition 4.8 and the fact that the morphism $\varphi = \{\varphi_k; k \in \text{Col}\}$ giving the planar isomorphism $P(\bar{\Gamma}) \cong^- P(\Gamma)$ in Theorem 4.10 is in fact a G -map, i.e., $\varphi_k(gx) = g\varphi_k(x), \forall g \in G, x \in P_k, k \in \text{Col}$. \square

We list some easily verifiable facts; which are also suggested in [5].

Lemma 4.13. Let Γ and G be as in Theorem 4.12. Then $P(\Gamma)^G$

- (1) is connected iff G acts transitively on \mathcal{U}^\pm ,
- (2) has modulus $\|\Gamma\|$ if in addition to (1), $(\mu_u^2)_{u \in \mathcal{U}}$ is the Perron–Frobenius eigenvector of the adjacency matrix A_Γ of Γ , where $\|\Gamma\|$ is the norm of A_Γ ,
- (3) is irreducible iff G acts transitively on the set $\{2\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^+\}$.

5. Planar algebra of the subgroup-subfactor

5.1 The bipartite graph \star_n

Consider the bipartite graph $\star_n = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$, as in figure 6, where $\mathcal{U}^+ := \{Hg_1, \dots, Hg_n\}$, $\mathcal{U}^- := \{*\}$ and the edge set $\mathcal{E} := \{\epsilon_1, \dots, \epsilon_n\}$, where ϵ_i is the edge

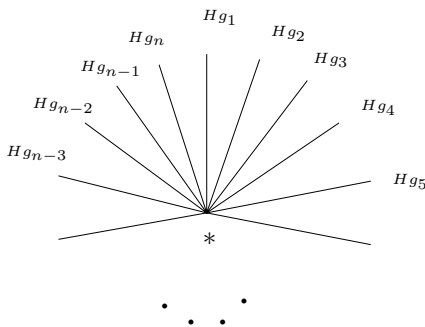


Figure 6. The bipartite graph \star_n .

joining the vertices Hg_i and $*$, $1 \leq i \leq n$. Let $\mu: \mathcal{U}^+ \sqcup \mathcal{U}^- \rightarrow (0, \infty)$ be the spin function given by $\mu(*) = n^{1/4}$ and $\mu(Hg_i) = 1$ for all $i \in I$.

Note that $(\mu_u^2)_{u \in \mathcal{U}^+ \sqcup \mathcal{U}^-}$ is the Perron–Frobenius eigenvector of the adjacency matrix of the bipartite graph \star_n , and the norm of this adjacency matrix is $\|\star_n\| = \sqrt{n}$. Further, with this set up, there is a natural action of G on (\star_n, μ) .

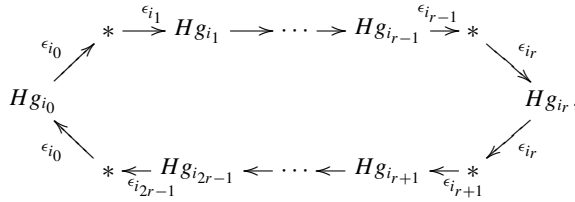
G acts trivially on \mathcal{U}^- ; on \mathcal{U}^+ it acts by the natural left action, i.e., $g \cdot Hg_i = Hg_j \Leftrightarrow Hg_i g^{-1} = Hg_j$; and on \mathcal{E} its action is induced by that on \mathcal{U}^+ , i.e., $g\epsilon_i = \epsilon_j \Leftrightarrow g \cdot Hg_i = Hg_j$, for $g \in G, i, j \in I$.

Remark 5.1. By Lemma 4.13, the planar algebra $P(\star_n)^G$ is connected and has positive modulus \sqrt{n} .

There being only one odd vertex and no multiple edges in \star_n , we can ignore the $*$ -vertex and the edge joining it to an even vertex in the notation of a loop on \star_n . Thus if k is even, say $k = 2r$, we simply write

$$\begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_{r-1}} & \\ Hg_{i_0} & Hg_{i_r} \\ Hg_{i_{2r-1}}, \dots, Hg_{i_{r+1}} & \end{pmatrix}$$

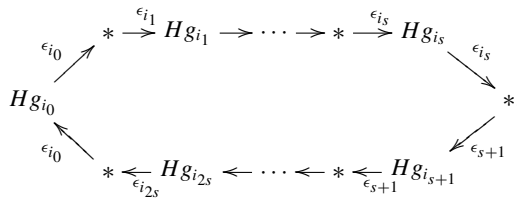
for the $2k$ -loop



and if k is odd, say $k = 2s + 1$, we denote by

$$\begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_s} & \\ Hg_{i_0} & Hg_{i_{s+1}} \\ Hg_{i_{2s}}, \dots, Hg_{i_{s+1}} & \end{pmatrix}$$

the $2k$ -loop



for all $\underline{i} = (i_0, i_1, \dots, i_{k-1}) \in I^k$.

We shall also view these new descriptions of $2k$ -loops as elements of $(H \setminus G)^k$. Also note that the action β^1 of G on I is basically the natural left action of G on the set⁵ $H \setminus G = \{Hg_1, \dots, Hg_n\}$.

With these simplified notations we have the following set of bases.

⁵Our running notation for orbit representatives is not be confused with $H \setminus G$.

Lemma 5.2. $\{\sum_{i \in I} Hg_i\}$ and $\{*\}$ form bases for $P_{0_+}(\star_n)^G$ and $P_{0_-}(\star_n)^G$, respectively. For every choice of orbit representatives $G \setminus (H \setminus G)^k$ under diagonal β^1 -action,

$$\left\{ \sum_{g \in G} g \begin{pmatrix} Hg_{i_0} & Hg_{i_1}, \dots, Hg_{i_{r-1}} & Hg_{i_r} \\ Hg_{i_{2r-1}}, \dots, Hg_{i_{r+1}} \end{pmatrix} : \begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_{r-1}} & Hg_{i_r} \\ Hg_{i_0} & Hg_{i_{2r-1}}, \dots, Hg_{i_{r+1}} \end{pmatrix} \in G \setminus (H \setminus G)^{2r} \right\}$$

forms a basis for $P_{2r}(\star_n)^G$ for all $r \geq 1$; and

$$\left\{ \sum_{g \in G} g \begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_r} \\ Hg_{i_0} & Hg_{i_{2r}}, \dots, Hg_{i_{r+1}} \end{pmatrix} : \begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_r} \\ Hg_{i_0} & Hg_{i_{2r}}, \dots, Hg_{i_{r+1}} \end{pmatrix} \in G \setminus (H \setminus G)^{2r+1} \right\}$$

forms a basis for $P_{2r+1}(\star_n)^G$ for all $r \geq 0$.

