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Harmonic Riemannian maps on locally conformal Kaehler manifolds

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Abstract. We study harmonic Riemannian maps on locally conformal Kaehler manifolds (lcK manifolds). We show that if a Riemannian holomorphic map between lcK manifolds is harmonic, then the Lee vector field of the domain belongs to the kernel of the Riemannian map under a condition. When the domain is Kaehler, we prove that a Riemannian holomorphic map is harmonic if and only if the lcK manifold is Kaehler. Then we find similar results for Riemannian maps between lcK manifolds and Sasakian manifolds. Finally, we check the constancy of some maps between almost complex (or almost contact) manifolds and almost product manifolds.

Keywords. Kaehler manifold; Sasakian manifold; locally conformal Kaehler manifold; harmonic map, Riemannian map; holomorphic map.

1. Introduction

A map between Riemannian manifolds is called harmonic if the divergence of its differential vanishes. The theory of harmonic maps between Riemannian manifolds has been considered extensively. For Riemannian manifolds with a differential structure, it is known that a smooth holomorphic map between Kaehler manifolds is harmonic [4]. This result has been extended to the other Riemannian manifolds. For example, in [8], it was shown that any $\mp\phi$ holomorphic map between contact metric manifolds is harmonic. In [7], it was shown that any CR-map between two strongly pseudoconvex CR-manifolds is a harmonic map. In the same paper, the authors proved that any (ϕ, J)-holomorphic map between a strongly pseudo-convex CR-manifold of dimension $2m+1$ and a Kaehler manifold is also harmonic.

On the other hand, Riemannian maps were introduced by Fischer in [5] as a generalization of Riemannian submersions and immersions. Later, these maps are widely studied in differential geometry [6]. Note that a Riemannian map may not be a harmonic map and vice versa.

In this paper, we consider locally conformal Kaehler manifolds and investigate harmonic maps on such manifolds. Let (M, J, g) and $(\bar{M}, \bar{J}, \bar{g})$ be lcK manifolds and $\varphi\colon M \longrightarrow \bar{M}$ a Riemannian holomorphic map between them. We first prove that if the Lee vector field of the codomain belongs to $(\mathrm{Im}\, \mathrm{d}\varphi)^{\perp}$ and φ is harmonic, then the Lee vector field of the domain belongs to $\mathrm{Ker}\, \mathrm{d}\varphi$. We also show that if the Lee vector field of the domain belongs to $\mathrm{Ker}\, \mathrm{d}\varphi$, then the tension field of φ can be expressed by the Lee vector field of the codomain. If the codomain is Kaehler, we obtain Ianus, Ornea and Vuletescu's result [9]. Moreover, we obtain a necessary and sufficient condition for a Riemannian holomorphic map between Kaehler manifold and lcK manifold to be harmonic. In fact, we show that lcK manifold should be Kaehler. Then, we investigate harmonic maps between lcK manifolds and Sasakian manifolds under some conditions. Finally, we show that (J, F) (resp., (ϕ, J)-) maps between almost complex manifolds (resp. Sasakian manifolds) and almost product manifolds are constant.

2. Preliminaries

In this section, we give a brief information for Sasakian, lcK, almost product manifolds and harmonic maps. We note that throughout this paper, all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞. A map is always a C^∞ map between manifolds.

2.1 *Sasakian manifolds*

An odd dimensional Riemannian manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [2] if there is a $(1, 1)$ tensor field ϕ, a vector field ξ, called the characteristic vector field and its 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \qquad \bar{g}(\xi, \xi) = 1,$$

$$\phi^2(X) = -X + \eta(X)\xi, \qquad \bar{g}(X, \xi) = \eta(X),$$

$$\mathrm{d}\eta(X, Y) = \bar{g}(X, \phi Y), \qquad \forall X, Y \in \Gamma(TM).$$

It follows that $\phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1$. Then $(\phi, \xi, \eta, \bar{g})$ is called contact metric structure of $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_\phi + 2\mathrm{d}\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [11]. A normal contact metric manifold is called a Sasakian manifold for which we have

$$\bar{\nabla}_X \xi = -\phi X, \tag{2.1}$$

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)\xi - \eta(Y)X, \tag{2.2}$$

where $\bar{\nabla}$ is the Levi–Civita connection of $\bar{M}$.