Remark 5.1, Lemma 5.2, Lemma 4.4, and Remark 4.5 give the following:

Lemma 5.3. With notations as in §5.1, the planar algebra $P(\star_n)^G$ is irreducible, spherical and admits a global trace.

Moving towards the desired identification, we define $\varphi_{0_\pm}: P_{0_\pm}^{R \times H \subset R \times G} \rightarrow P_{0_\pm}(\star_n)^G$ by

$$P_{0_\pm}^{R \times H \subset R \times G} := \mathbb{C} \ni \lambda \xrightarrow{\varphi_{0_\pm}} \lambda 1_{0_\pm} \in P_{0_\pm}(\star_n)^G, \quad (33)$$

where $1_{0_+} = \sum_{i \in I} Hg_i$ and $1_{0_-} = *$ are the respective multiplicative units of $P_{0_\pm}(\star_n)^G$; and $\varphi_1: P_1^{R \times H \subset R \times G} = \mathbb{C} \rightarrow P_1(\star_n)^G$ by

$$P_1^{R \times H \subset R \times G} = \mathbb{C} \ni \lambda \xrightarrow{\varphi_1} \lambda 1_1 \in P_1(\star_n)^G, \quad (34)$$

where 1_1 is the multiplicative unit $\sum_{i \in I} (Hg_i)$ of $P_1(\star_n)^G$.

For $k \geq 2$, we define $\varphi_k: P_k^{R \times H \subset R \times G} \rightarrow P_k(\star_n)^G$ on the basis vectors described in Lemma 3.6 by

$$P_{2k+1}^{R \times H \subset R \times G} = N' \cap M_{2k} \ni \sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \xrightarrow{\varphi_{2k+1}} \sum_{g \in G} g \begin{pmatrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \dots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \dots, H \sqcap g_{\underline{j}} \end{pmatrix} \in P_{2k+1}(\star_n)^G, \quad (35)$$

for all $(\underline{i}, \underline{j}) \in H \setminus (I^k \times I^k)$, $k \geq 1$; and

$$P_{2k}^{R \times H \subset R \times G} = N' \cap M_{2k-1} \ni \sum_{h \in H} h[\underline{i}, \underline{j}]^{od} \xrightarrow{\varphi_{2k}} \sum_{g \in G} g \begin{pmatrix} Hg_{i_k}, \dots, Hg_{i_2} \dots g_{i_k} & H \sqcap g_{\underline{i}} \\ Hg_{j_k}, \dots, Hg_{j_2} \dots g_{j_k} \end{pmatrix} \in P_{2k}(\star_n)^G, \quad (36)$$

for all $(\underline{i}, \underline{j}) \in H \setminus Y_k$, $k \geq 1$.

We want to show that

$$\{\varphi_k : k \in \text{Col}\} =: \varphi : P^{R \rtimes H \subset R \rtimes G} \rightarrow P(\star_n)^G$$

is a planar algebra isomorphism.

5.2 *Ingredients of the proof*

Lemma 5.4. *With running notations, we have*

$$\dim P_k^{R \rtimes H \subset R \rtimes G} = \dim P_k(\star_n)^G, \forall k \in \text{Col}.$$

Proof. We have already seen that $P_{0_{\pm}}(\star_n)^G, P_1(\star_n)^G, P_{0_{\pm}}^{R \rtimes H \subset R \rtimes G}$ and $P_1^{R \rtimes H \subset R \rtimes G}$ are all one dimensional.

For each $k \geq 1$, by Lemma 3.6, $\dim N' \cap M_{2k}$ (resp., $\dim N' \cap M_{2k-1}$) is same as the number of H -orbits of $I^k \times I^k$ (resp., Y_k) under the diagonal β^k -action. On the other hand, by Lemma 5.2, $\dim P_{2k+1}(\star_n)^G$ (resp., $\dim P_{2k}(\star_n)^G$) is equal to the number of G -orbits of $(H \setminus G)^{2k+1}$ (resp., $(H \setminus G)^{2k}$) under the diagonal β^1 -action.

The corresponding dimensions are same because the number of H -orbits of $I^k \times I^k$ (resp., Y_k) under the diagonal β^k -action is equal to the number of G -orbits of $(H \setminus G)^{2k+1}$ (resp., $(H \setminus G)^{2k}$) under the diagonal β^1 -actions. We include a proof of the assertion in parentheses, and the other follows similarly.

Note that the number of G -orbits of $(H \setminus G)^{2k}$ is same as the number of H -orbits of $(H \setminus G)^{2k-1}$ under the diagonal actions mentioned in the previous paragraph. This is true because any element of $(H \setminus G)^{2k}$ lies in the G -orbit of an element of the type $(H, Hg_{i_1}, \dots, Hg_{i_{2k-1}})$; and

$$g(H, Hg_{i_1}, \dots, Hg_{i_{2k-1}}) = (H, Hg_{j_1}, \dots, Hg_{j_{2k-1}})$$

if and only if $g \in H$ and $g(Hg_{i_1}, \dots, Hg_{i_{2k-1}}) = (Hg_{j_1}, \dots, Hg_{j_{2k-1}})$. And thus the assertion in the parentheses follows from the facts that the correspondence

$$Y_k \ni (\underline{i}, \underline{j}) \mapsto \left(\begin{array}{l} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \dots, Hg_{i_2} \cdots g_{i_k}, \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \dots, Hg_{j_2} \cdots g_{j_k}, \end{array} H \cap g_{\underline{i}} \right) \in (H \setminus G)^{2k-1},$$

by the definition of the action β^k , is an H -injection, and that $|Y_k| = |X^{2k-1}|$. □

In particular, this gives:

Lemma 5.5. *With notations as in eqs (33)–(36), for each $k \in \text{Col}$, the map $\varphi_k : P_k^{R \rtimes H \subset R \rtimes G} \rightarrow P_k(\star_n)^G$ is a linear isomorphism.*

We conclude this subsection with two lemmas that we shall need for verifying that the φ_k 's are equivariant with respect to the inclusion and conditional expectation tangle maps.

In order to understand how the inclusion tangles act on the subgroup-subfactor planar algebra, and the way we defined the maps φ_k , we need to analyse the inclusions in terms of the bases that we obtained in Lemma 3.6. For k odd, say $k = 2r - 1$, it is the usual matrix algebra inclusion $\theta^{(r)}(N)' \cap M_{1r}(N) \subset \theta^{(r)}(N)' \cap M_{1r}(M)$, whereas for k even, the inclusion is given by an appropriate Θ_r .

Lemma 5.6. For each $k \geq 1$, the inclusion $N' \cap M_{2k} \subset N' \cap M_{2k+1}$ is given as under:

$$\begin{aligned} \theta^{(k)}(N)' \cap M_{I^k}(M) &\ni \sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \\ &\xrightarrow{\Theta_{k+1}} \sum_{\substack{h \in H, r, s \in I: \\ ((r, \underline{i}), (s, \underline{j})) \in Y_{k+1}}} h[(r, \underline{i}), (s, \underline{j})]^{od} \in \theta^{(k+1)}(N)' \cap M_{I^{k+1}}(N), \end{aligned} \quad (37)$$

for all $(\underline{i}, \underline{j}) \in H \setminus (I^k \times I^k)$.

Proof. First we note that, for each $k \geq 1$ and $(\underline{i}, \underline{j}) \in I^k \times I^k$,

$$\Theta_{k+1}([\underline{i}, \underline{j}]^{ev}) = \sum_{\substack{r, s \in I: \\ ((r, \underline{i}), (s, \underline{j})) \in Y_{k+1}}} [(r, \underline{i}), (s, \underline{j})]^{od}.$$

Indeed, by the description of Θ_{k+1} as in Lemma 3.4, we have

$$\begin{aligned} \Theta_{k+1}([\underline{i}, \underline{j}]^{ev})_{\underline{u}, \underline{v}} &= \theta_{u_1, v_1}([\underline{i}, \underline{j}]_{u_2, v_2}^{ev}) \\ &= \theta_{u_1, v_1}(\delta_{u_2}^{\underline{i}} \delta_{v_2}^{\underline{j}} u_{(\cap g_{\underline{i}})(\cap g_{\underline{j}})^{-1}}) \\ &= \delta_{u_2}^{\underline{i}} \delta_{v_2}^{\underline{j}} 1_H (g_{u_1} (\cap g_{\underline{i}}) (\cap g_{\underline{j}})^{-1} g_{v_1}^{-1}) u_{(\cap g_{(u_1, \underline{i})}(\cap g_{(v_1, \underline{j})})^{-1}}, \end{aligned}$$

$\forall \underline{u}, \underline{v} \in I^{k+1}$. Thus we just need to show that

$$\Theta_{k+1}(h[\underline{i}, \underline{j}]^{ev}) = \sum_{\substack{r, s \in I: \\ ((r, \underline{i}), (s, \underline{j})) \in Y_{k+1}}} h[(r, \underline{i}), (s, \underline{j})]^{od}, \quad \forall h \in H,$$

i.e.,

$$\begin{aligned} &\sum_{\substack{x, y \in I: \\ ((x, \beta_h^k(\underline{i})), (y, \beta_h^k(\underline{j}))) \in Y_{k+1}}} [(x, \beta_h^k(\underline{i})), (y, \beta_h^k(\underline{j}))]^{od} \\ &= \sum_{\substack{r, s \in I: \\ ((r, \underline{i}), (s, \underline{j})) \in Y_{k+1}}} [\beta_h^{k+1}(r, \underline{i}), \beta_h^{k+1}(s, \underline{j})]^{od}, \quad \forall h \in H. \end{aligned} \quad (38)$$

For each $h \in H$ and $(\underline{u}, \underline{v}) \in Y_{k+1}$, the coefficient of $[\underline{u}, \underline{v}]^{od}$ on l.h.s. (resp., r.h.s.) of eq. (38) is the number of elements in the set

$$L_{\underline{u}, \underline{v}} := \{(x, y) \in I \times I: ((x, \beta_h^k(\underline{i})), (y, \beta_h^k(\underline{j}))) \in Y_{k+1},$$

$$[(x, \beta_h^k(\underline{i})), (y, \beta_h^k(\underline{j}))]^{od} = [\underline{u}, \underline{v}]^{od}\}$$

$$\text{(resp., } R_{\underline{u}, \underline{v}} := \{(r, s) \in I \times I: r, s \in I: ((r, \underline{i}), (s, \underline{j})) \in Y_{k+1},$$

$$[\beta_h^{k+1}(r, \underline{i}), \beta_h^{k+1}(s, \underline{j})]^{od} = [\underline{u}, \underline{v}]^{od}\}).$$

Viewing the two sides of eq. (38) as elements of $R' \cap M_{2k+1}$, we shall be done once we show that these coefficients are same.

Note that if $(\beta_h^{k+1}((r, \underline{i})), \beta_h^{k+1}((s, \underline{j}))) = (\underline{u}, \underline{v})$, for some $h \in H$ and $((r, \underline{i}), (s, \underline{j})) \in Y_{k+1}$, then $\underline{u}_{[2]} = \beta_h^k(\underline{i})$, $\underline{v}_{[2]} = \beta_h^k(\underline{j})$ and

$$((u_1, \beta_h^k(\underline{i})), (v_1, \beta_h^k(\underline{j}))) \in Y_{k+1}.$$

Consider the maps $L_{\underline{u}, \underline{v}} \xrightleftharpoons[\psi]{\phi} R_{\underline{u}, \underline{v}}$ given as under:

For each $(x, y) \in L_{\underline{u}, \underline{v}}$ (resp., $(r, s) \in R_{\underline{u}, \underline{v}}$), $\phi((x, y)) = (p, q) \in R_{\underline{u}, \underline{v}}$ (resp., $\psi((r, s)) = (w, z)$), where (r, s) (resp., (w, z)) is given by the equations

$$\begin{cases} Hg_p = Hg_x(\sqcap g_{\beta_h^k(\underline{i})})h(\sqcap g_{\underline{i}})^{-1} \text{ and} \\ Hg_q = Hg_y(\sqcap g_{\beta_h^k(\underline{j})})h(\sqcap g_{\underline{j}})^{-1} \end{cases}$$

(resp., $\beta_h^{k+1}((r, \underline{i})) = (w, \beta_h^k(\underline{i}))$ and $\beta_h^{k+1}((s, \underline{j})) = (z, \beta_h^k(\underline{j}))$).

It can be seen that the maps ϕ and ψ are inverses of each other. Thus the coefficient of $[\underline{u}, \underline{v}]^{od}$ is same on both sides of eq. (38), and we are done. \square

The following lemma is a consequence of the uniqueness of trace preserving conditional expectation.