2.2 *Locally conformal Kaehler manifolds*

Let (N, J, h) be a complex m-dimensional Hermitian manifold, where J denotes its complex structure and h its Hermitian metric. (N, J, h) is called locally conformal Kaehler (lcK) manifold if there is an open cover $\{U_i\}_{i\in I}$ of N and family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i\colon U_i \to R$ so that each local metric

$$h_i = \exp(-f_i)h|_{U_i}$$

is Kaehlerian (for lcK manifolds, see [10] and [3]). Here $h|_{U_i} = \iota_i^*(h)$ where $\iota\colon U_i \to N$ is the inclusion. It is well-known that a Hermitian manifold (N, J, h) is a lcK manifold if and only if

$$(\nabla_X J)Y = \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - h(X, Y)\mathcal{A} - h(X, JY)\mathcal{B}\}, \tag{2.3}$$

for any vector fields X, Y on M (page 4 of [3]), where ∇ is the Levi–Civita connection of N, θ and ω are 1-forms, $\mathcal{B} = \omega^\sharp$, here $\sharp$ denotes the raising of indices with respect to h, i.e., $h(X, \omega^\sharp) = \omega(X)$. Moreover $\theta = \omega o J$ and $\mathcal{A} = -J\mathcal{B}$. We note that $\mathcal{B}$ is called the Lee vector field.

2.3 *Almost product manifolds*

Let M be an n-dimensional manifold with a tensor of type (1,1) such that

$$F^2 = I, \tag{2.4}$$

where I denotes the identity transformation. Then we say that M is an almost product manifold with almost product structure F. We put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F). \tag{2.5}$$

Then we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \tag{2.6}$$

and

$$F = P - Q. \tag{2.7}$$

If an almost product manifold M admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y)$$

for any vector fields X and Y on M, then M is called an almost product Riemannian manifold [11].

2.4 *Harmonic maps*

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi \colon M \longrightarrow N$ is a smooth mapping between them. Then the differential $\mathrm{d}\varphi$ of φ can be viewed a section of the bundle $\mathrm{Hom}(TM, \varphi^{-1}TN) \longrightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $\mathrm{Hom}(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi–Civita connection $\overset{M}{\nabla}$ and the pullback connection. Then the second fundamental form of φ is given by

$$\nabla \mathrm{d}\varphi(X, Y) = \overset{\varphi}{\nabla}_X \mathrm{d}\varphi(Y) - \mathrm{d}\varphi(\overset{M}{\nabla}_X Y) \tag{2.8}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi \colon (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if trace $\nabla \mathrm{d}\varphi = 0$. The tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \operatorname{div} \mathrm{d}\varphi = \sum_{i=1}^{m} \nabla \mathrm{d}\varphi(e_i, e_i), \tag{2.9}$$

where $e_1, \ldots, e_m$ is a local orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$. For more information, see [1].

Let (M, g_M) and (N, g_N) be Riemannian manifolds of dimensions m and n, respectively. Let $f \colon M \longrightarrow N$ be a smooth map and $\mathrm{d}f_p \colon T_pM \longrightarrow T_{f(p)}N$ be the tangent map of f at $p \in M$. Then f is called Riemannian at p if $\mathrm{d}f_p \colon (\mathrm{Ker}\, \mathrm{d}f_p)^\perp \longrightarrow (\mathrm{Im}\, \mathrm{d}f_p)$ is isometric. A smooth map $f \colon M \longrightarrow N$ is called Riemannian if f is Riemannian at each $p \in M$ [5].

3. Harmonic maps on locally conformal Kaehler manifolds

Let (M, J) and (N, J') be almost complex manifolds. Then a map $\varphi\colon M \longrightarrow N$ is called a holomorphic map if $\mathrm{d}\varphi o J = J' o \mathrm{d}\varphi$, where $\mathrm{d}\varphi$ is the tangent map or the differential of φ.