Lemma 5.7. Let $A_0 \subset A_1$ and $B_0 \subset B_1$ be inclusions of finite dimensional C^* -algebras. Suppose tr_{A_1} and tr_{B_1} are faithful traces on A_1 and B_1 , respectively. Let $\phi_i: A_i \rightarrow B_i$, $i = 0, 1$ be C^* -isomorphisms preserving the traces such that $\phi_1/A_0 = \phi_0$. Then $E_{B_0}^{B_1} \circ \phi_1 = \phi_0 \circ E_{A_0}^{A_1}$, where $E_{X_0}^{X_1}: X_1 \rightarrow X_0$ is the unique tr_{X_1} -trace preserving conditional expectation for $X_i \in \{A_i, B_i\}$, $i = 0, 1$.

5.3 The identification

We need to show that φ , as defined in eqs (33)–(36), commutes with all the tangle maps.

For notational convenience, in what follows, we shall simply write P (resp., Q) for the subgroup-subfactor planar algebra $P^{R \times H \subset R \times G}$ (resp., the invariant planar subalgebra $P(\star_n)^G$).

Lemma 5.8. $\varphi_{k+1} \circ Z_{I_k^{k+1}}^P = Z_{I_k^{k+1}}^Q \circ \varphi_k$, $\forall k \in \text{Col}$.

Proof. We clearly have $\varphi_1 \circ Z_{I_{0\pm}^1}^P = Z_{I_{0\pm}^1}^Q \circ \varphi_{0\pm}$, by (33) and (34).

Now for $k \geq 1$, we give a proof for odd k , the even case is similar. Suppose $k = 2r + 1$ for some $r \geq 0$. Then, by Lemma 5.6, $Z_{I_{2r+1}^{2r+2}}^P: N' \cap M_{2r} \hookrightarrow N' \cap M_{2r+1}$ is given by the inclusion map Θ_{r+1} . On the other hand, by (28) for the bipartite graph \star_n , we have

$$Q_{2r+1} \ni \sum_{g \in G} g \begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_r} \\ Hg_{i_0} \quad Hg_{i_{2r}}, \dots, Hg_{i_{r+1}} \end{pmatrix}$$

$$\xrightarrow{Z_{I_{2r+1}^{2r+2}}^Q} \sum_{j \in I, g \in G} g \begin{pmatrix} Hg_{i_1}, \dots, Hg_{i_r} & Hg_j \\ Hg_{i_0} \quad Hg_{i_{2r-1}}, \dots, Hg_{i_{r+1}} \end{pmatrix} \in Q_{2r+2}.$$

If $r = 0$, then for each $\lambda \in \mathbb{C} = P_1$, by (34) and (36), we have

$$\begin{aligned} (Z_{I_1}^Q \circ \varphi_1)(\lambda) &= Z_{I_1}^Q \left(\lambda \sum_{i \in I} (H g_i) \right) \\ &= Z_{I_1}^Q \left(\frac{\lambda}{|\text{Iso}_H(H)|} \sum_{g \in G} g(H) \right) \\ &= \frac{\lambda}{|H|} \sum_{i \in I, g \in G} g(H, H g_i) \end{aligned}$$

and

$$\begin{aligned} (\varphi_2 \circ Z_{I_1}^P)(\lambda) &= \varphi_2 \left(\frac{\lambda}{|\text{Iso}_H((1, 1))|} \sum_{g \in G} g[1, 1]^{od} \right) \\ &= \frac{\lambda}{|H|} \varphi_2 \left(\sum_{i \in I, h \in H} h g_i [1, 1]^{od} \right) \\ &= \frac{\lambda}{|H|} \sum_{i \in I, g \in G} g(H, H g_i), \end{aligned}$$

showing that $\varphi_2 \circ Z_{I_1}^P = Z_{I_1}^Q \circ \varphi_1$, where for an H -set X and $x \in X$, $\text{Iso}_H(x) := \{h \in H : hx = x\}$.

And for $r \geq 1$, for each $(\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r)$, by Lemma 5.6, we have

$$\begin{aligned} (\varphi_{2r+2} \circ Z_{I_{2r+1}}^P) \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) &= \varphi_{2r+2} \left(\sum_{\substack{h \in H, x, y \in I: \\ ((x, \underline{i}), (y, \underline{j})) \in Y_{r+1}}} h[(x, \underline{i}), (y, \underline{j})]^{od} \right) \\ &= \sum_{g \in G, x \in I} g \left(\begin{matrix} H g_{i_r}, \dots, H g_{i_1} \dots g_{i_r} & H \sqcap g_{(x, \underline{i})} \\ H g_{j_r}, \dots, H g_{j_1} \dots g_{j_r} & \end{matrix} \right) \end{aligned}$$

and

$$\begin{aligned} (Z_{I_{2r+1}}^Q \circ \varphi_{2r+1}) \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) &= Z_{I_{2r+1}}^Q \left(\sum_{g \in G} g \left(\begin{matrix} H g_{i_k}, \dots, H \sqcap g_{\underline{i}} \\ H g_{j_k}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{x \in I, g \in G} g \left(\begin{matrix} Hg_{i_k}, \dots, H \sqcap g_{\underline{i}} \\ H \\ Hg_{j_k}, \dots, H \sqcap g_{\underline{j}} \end{matrix} Hg_x \right) \\
 &= \sum_{j \in I, g \in G} g \left(\begin{matrix} Hg_{i_k}, \dots, Hg_{i_1} \dots g_{i_k} \\ H \\ Hg_{j_k}, \dots, Hg_{j_1} \dots g_{j_k} \end{matrix} H \sqcap g_{(j, \underline{i})} \right).
 \end{aligned}$$

Thus $\varphi_{k+1} \circ Z_{I_k^{k+1}}^P = Z_{I_k^{k+1}}^Q \circ \varphi_k$. □

Lemma 5.9. $\varphi_k \circ Z_{M_k}^P = Z_{M_k}^Q \circ (\varphi_k \otimes \varphi_k), \forall k \in \text{Col}.$

Proof. There is nothing to prove for $k = 0_{\pm}$. Both P and Q being irreducible, there is nothing to be proved for $k = 1$ either. For $k \geq 2$, we give a proof for the case when k is odd, say $k = 2r + 1$, for some $r \geq 1$, and the other case can be proved on exactly similar lines. Let

$$X_{2r+1} = \left\{ \left(\begin{matrix} Hg_{i_1}, \dots, Hg_{i_r} \\ H \\ Hg_{j_1}, \dots, Hg_{j_r} \end{matrix} \right) \in (H/G)^{2r+1} : \underline{i}, \underline{j} \in I^r \right\}.$$

Then X_{2r+1} is invariant under the diagonal β^1 -action of H on $(H \setminus G)^{2r+1}$ and it can also be identified with $(H \setminus G)^{2r}$ as H -sets. Thus, as in the proof of Lemma 5.14, the correspondence

$$I^r \times I^r \ni (\underline{i}, \underline{j}) \mapsto \left(\begin{matrix} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \dots, H \sqcap g_{\underline{i}} \\ H \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) \in X_{2r+1}$$

is an H -bijection; and a set of representatives of H -orbits of X_{2r+1} is also a set of representatives of G -orbits of $(H \setminus G)^{2r+1}$. This shows that

$$\left\{ \left(\begin{matrix} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \dots, H \sqcap g_{\underline{i}} \\ H \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) : (\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r) \right\}$$

is a set of representatives of G -orbits of $(H \setminus G)^{2r+1}$. In particular, by Lemma 5.2,

$$\left\{ \sum_{g \in G} g \left(\begin{matrix} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \dots, H \sqcap g_{\underline{i}} \\ H \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) : (\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r) \right\}$$

forms a basis for Q_{2r+1} .