Lemma 3.1. *Let (M, J, g) and (N, J', g') be lcK manifolds. If $\varphi\colon M \longrightarrow N$ is a Riemannian holomorphic map, then* $\mathrm{Ker}\,\mathrm{d}\varphi$ *and* $(\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp}$ *are invariant with respect to the almost complex structure of M.*

Proof. Since φ is holomorphic, for $X \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)$, we have $\mathrm{d}\varphi(JX) = J'\mathrm{d}\varphi(X) = 0$, which shows that $JX \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)$. In a similar way, for $X \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)$, $Y \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp}$, we get $g(JY, X) = -g(Y, JX) = 0$ due to $JX \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)$. Hence, $JY \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp}$.

Theorem 3.1. *Let (M, J, g) and $(\bar{M}, \bar{J}, \bar{g})$ be lcK manifolds such that* $\dim M \neq 2$. *Suppose that $\varphi\colon M \longrightarrow \bar{M}$ is a Riemannian holomorphic map such that the Lee vector field $\bar{\mathcal{B}}$ of $\bar{M}$ belongs to* $(\mathrm{Im}\,\mathrm{d}\varphi)^{\perp}$. *If φ is harmonic, then the Lee vector field $\mathcal{B}$ of M belongs to* $\mathrm{Ker}\,\mathrm{d}\varphi$. *Conversely, if the Lee vector field $\mathcal{B}$ of M belongs to* $\mathrm{Ker}\,\mathrm{d}\varphi$, *then $\tau(\varphi) = -\frac{n_2}{2}\bar{\mathcal{B}}$, where $n_2 = \dim(\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp}$.*

Proof. From (2.3) and (2.8) we have

$$\begin{aligned}\nabla\mathrm{d}\varphi(X, JY) = \bar{J}\nabla\mathrm{d}\varphi(X, Y) + \frac{1}{2}\{&\bar{\theta}(\mathrm{d}\varphi(Y))\mathrm{d}\varphi(X) - \theta(Y)\mathrm{d}\varphi(X)\\ &- \bar{\omega}(\mathrm{d}\varphi(Y))\bar{J}\mathrm{d}\varphi(X) + \omega(Y)\bar{J}\mathrm{d}\varphi(X) - \bar{g}(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\bar{\mathcal{A}}\\ &+ g(X, Y)\mathrm{d}\varphi(\mathcal{A}) - \bar{\Omega}(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\bar{\mathcal{B}} + \Omega(X, Y)\mathrm{d}\varphi(\mathcal{B})\},\end{aligned}$$

where $\bar{\omega}$ and $\bar{\mathcal{B}}$ are the Lee form and the Lee vector field on $\bar{M}$, respectively. Since $\nabla\mathrm{d}\varphi$ is symmetric, we obtain

$$\begin{aligned}\nabla\mathrm{d}\varphi(X, JY) - \nabla\mathrm{d}\varphi(Y, JX) = \frac{1}{2}\{&\bar{\theta}(\mathrm{d}\varphi(Y))\mathrm{d}\varphi(X) - \bar{\theta}(\mathrm{d}\varphi(X))\mathrm{d}\varphi(Y)\\ &- \theta(Y)\mathrm{d}\varphi(X) + \theta(X)\mathrm{d}\varphi(Y) - \bar{\omega}(\mathrm{d}\varphi(Y))\bar{J}\mathrm{d}\varphi(X)\\ &+ \bar{\omega}(\mathrm{d}\varphi(X))\bar{J}\mathrm{d}\varphi(Y) + \omega(Y)\bar{J}\mathrm{d}\varphi(X) - \omega(X)\bar{J}\mathrm{d}\varphi(Y)\\ &- 2\bar{\Omega}(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\bar{\mathcal{B}} + 2\Omega(X, Y)\mathrm{d}\varphi(\mathcal{B})\}.\end{aligned}$$

Then, for $X = JY$, we derive

$$\begin{aligned}\nabla\mathrm{d}\varphi(JY, JY) + \nabla\mathrm{d}\varphi(Y, Y) = {}&(\bar{\theta}(\mathrm{d}\varphi(Y)) - \theta(Y))\mathrm{d}\varphi(JY)\\ &+ (\bar{\omega}(\mathrm{d}\varphi(Y)) - \omega(Y))\mathrm{d}\varphi(Y)\\ &- \bar{g}(\mathrm{d}\varphi(Y), \mathrm{d}\varphi(Y))\bar{\mathcal{B}} + g(Y, Y)\mathrm{d}\varphi(\mathcal{B}).\end{aligned}$$

Since $\bar{\mathcal{B}}$ belongs to $(\mathrm{Im}\,\mathrm{d}\varphi)^{\perp}$, we arrive at

$$\begin{aligned}\nabla\mathrm{d}\varphi(JY, JY) + \nabla\mathrm{d}\varphi(Y, Y) = {}&-\theta(Y)\mathrm{d}\varphi(JY) - \omega(Y)\mathrm{d}\varphi(Y)\\ &- \bar{g}(\mathrm{d}\varphi(Y), \mathrm{d}\varphi(Y))\bar{\mathcal{B}} + g(Y, Y)\mathrm{d}\varphi(\mathcal{B}). \qquad (3.1)\end{aligned}$$

Now, we choose a local orthonormal frame

$$\{e_1, \ldots, e_p, J(e_1), \ldots, J(e_p),$$
$$e'_1, \ldots, e'_q, J(e'_1), \ldots, J(e'_q)\},\ 2p + 2q = n_1 + n_2 = n$$

in such a way that $\{e_1, \ldots, e_p, J(e_1), \ldots, J(e_p)\}$ form a local orthonormal frame of $\mathrm{Ker}\, \mathrm{d}\varphi$ and $\{e'_1, \ldots, e'_q, J(e'_1), \ldots, J(e'_q)\}$ form a local orthonormal frame of $(\mathrm{Ker}\, \mathrm{d}\varphi)^{\perp}$. Then, using (3.1), we get

$$\tau(\varphi) = \frac{n}{2}\mathrm{d}\varphi(\mathcal{B}) + \sum_{j=1}^{q}\{-\theta(e'_j)\mathrm{d}\varphi(Je'_j) - \omega(e'_j)\mathrm{d}\varphi(e'_j)\} - \frac{n_2}{2}\bar{\mathcal{B}}. \tag{3.2}$$

On the other hand, we have

$$\sum_{j=1}^{q}\{-\theta(e'_j)\mathrm{d}\varphi(Je'_j) - \omega(e'_j)\mathrm{d}\varphi(e'_j)\}$$
$$= \sum_{j=1}^{q} -g(Je'_j, \mathcal{B})\mathrm{d}\varphi(Je'_j) - g(e'_j, \mathcal{B})\mathrm{d}\varphi(e'_j).$$

Then, Riemannian holomorphic map φ implies that

$$\sum_{j=1}^{q}\{-\theta(e'_j)\mathrm{d}\varphi(Je'_j) - \omega(e'_j)\mathrm{d}\varphi(e'_j)\} = \sum_{j=1}^{q} -\bar{g}(\bar{J}\mathrm{d}\varphi(e'_j), \mathrm{d}\varphi(\mathcal{B}'))\bar{J}\mathrm{d}\varphi(e'_j)$$
$$- \bar{g}(\mathrm{d}\varphi(e'_j), \mathrm{d}\varphi(\mathcal{B}'))\mathrm{d}\varphi(e'_j),$$

where $\mathcal{B}'$ is the component of $\mathcal{B}$ along $(\mathrm{Ker}\, \mathrm{d}\varphi)^{\perp}$. Furthermore, Riemannian map φ implies that $\{\mathrm{d}\varphi(e'_1), \ldots, \mathrm{d}\varphi(e'_q), \bar{J}\mathrm{d}\varphi(e'_1), \ldots, \bar{J}\mathrm{d}\varphi(e'_q)\}$ is a local orthonormal frame for $\mathrm{Im}\, \mathrm{d}\varphi$. Thus, we obtain

$$\sum_{j=1}^{q}\{-\theta(e'_j)\mathrm{d}\varphi(Je'_j) - \omega(e'_j)\mathrm{d}\varphi(e'_j)\} = -\mathrm{d}\varphi(\mathcal{B}').$$

Then, (3.2) becomes

$$\tau(\varphi) = \frac{n-2}{2}\mathrm{d}\varphi(\mathcal{B}) - \frac{n_2}{2}\bar{\mathcal{B}},$$

where $\mathrm{d}\varphi(\mathcal{B})$ and $\bar{\mathcal{B}}$ are $\bar{g}$-orthogonal. Now the statement follows.