We now prove that φ commutes with the tangle map for the multiplication tangle M_{2r+1} . For each pair of representatives $(\underline{i}, \underline{j}), (\tilde{i}, \tilde{j}) \in H \setminus (I^r \times I^r)$, we have

$$\begin{aligned} & \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) \left(\sum_{\tilde{h} \in H} \tilde{h}[\tilde{i}, \tilde{j}]^{ev} \right) \\ &= \sum_{h, \tilde{h} \in H} \delta_{\beta_h^r(\tilde{i})}^{\beta_h^r(j)} [\beta_h^r(\underline{i}), \beta_h^r(\tilde{j})]^{ev} \\ &= \sum_{\hat{h} \in H, (\hat{i}, \hat{j}) \in H \setminus (I^r \times I^r)} C_{\hat{i}, \hat{j}} \hat{h}[\hat{i}, \hat{j}]^{ev} \\ &\xrightarrow{\varphi_{2r+1}} \sum_{\hat{g} \in G, (\hat{i}, \hat{j}) \in H \setminus (I^r \times I^r)} C_{\hat{i}, \hat{j}} \hat{g} \left(\begin{array}{c} H g_{i_r}, H g_{i_{r-1}} g_{i_r}, \dots, H \cap g_{\underline{i}} \\ H g_{j_r}, H g_{j_{r-1}} g_{j_r}, \dots, H \cap g_{\underline{j}} \end{array} \right), \end{aligned}$$

where $C_{\hat{i}, \hat{j}}$ is the number of elements in the set

$$\tilde{C}_{\hat{i}, \hat{j}} := \{(h, \tilde{h}) \in H \times H : \beta_h^r(j) = \beta_{\tilde{h}}^r(\tilde{i}), \beta_h^r(\underline{i}) = \hat{i} \text{ and } \beta_{\tilde{h}}^r(\tilde{j}) = \hat{j}\}.$$

We have used the fact that since the above product is in $N' \cap M_{2r}$, the coefficient of $\hat{h}[\hat{i}, \hat{j}]^{ev}$ in it is the same as that of $[\hat{i}, \hat{j}]^{ev}$ for all $\hat{h} \in H$.

On the other hand, by (29) for the bipartite graph \star_n , we have

$$\begin{aligned} & \varphi_{2r+1} \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) \varphi_{2r+1} \left(\sum_{\tilde{h} \in H} \tilde{h}[\tilde{i}, \tilde{j}]^{ev} \right) \\ &= \sum_{g \in G} g \left(\begin{array}{c} H g_{i_r}, \dots, H \cap g_{\underline{i}} \\ H g_{j_r}, \dots, H \cap g_{\underline{j}} \end{array} \right) \sum_{\tilde{g} \in G} \tilde{g} \left(\begin{array}{c} H g_{\tilde{i}_r}, \dots, H \cap g_{\tilde{i}} \\ H g_{\tilde{j}_r}, \dots, H \cap g_{\tilde{j}} \end{array} \right) \\ &= \sum_{g, \tilde{g} \in G} \delta_{\tilde{g}(H, H g_{\tilde{i}_r}, \dots, H \cap g_{\tilde{i}})}^{g(H, H g_{j_r}, \dots, H \cap g_{\underline{j}})} \left(\begin{array}{c} g \cdot H g_{i_r}, \dots, g \cdot H \cap g_{\underline{i}} \\ \tilde{g} \cdot H g_{j_r}, \dots, \tilde{g} \cdot H \cap g_{\underline{j}} \end{array} \right) \\ &= \sum_{\hat{g} \in G, (\hat{i}, \hat{j}) \in H \setminus (I^r \times I^r)} D_{\hat{i}, \hat{j}} \hat{g} \left(\begin{array}{c} H g_{i_r}, \dots, H \cap g_{\underline{i}} \\ H g_{j_r}, \dots, H \cap g_{\underline{j}} \end{array} \right), \end{aligned}$$

where $D_{\hat{i}, \hat{j}}$ is the number of elements in the set

$$\tilde{D}_{\hat{i}, \hat{j}} := \left\{ \left(\begin{array}{l} (g, \tilde{g}) \in G \times G: \\ g(H, H g_{j_r}, \dots, H \cap g_{\underline{j}}) = \tilde{g}(H, H g_{\tilde{i}_r}, \dots, H \cap g_{\tilde{i}}), \\ g(H, H g_{i_r}, \dots, H \cap g_{\underline{i}}) = (H, H g_{\tilde{i}_r}, \dots, H \cap g_{\tilde{i}}) \text{ and} \\ \tilde{g}(H, H g_{\tilde{j}_r}, \dots, H \cap g_{\tilde{j}}) = (H, H g_{j_r}, \dots, H \cap g_{\underline{j}}) \end{array} \right) \right\},$$

and the coefficients are constant on each orbit as in the former case.

It can be seen that, for each $(\hat{i}, \hat{j}) \in H \setminus (I^r \times I^r)$, the sets $\tilde{C}_{\hat{i}, \hat{j}}$ and $\tilde{D}_{\hat{i}, \hat{j}}$ are same.

Thus we conclude that $\varphi_{2r+1} \circ Z_{M_{2r+1}}^P = Z_{M_{2r+1}}^Q \circ (\varphi_{2r+1} \otimes \varphi_{2r+1})$.

Note that $Q_{1,k} = [P(\star_n)^G]_{1,k} = P_{1,k}(\star_n)^G, \forall k \geq 1$.

Lemma 5.10. With running notations, we have

$$\varphi_k \circ Z_{E_k}^P = Z_{E_k}^Q \circ \varphi_{k+1}, \text{ and} \tag{39}$$

$$\varphi_{k+1} \circ Z_{(E')_{k+1}}^P = Z_{(E')_{k+1}}^Q \circ \varphi_{k+1}, \forall k \in \text{Col}. \tag{40}$$

Proof. First note that the maps $\varphi_k, k \in \text{Col}$ are all $*$ -preserving, where the $*$ -structure on Q is given as in the last paragraph of §4.1; and, by Lemma 5.9, they are algebra homomorphisms as well.

We next show that these maps preserve the traces as well, where Q is equipped with the global pictorial trace as given in Lemma 5.3. Then by the fact that they are $*$ -preserving algebra isomorphisms, it follows that the global pictorial trace on Q is in fact faithful. Thus (39) holds by Lemma 5.7, and also by the same result, (40) will hold once we show that $\varphi_k(P_{1,k}) = Q_{1,k} := \text{Image}(Z_{(E')_k}^Q), \forall k \geq 1$.

We calculate the trace on Q_k for odd k , say $k = 2r + 1$ for some $r \geq 1$. In the planar algebra $P(\star_n)$, given $\underline{i}, \underline{j} \in I^r$ and $x, y \in I$, we have

$$Z_{E_{2r+1}}^{P(\star_n)} \left(\left(\begin{array}{c} Hg_{i_1}, \dots, Hg_{i_r} \\ Hg_x \\ Hg_{j_1}, \dots, Hg_{j_r} \end{array} \right) \right)_{Hg_y} = \text{Diagram} = \delta_{\underline{j}}^i \delta_y^x \mu_*^2$$

Thus the pictorial trace on $P_{2r+1}(\star_n)^G$ is given by

$$\begin{aligned} \text{tr}_{2r+1} \left(\sum_{g \in G} g \left(\begin{array}{c} Hg_{i_1}, \dots, Hg_{i_r} \\ Hg_x \\ Hg_{j_1}, \dots, Hg_{j_r} \end{array} \right) \right) &= \frac{1}{(\sqrt{n})^{2r+1}} \mu_*^2 \delta_{\underline{j}}^i |\text{Iso}_G(Hg_x)| \\ &= \frac{|H|}{n^r} \delta_{\underline{j}}^i, \end{aligned}$$

for all $x \in I$ and $\underline{i}, \underline{j} \in I^r$, where we have used the fact that, G acting transitively on $H \setminus G, \text{Iso}_G(Hg_x) \cong \text{Iso}_G(H) = H$, for all $x \in I$.

On the other hand, for each $(\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r)$, we have $\text{tr}_{M_{I^r}(M)}([\underline{i}, \underline{j}]^{ev}) = \delta_{\underline{j}}^{\underline{i}} \frac{1}{n^r}$; so that

$$\text{tr}_{M_{I^r}(M)} \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) = \sum_{h \in H} \delta_{\beta_h(\underline{j})}^{\beta_h(\underline{i})} \frac{1}{n^r} = \frac{|H|}{n^r} \delta_{\underline{j}}^{\underline{i}},$$

and thus

$$\begin{aligned} \text{tr}_{2r+1} \left(\varphi_{2r+1} \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) \right) &= \text{tr}_{2r+1} \left(\sum_{g \in G} g \left(\begin{array}{c} H g_{i_1}, \dots, H g_{i_r} \\ H g_{j_1}, \dots, H g_{j_r} \end{array} \right) \right) \\ &= \frac{|H|}{n^r} \delta_{\underline{j}}^{\underline{i}}. \end{aligned}$$

This shows that φ_{2r+1} preserves trace. With exactly similar calculations we can show that φ_k preserves trace for even k as well.

It only remains to show that $\varphi_k(P_{1,k}) = Q_{1,k}$ for all $k \geq 1$. Again, there is nothing to prove for $k = 1$. We prove the assertion for odd k , say $k = 2r + 1$, for some $r \geq 1$. Note that

$$\left\{ \sum_{g \in G} \left(\begin{array}{c} H g_{i_r}, H g_{i_{r-1}} g_{i_r}, \dots, H \cap g_{\underline{i}} \\ H g_{j_r}, H g_{j_{r-1}} g_{j_r}, \dots, H \cap g_{\underline{j}} \end{array} \right) : (\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r) \right\}$$

forms a basis for Q_{2r+1} .

By (30) for the bipartite graph \star_n , for each $(\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r)$, we have

$$\begin{aligned} Z_{(E')^k}^Q \left(\sum_{g \in G} g \left(\begin{array}{c} H g_{i_r}, H g_{i_{r-1}} g_{i_r}, \dots, H \cap g_{\underline{i}} \\ H g_{j_r}, H g_{j_{r-1}} g_{j_r}, \dots, H \cap g_{\underline{j}} \end{array} \right) \right) \\ = \frac{1}{\sqrt{n}} \sum_{z \in I, g \in G} \left(\begin{array}{c} g \cdot H g_{i_r}, g \cdot H g_{i_{r-1}} g_{i_r}, \dots, g \cdot H \cap g_{\underline{i}} \\ H g_z, g \cdot H g_{j_r}, g \cdot H g_{j_{r-1}} g_{j_r}, \dots, g \cdot H \cap g_{\underline{j}} \end{array} \right) \\ = \frac{1}{\sqrt{n}} \sum_{x \in I, g \in G} g \left(\begin{array}{c} H g_{i_r}, H g_{i_{r-1}} g_{i_r}, \dots, H \cap g_{\underline{i}} \\ H g_x, H g_{j_r}, H g_{j_{r-1}} g_{j_r}, \dots, H \cap g_{\underline{j}} \end{array} \right); \end{aligned}$$

and these elements generate $Q_{1,2r+1}$ as a vector space.

Further, for each $(\underline{i}, \underline{j}) \in H \setminus (I^r \times I^r)$, $\sum_{g \in G} g[\underline{i}, \underline{j}]^{ev} \in M' \cap M_{2r}$ and

$$\begin{aligned} \sum_{g \in G} g[\underline{i}, \underline{j}]^{ev} &= \sum_{x \in I, h \in H} h g_x[\underline{i}, \underline{j}]^{ev} \\ &\xrightarrow{\varphi_{2r+1}} \sum_{x \in I, g \in G} g \left(\begin{array}{c} H g_{i_r} g_x^{-1}, H g_{i_{r-1}} g_{i_r} g_x^{-1}, \dots, H(\cap g_{\underline{i}}) g_x^{-1} \\ H g_{j_r} g_x^{-1}, H g_{j_{r-1}} g_{j_r} g_x^{-1}, \dots, H(\cap g_{\underline{j}}) g_x^{-1} \end{array} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{x \in I, g \in G} gg_x \left(\begin{matrix} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \dots, H \sqcap g_{\underline{i}} \\ Hg_x \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) \\
 &= \sum_{x \in I, \tilde{g} \in G} \tilde{g} \left(\begin{matrix} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \dots, H \sqcap g_{\underline{i}} \\ Hg_x \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right)
 \end{aligned}$$

Thus the elements of the above generating set for $Q_{1,2r+1}$ are in the space $\varphi_{2r+1}(M' \cap M_{2r})$, and we know that $P_{1,2r+1} = E_{M' \cap M_{2r}}(N' \cap M_{2r}) = M' \cap M_{2r}$. This proves our second claim. \square

Lemma 5.11. $(\varphi_{k+1} \circ Z_{\mathcal{E}^{k+1}}^P) = Z_{\mathcal{E}^{k+1}}^Q, \forall k \geq 1$.