From Theorem 3.1, we have the following result.

COROLLARY 3.1

Let (M, J) be a lcK manifold and (N, J') be a Kaehler manifold such that $\dim M \neq 2$. *Suppose that $\varphi\colon M \longrightarrow N$ is a holomorphic map. Then φ is harmonic if and only if the Lee vector field belongs to the kernel of* $\mathrm{d}\varphi$.

Remark 1. We note that Corollary 3.1 was proved, in a different way, by Ianus, Ornea and Vuletescu in [9].

Theorem 3.2. *Let* (M, J, g) *be a Kaehler manifold and* (N, J') *be a lcK manifold. Suppose that* $\varphi\colon M \longrightarrow N$ *is a Riemannian holomorphic map such that the Lee vector field of N belongs to* $(\mathrm{Im}\,\mathrm{d}\varphi)^{\perp}$. *Then* φ *is harmonic if and only if N is a Kaehler manifold.*

Proof. Following the proof of Theorem 3.1, for $X \in \Gamma(TM)$, we obtain

$$\nabla \mathrm{d}\varphi(X, X) + \nabla \mathrm{d}\varphi(JX, JX) = -g_N(\mathrm{d}\varphi(X), \mathrm{d}\varphi(X))\mathcal{B},$$

where $\mathcal{B}$ and g_N are the Lee vector field and the Riemannian metric of N, respectively. Since φ is a Riemannian map, we get

$$\tau(\varphi) = -\sum_{i=1}^{q} g(e'_i, e'_i)\mathcal{B}.$$

Hence, we have

$$\tau(\varphi) = -\frac{n_2}{2}\mathcal{B},$$

which proves the theorem.

Lemma 3.2. *Let* (M, J, g_M) *be a lcK manifold and* $(N, \phi, \eta, \xi, g_N)$ *be a Sasakian manifold. Suppose that* φ *is a smooth Riemannian* (J, ϕ)*-map. Then* ξ *does not belong to* $(\mathrm{Im}\,\mathrm{d}\varphi)$.

Proof. First note that if φ is a Riemannian (J, ϕ) map, then $\mathrm{Ker}\,\mathrm{d}\varphi$ and $(\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp}$ are invariant with respect to J. Indeed, assume that $X \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)$, then $\mathrm{d}\varphi(JX) = \phi\mathrm{d}\varphi(X) = \phi(0) = 0$, hence $JX \in \Gamma(\mathrm{Ker}\,\mathrm{d}\varphi)$. In a similar way, one can show that $(\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp}$ is also invariant. For $Y \in \Gamma((\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp})$, suppose that $\mathrm{d}\varphi(Y) = \xi$. Then applying ϕ to this equation and taking into account that φ is (J, ϕ)-map, we get $\mathrm{d}\varphi(JY) = 0$. Hence, it follows that JY belongs to $\mathrm{Ker}\,\mathrm{d}\varphi$ while $Y \in \Gamma((\mathrm{Ker}\,\mathrm{d}\varphi)^{\perp})$, which is not possible. Hence we conclude that ξ does not belong to $(\mathrm{Im}\,\mathrm{d}\varphi)$.

Theorem 3.3. *Let* (M, J, g) *be a lcK manifold and* $(N, \phi, \eta, \xi, g_N)$ *be a Sasakian manifold such that* $\dim M \neq 2$. *Suppose that* $\varphi\colon M \longrightarrow N$ *is a Riemannian* (J, ϕ)*-map such that* $\xi \in \Gamma((\mathrm{Im}\,\mathrm{d}\varphi)^{\perp})$. *Then* φ *is harmonic if and only if the Lee vector field belongs to the kernel of* $\mathrm{d}\varphi$.