Proof. We first prove the assertion for even k .

We need to consider some H and G invariant subsets of $I^k \times I^k$ and $(H \setminus G)^{2k+1}$ for all $k \geq 1$. For each $k \geq 1$, we set $W_k = \{(\underline{i}, \underline{j}) \in I^k \times I^k : \underline{i}_{[2]} = \underline{j}_{[2]}\}$,

$$F_{2k+1} = \left\{ \left(\begin{matrix} Hg_{i_1}, \dots, Hg_{i_k} \\ Hg_x \\ Hg_{j_1}, \dots, Hg_{j_k} \end{matrix} \right) : x \in I, \underline{i}, \underline{j} \in I^k \text{ with } \underline{i}_{[k-1]} = \underline{j}_{[k-1]} \right\}$$

and

$$G_{2k+1} = \left\{ \left(\begin{matrix} Hg_{i_1}, \dots, Hg_{i_k} \\ H \\ Hg_{j_1}, \dots, Hg_{j_k} \end{matrix} \right) : \underline{i}, \underline{j} \in I^k \text{ with } \underline{i}_{[k-1]} = \underline{j}_{[k-1]} \right\}.$$

Then W_k (resp., F_{2k+1}) is G invariant under the diagonal β^k (resp., β^1)-action, and $G_{2k+1} \subset F_{2k+1}$ is H invariant under the restricted action.

Further, the correspondence

$$W_k \ni (\underline{i}, \underline{j}) \mapsto \left(\begin{matrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \dots, H \sqcap g_{\underline{i}} \\ H \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) \in G_{2k+1}$$

is an H -bijection. So

$$\left\{ \left(\begin{matrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \dots, H \sqcap g_{\underline{i}} \\ H \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \dots, H \sqcap g_{\underline{j}} \end{matrix} \right) : (\underline{i}, \underline{j}) \in H \setminus W_k \right\}$$

is a set of representatives of H orbits of G_{2k+1} . Clearly this is also a set of representatives of G orbits of F_{2k+1} .

Now we note that, by (31) for the graph \star_n , we have

$$Z_{\mathcal{E}^{2k+1}}^Q(1) = \frac{1}{\sqrt{n}} \sum_{x,y,z \in I, \underline{i} \in I^{k-1}} \left(\begin{matrix} Hg_{i_1}, \dots, Hg_{i_{k-1}}, Hg_y \\ Hg_x \\ Hg_{i_1}, \dots, Hg_{i_{k-1}}, Hg_z \end{matrix} \right).$$

Recall, from Theorem 4.6, that $Z_{\mathcal{E}^{2k+1}}^P(1) = \sqrt{n} \tilde{e}_{2k}$, where the Jones projection $\tilde{e}_{2k} \in M_{I^k}(M)$ is given, as in Corollary 3.3, by

$$(\tilde{e}_{2k})_{\underline{i}, \underline{j}} = n^{-1} \delta_{\underline{i}_{[2]} \underline{j}_1} g_{i_1} g_{j_1}^{-1}, \forall \underline{i}, \underline{j} \in I^k.$$

Thus

$$\begin{aligned}
 \tilde{e}_{2k} &= \frac{1}{n} \sum_{(\underline{i}, \underline{j}) \in W_k} [\underline{i}, \underline{j}]^{ev} \\
 &= \frac{1}{n} \sum_{\substack{h \in H, \\ (\underline{i}, \underline{j}) \in H \setminus W_k}} \frac{1}{|\text{Iso}_H(\underline{i}, \underline{j})|} h[\underline{i}, \underline{j}]^{ev} \\
 &\xrightarrow{\varphi_{2k+1}} \frac{1}{n} \sum_{\substack{g \in G, \\ (\underline{i}, \underline{j}) \in H \setminus W_k}} \frac{1}{|\text{Iso}_H(\underline{i}, \underline{j})|} g \left(\begin{array}{c} Hg_{i_k}, \dots, H \cap g_{\underline{i}} \\ H \\ Hg_{j_k}, \dots, H \cap g_{\underline{j}} \end{array} \right) \\
 &= \frac{1}{n} \sum_{\substack{g \in G, \\ (\underline{i}, \underline{j}) \in H \setminus W_k}} \frac{1}{\left| \text{Iso}_G \left(\begin{array}{c} Hg_{i_k}, \dots, H \cap g_{\underline{i}} \\ H \\ Hg_{j_k}, \dots, H \cap g_{\underline{j}} \end{array} \right) \right|} \\
 &\quad \times g \left(\begin{array}{c} Hg_{i_k}, \dots, H \cap g_{\underline{i}} \\ H \\ Hg_{j_k}, \dots, H \cap g_{\underline{j}} \end{array} \right) \\
 &= \frac{1}{n} \sum_{x, y, z \in I, \underline{i} \in I^{k-1}} \left(\begin{array}{c} Hg_{i_1}, \dots, Hg_{i_{k-1}}, Hg_y \\ Hg_x \\ Hg_{i_1}, \dots, Hg_{i_{k-1}}, Hg_z \end{array} \right),
 \end{aligned}$$

where we have used the fact that

$$\text{Iso}_H((\underline{i}, \underline{j})) = \text{Iso}_G \left(\begin{array}{c} Hg_{i_k}, \dots, H \cap g_{\underline{i}} \\ H \\ Hg_{j_k}, \dots, H \cap g_{\underline{j}} \end{array} \right).$$

This shows that $Z_{\mathcal{E}^{2k+1}}^Q = \varphi_{2k+1} \circ Z_{\mathcal{E}^{2k+1}}^P$.