Proof. For $X, Y \in \Gamma(TM)$, from (2.3) we have

$$\nabla \mathrm{d}\varphi(X, JY) = \nabla^{\varphi}_X \phi \mathrm{d}\varphi(Y) - \phi \mathrm{d}\varphi(\nabla_X Y) - \frac{1}{2}\{\theta(Y)\mathrm{d}\varphi(X) - \omega(Y)\mathrm{d}\varphi(JX) - g(X, Y)\mathrm{d}\varphi(\mathcal{A}) - \Omega(X, Y)\mathrm{d}\varphi(\mathcal{B})\},$$

where ∇ is the Levi–Civita connection on M. Then, using (2.2) and (2.8) we obtain

$$\nabla \mathrm{d}\varphi(X, JY) = \phi\nabla \mathrm{d}\varphi(X, Y) + g_N(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\xi - \eta(\mathrm{d}\varphi(Y))\mathrm{d}\varphi(X) - \frac{1}{2}\{\theta(Y)\mathrm{d}\varphi(X) - \omega(Y)\mathrm{d}\varphi(JX) - g(X, Y)\mathrm{d}\varphi(\mathcal{A}) - \Omega(X, Y)\mathrm{d}\varphi(\mathcal{B})\}.$$

Since ξ belongs to $(\mathrm{Im}\, \mathrm{d}\varphi)^{\perp}$, $\eta(\mathrm{d}\varphi(Y))\mathrm{d}\varphi(X) = 0$. Hence

$$\nabla \mathrm{d}\varphi(X, JY) = \phi \nabla \mathrm{d}\varphi(X, Y) + g_N(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\xi - \frac{1}{2}\{\theta(Y)\mathrm{d}\varphi(X) - \omega(Y)\mathrm{d}\varphi(JX) - g(X, Y)\mathrm{d}\varphi(\mathcal{A}) - \Omega(X, Y)\mathrm{d}\varphi(\mathcal{B})\}.$$

Then symmetric $\nabla \mathrm{d}\varphi$ implies that

$$\nabla \mathrm{d}\varphi(X, JY) - \nabla \mathrm{d}\varphi(JX, Y) = -\frac{1}{2}\{\theta(Y)\mathrm{d}\varphi(X) - \theta(X)\mathrm{d}\varphi(Y) - \omega(Y)\mathrm{d}\varphi(JX) + \omega(X)\mathrm{d}\varphi(JY) - 2\Omega(X, Y)\mathrm{d}\varphi(\mathcal{B})\}.$$

Hence, we obtain

$$\nabla \mathrm{d}\varphi(JX, JX) + \nabla \mathrm{d}\varphi(X, X) = -\theta(X)\mathrm{d}\varphi(JX) - \omega(X)\mathrm{d}\varphi(X) + g(X, X)\mathrm{d}\varphi(\mathcal{B}).$$

Thus, choosing a local orthonormal frame adapted to the decomposition $\mathrm{Ker}\, \mathrm{d}\varphi \oplus (\mathrm{Ker}\, \mathrm{d}\varphi)^{\perp}$, we get

$$\tau(\varphi) = \frac{n}{2}\mathrm{d}\varphi(\mathcal{B}) + \sum_{i=1}^{q} -\theta(e'_i)\mathrm{d}\varphi(Je'_i) - \omega(e'_i)\mathrm{d}\varphi(e'_i), \tag{3.3}$$

where $\{e'_1, \ldots e'_q, J(e'_1), \ldots, J(e'_q)\}$ is a local orthonormal frame for $(\mathrm{Ker}\, \mathrm{d}\varphi)^{\perp}$. Then we have

$$\sum_{i=1}^{q} -\theta(e'_i)\mathrm{d}\varphi(Je'_i) - \omega(e'_i)\mathrm{d}\varphi(e'_i) = \sum_{i=1}^{q} -g(Je'_i, \mathcal{B})\mathrm{d}\varphi(Je'_i) - g(e'_i, \mathcal{B})\mathrm{d}\varphi(e'_i).$$

On the other hand, Riemannian (J, ϕ)-map implies that

$$\sum_{i=1}^{q} -\theta(e'_i)\mathrm{d}\varphi(Je'_i) - \omega(e'_i)\mathrm{d}\varphi(e'_i) = \sum_{i=1}^{q} -g_N(\phi\mathrm{d}\varphi(e'_i), \mathrm{d}\varphi(\mathcal{B}')\phi\mathrm{d}\varphi(e'_i) - g_N(\mathrm{d}\varphi(e'_i), \mathrm{d}\varphi(\mathcal{B}'))\mathrm{d}\varphi(e'_i),$$

where $\mathcal{B}'$ is the component of $\mathcal{B}$ along $(\mathrm{Ker}\, \mathrm{d}\varphi)^{\perp}$. Hence, we get

$$\sum_{i=1}^{q} -\theta(e'_i)\mathrm{d}\varphi(Je'_i) - \omega(e'_i)\mathrm{d}\varphi(e'_i) = -\mathrm{d}\varphi(\mathcal{B}').$$