The case for odd k follows on similar lines by taking analogues of W_k , F_{2k+1} and G_{2k+1} to be the sets Z_k , F_{2k} , G_{2k} , respectively, which are given by

$$\begin{aligned}
 Z_k &= \{(\underline{i}, \underline{j}) \in I^k \times I^k : \underline{i} = \underline{j}, i_1 = 1\}, \\
 F_{2k} &= \left\{ \left(\begin{array}{c} Hg_{i_1}, \dots, Hg_{i_{k-1}} \\ Hg_x \\ Hg_{i_1}, \dots, Hg_{i_{k-1}} \end{array} \right) : x, y \in I, \underline{i} \in I^{k-1} \text{ with } y = i_{k-1} \right\}
 \end{aligned}$$

and

$$G_{2k} = \left\{ \left(\begin{array}{c} Hg_{i_1}, \dots, Hg_{i_{k-1}} \\ H \\ Hg_{i_1}, \dots, Hg_{i_{k-1}} \end{array} \right) : y \in I, \underline{i} \in I^{k-1} \text{ with } y = i_{k-1} \right\}.$$

□

Finally, all the calculative work being done, we collect some of the lemmas proved above to give a complete proof of the main theorem.

Theorem 5.12. *Given a finite group G , a subgroup H of index, say n , and an outer action α of G on the hyperfinite II_1 -factor R , the planar algebra of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ is isomorphic to the G -invariant planar subalgebra of $P(\star_n)$, i.e.,*

$$P^{R \rtimes H \subset R \rtimes G} \cong P(\star_n)^G.$$

Proof. Let $\varphi_k, k \in \text{Col}$ be the maps defined as in eqs (33)–(36). We claim that

$$\{\varphi_k: k \in \text{Col}\} =: \varphi: P^{R \rtimes H \subset R \rtimes G} \rightarrow P(\star_n)^G$$

is a planar algebra isomorphism.

We already know, by Lemma 5.5, that the maps $\varphi_k, k \in \text{Col}$ are all linear isomorphisms. Thus what remains to be shown is that φ is a planar algebra morphism.

Let \mathcal{T} be the collection of coloured tangles T which commute with φ . Then, by Theorem 4.1, it is enough to show that \mathcal{T} is closed under composition of tangles, whenever it makes sense, and that it contains the generating set of tangles $\mathcal{G}_0 = \{1^{0\pm}\} \cup \{E_{k+1}^k, M_k, I_k^{k+1}: k \in \text{Col}\} \cup \{\mathcal{E}^{k+1}, (E')_k^k: k \geq 1\}$.

The first assertion follows from Lemma 4.3. Then, it readily follows from definitions that $\varphi_{0\pm} \circ Z_{1^{0\pm}}^P = Z_{1^{0\pm}}^{P(\star_n)^G}$. Thus $\{1^{0+}, 1^{0-}\} \subset \mathcal{T}$. Finally, Lemmas 5.8–5.11 show that the collection $\{I_k^{k+1}, M_k: k \in \text{Col}\} \cup \{E_{k+1}^k, (E')_{k+1}^{k+1}: k \in \text{Col}\} \cup \{\mathcal{E}^{k+1}: k \geq 1\}$ is also contained in \mathcal{T} . Thus $\mathcal{G}_0 \subset \mathcal{T}$.

This completes the proof. □

What follows immediately is the following fact, which, however, already exists in literature.

COROLLARY 5.13

Given a finite group G and a subgroup H , the subgroup-subfactor $R \rtimes_{\alpha/H} H \subset R \rtimes_{\alpha} G$ does not depend upon the outer action α of G on the hyperfinite II_1 -factor R .

Proof. It follows from Theorem 5.12 that the planar algebra of the subgroup-subfactor $R \rtimes_{\alpha/H} H \subset R \rtimes_{\alpha} G$ is independent of the outer action α . In particular, it follows that the standard invariant of the subfactor is independent of α . $R \rtimes_{\alpha} G$ being hyperfinite, the standard invariant of $R \rtimes_{\alpha/H} H \subset R \rtimes_{\alpha} G$ is a complete invariant (c.f. [11]). Thus the subgroup-subfactor $R \rtimes_{\alpha/H} H \subset R \rtimes_{\alpha} G$ is independent of the outer action α . □

By repeated applications of the ‘Not Burnside’s lemma’, the coefficients in the Poincaré series of the subgroup-subfactor are given by:

COROLLARY 5.14

For each $k \geq 1$,

$$\dim P_k^{R \rtimes H \subset R \rtimes G} = \frac{1}{|G|} \sum_{C \in \mathcal{C}_G} |C| \left(\frac{|C \cap H| |G|}{|C| |H|} \right)^k, \tag{41}$$

where \mathcal{C}_G is the set of conjugacy classes of G .

As promised above, we note that since the automorphism group of the bipartite graph \star_n is S_n , we have the following:

COROLLARY 5.15

Given any pair of finite groups $H \subset G$ with index n , the planar algebra $P^{R \rtimes H \subset R \rtimes G}$ is a planar subalgebra of the planar algebra of the bipartite graph \star_n and contains the planar algebra $P(\star_n)^{S_n} \cong P^{R \rtimes S_{n-1} \subset R \rtimes S_n}$.

We can also identify the planar algebra of the fixed subalgebra with the G -invariant planar subalgebra of the planar algebra of the flip of \star_n .

COROLLARY 5.16

Given a finite group G , a subgroup H of index, say n , and an outer action α of G on the hyperfinite II_1 -factor R , the planar algebra $P^{R^G \subset R^H}$ is isomorphic to the G -invariant planar subalgebra of $P(\overline{\star}_n)$, i.e.,

$$P^{R^G \subset R^H} \cong P(\overline{\star}_n)^G,$$

where $\overline{\star}_n$ is the flip of the bipartite graph \star_n . In particular, the subfactor $R^G \subset R^H$ is independent of the outer action α .

Proof. Recall, from Proposition 3.1, that if $N := R \rtimes_{\alpha/H} H$ and $M := R \rtimes_{\alpha} G$, then $P^{R^G \subset R^H} \cong P^{M \subset M_1}$, where M_1 is the II_1 -factor obtained by the basic construction of the subfactor $N \subset M$. Further, by Proposition 4.17 of [8], $P^{M \subset M_1}$ is isomorphic to the dual planar algebra $-P^{N \subset M}$. Thus, by Theorems 4.12 and 5.12, we conclude that

$$P^{R^G \subset R^H} \cong P(\overline{\star}_n)^G.$$

This shows that the planar algebra $P^{R^G \subset R^H}$ is independent of the outer action α . Further, R^H , being a II_1 -factor sitting in R , is itself hyperfinite – see [2]. Thus, as in the preceding Corollary, the subfactor $R^G \subset R^H$ is independent of the action α .

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