Then, (3.3) becomes

$$\tau(\varphi) = \frac{n-2}{2}\mathrm{d}\varphi(\mathcal{B}),$$

which shows that φ is harmonic if and only if $\mathrm{d}\varphi(\mathcal{B}) = 0$.

Lemma 3.3. *Let $(M, \phi, \eta, \xi, g_M)$ be a Sasakian manifold and (N, J, g_N) be a lcK manifold. Suppose that $\varphi\colon M \longrightarrow N$ is a Riemannian (ϕ, J)-map. Then*

$$\xi \in \Gamma(\mathrm{Ker}\ \mathrm{d}\varphi) \tag{3.4}$$

and

$$\nabla \mathrm{d}\varphi(X, \xi) = J\mathrm{d}\varphi(X) \tag{3.5}$$

for $X \in \Gamma(TM)$.

Proof. Since φ is a (ϕ, J)-map, we have $0 = \mathrm{d}\varphi(\phi\xi) = J\mathrm{d}\varphi(\xi)$. Then non-singular J implies that $\mathrm{d}\varphi(\xi) = 0$. On the other hand, from (2.8) and (3.4), we have $\nabla\mathrm{d}\varphi(X, \xi) = -\mathrm{d}\varphi(\nabla_X \xi)$ for $X \in \Gamma(TM)$. Using (2.1) we obtain $\nabla\mathrm{d}\varphi(X, \xi) = \mathrm{d}\varphi(\phi X)$. Then (ϕ, J)-map implies that $\nabla\mathrm{d}\varphi(X, \xi) = J\mathrm{d}\varphi(X)$.

Theorem 3.4. *Let $(M^{2n+1}, \phi, \eta, \xi, g_M)$ be a Sasakian manifold and (N, J, g_N) be a lcK manifold. Suppose that $\varphi\colon M \longrightarrow N$ is a Riemannian (ϕ, J)-map such that the Lee vector field $\mathcal{B}$ of N belongs to* $(\mathrm{Im}\ \mathrm{d}\varphi)^{\perp}$. *Then φ is harmonic if and only if N is a Kaehler manifold.*

Proof. We first note that a Sasakian (in fact, any almost contact manifold) has the following decomposition

$$TM = D \oplus \mathrm{span}\{\xi\},$$

where $D = \{X \in \Gamma(TM) | g_M(X, \xi) = 0\}$. It is known that the distribution D is an invariant distribution with respect to ϕ. By direct computations, using (2.1), (2.3) and (2.8), we get

$$\nabla\mathrm{d}\varphi(\phi X, \phi X) + \nabla\mathrm{d}\varphi(X, X) = \theta(\mathrm{d}\varphi(X))\mathrm{d}\varphi(\phi X) + \omega(\mathrm{d}\varphi(X))\mathrm{d}\varphi(X) - g_N(\mathrm{d}\varphi(X), \mathrm{d}\varphi(X))\mathcal{B} \tag{3.6}$$

for $X \in \Gamma(D)$. Then the tension field of φ is

$$\tau(\varphi) = \sum_{i=1}^{n} \nabla\mathrm{d}\varphi(\phi e_i, \phi e_i) + \nabla\mathrm{d}\varphi(e_i, e_i) + \nabla\mathrm{d}\varphi(\xi, \xi).$$

Using (3.6), (3.5) and (3.4) we have

$$\tau(\varphi) = \sum_{i=1}^{n} \theta(\mathrm{d}\varphi(e_i))\mathrm{d}\varphi(\phi e_i) + \omega(\mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) - g_N(\mathrm{d}\varphi(e_i), \mathrm{d}\varphi(e_i))\mathcal{B}.$$

Since Ker $\mathrm{d}\varphi$ and $(\mathrm{Ker}\ \mathrm{d}\varphi)^{\perp}$ are ϕ-invariant, we can choose an adapted local orthonormal frame

$$\{e_1, \ldots, e_p, \phi(e_1), \ldots, \phi(e_p), \xi\}$$

for Ker $\mathrm{d}\varphi$ and

$$\{e'_1, \ldots, e'_p, \phi(e'_1), \ldots, \phi(e'_p)\}$$

for $(\mathrm{Ker}\ \mathrm{d}\varphi)^{\perp}$. Then since $\mathcal{B} \in \Gamma(\mathrm{Im}\ \mathrm{d}\varphi)^{\perp}$ and φ is Riemannian, we obtain

$$\tau(\varphi) = -\frac{n_2}{2}\mathcal{B},$$

$n_2 = 2q$ being the dimension of $(\mathrm{Ker}\ \mathrm{d}\varphi)^{\perp}$. Thus the proof is complete.

Let M be an almost complex manifold (almost contact manifold) and N be an almost product manifold, then we say that $\varphi\colon M \longrightarrow N$ is a (J, F)- (resp., (ϕ, F)-map) if $F o \mathrm{d}\varphi = \mathrm{d}\varphi o J$ (resp., $F o \mathrm{d}\varphi = \mathrm{d}\varphi o \phi$). In fact, the following results show that such maps are constant.

PROPOSITION 3.1

Let (M, J) be an almost complex manifold and (N, F) be an almost product manifold. Then, every smooth (J, F)-map φ between M and N is constant.

Proof. For arbitrary $X \in \Gamma(TM)$, (J, F)-map implies that $F\mathrm{d}\varphi(X) = \mathrm{d}\varphi(JX)$. Applying F to this equation and using (2.4) we obtain $\mathrm{d}\varphi(X) = \mathrm{d}\varphi(J^2 X)$. Then almost complex structure implies that $J^2 = -I$. Hence, $\mathrm{d}\varphi(X) = -\mathrm{d}\varphi(X)$. That is, $\mathrm{d}\varphi(X) = 0$ which proves our assertion.

PROPOSITION 3.2

Let (M, ϕ, η, ξ) be an almost contact manifold and (N, F) be an almost product manifold. Then, every smooth (ϕ, F)-map φ between M and N is constant.

Proof. Using (2.4), (2.1) and the definition of (ϕ, F)-map, we get

$$2\mathrm{d}\varphi(X) = \eta(X)\mathrm{d}\varphi(\xi).$$

Now, for $X = \xi$ we have $\mathrm{d}\varphi(\xi) = 0$ and then $2\mathrm{d}\varphi(X) = 0$ for any X. Thus φ is a constant map.

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References

[1] Baird P and Wood J C, Harmonic morphisms between Riemannian manifolds (Oxford: Clarendon Press) (2003)
[2] Blair D E, Riemannian geometry of contact and symplectic manifolds, PM 203 (Birkhäuser) (2002)
[3] Dragomir S and Ornea L, Locally conformal Kähler geometry (Birkhäuser) (1998)
[4] Eells J and Sampson H, Harmonic mapping of Riemannian manifolds, *Am. J. Math.* **86(1)** (1964) 109–160
[5] Fischer A E, Riemannian maps between Riemannian manifolds, *Contemp. Math.* **132** (1992) 331–366
[6] Garcia-Rio E and Kupeli D N, Semi-Riemannian maps and their applications (Kluwer Acad. Publ.) (1999)
[7] Gherghe C, Ianus S and Pastore A M, CR-manifolds, harmonic maps and stability, *J. Geom.* **71** (2001) 42–53
[8] Ianus S and Pastore A M, Harmonic maps on contact metric manifolds, *Ann. Math. Blaise Pascal* **2(2)** (1995) 43–55
[9] Ianus S, Ornea L and Vuletescu V, Holomorphic and harmonic maps of locally conformal Kaehler manifolds, *Bollettino U. M. I. (7)* **9A** (1995) 569–579
[10] Vaisman I, On locally conformal almost Kaehler manifolds, *Israel J. Math.* **24** (1976) 15–19
[11] Yano K and Kon M, Structures on manifolds (Ser. Pure Math. World Scientific) (1984)