Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 4, November 2008, pp. 569–572.

Hypersurfaces in simply connected space forms

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MS received 6 July 2007

Abstract. Let M be a hypersurface in a simply connected space form $\mathbb{M}(\kappa)$. We prove some rigidity results for M in terms of lower bounds on the Ricci curvature of the hypersurface M.

Keywords. Hypersurface; Ricci curvature; space forms; totally umbilic; horosphere.

1. Statement of theorems

Let $(\mathbb{M}(\kappa), \mathrm{d}s^2)$ denote the simply connected space form of constant curvature $\kappa = 0, 1$ or -1 and dimension $n + 1 \geq 3$. Let M be a hypersurface of $(\mathbb{M}(\kappa), \mathrm{d}s^2)$. For every point $m \in M$, let A: $T_m M \to T_m M$ be the Weingarten map of the hypersurface.

In this paper we prove the following results.

Theorem 1. *Let M be a connected hypersurface in the simply connected space form $(\mathbb{M}(\kappa), \mathrm{d}s^2)$ such that the Ricci curvature of M satisfies the inequality* $\mathrm{Ric}_M(u, u) \geq (n-1)\{\kappa + \langle Au, u\rangle^2\}$ *for every unit vector $u \in TM$. Then M is totally umbilic.*

COROLLARY 1

Let M be as in Theorem 1. *If M is complete, then we have the following*:

1. *If $\kappa = 0$, then M is either a hyperplane or a geodesic sphere in $\mathbb{R}^n$.*
2. *If $\kappa = 1$, then M is a geodesic sphere in S^n.*
3. *If $\kappa = -1$, then M is either a totally geodesic hyperplane, a horosphere or a geodesic sphere in $\mathbb{H}^n$.*

Theorem 2. *Let M be a connected hypersurface in the simply connected space form $(\mathbb{M}(\kappa), \mathrm{d}s^2)$. Let* Scal *denote the scalar curvature of M. Then*

$$\mathrm{Scal}_m \leq \frac{n(n-1)}{\mathrm{vol}(S^{n-1})} \int_{U_m M} [\kappa + \langle Au, u\rangle^2]$$

for every point $m \in M$. Furthermore, the equality holds iff M is totally umbilic.

COROLLARY 2

Let M be a connected compact surface in $\mathbb{M}(\kappa)$ of dimension 3. *Then*

$$\chi(M) \leq \frac{1}{4\pi^2} \int_{UM} [\kappa + \langle Au, u\rangle^2] \mathrm{d}\mu,$$

where $\chi(M)$ denotes the Euler characteristic of M and $\mathrm{d}\mu$ is the Liouville measure on the unit tangent bundle of M.

Furthermore equality holds iff M is a geodesic sphere in $\mathbb{R}^3$.

We refer to [1] and [2] for the basic Riemannian geometry used in this paper.

2. Proofs of the results

We start with the following key lemma.

Lemma 1. *Let $(V, \langle\,,\,\rangle)$ be a real inner product space of dimension $n \geq 2$ and $A\colon V \to V$ be a self-adjoint linear map. Then*

$$\int_{S^{n-1}} \{2\| Au \|^2 - (n+2)\,\langle Au, u\rangle^2 + \mathrm{Tr}(A)\,\langle Au, u\rangle\} = 0, \tag{1}$$

where S^{n-1} is the unit sphere in V.

Proof. Let $f\colon S^{n-1} \to \mathbb{R}$ be the map defined by $f(u) := \langle Au, u\rangle^2$. Then f is a smooth function on S^{n-1}.

Let Δ denote the Laplacian of S^{n-1}. We will now compute $(\Delta f)(u)$ for every point $u \in S^{n-1}$.

Let $u \in S^{n-1}$ and $\{e_2, \ldots, e_n\}$ be an orthonormal basis of $T_u S^{n-1}$. Let $\gamma_i : [-\pi, \pi] \to S^{n-1}$ be the curves defined by $\gamma_i(t) := \cos\, tu + \sin\, te_i$ for $2 \leq i \leq n$. Then

$$\begin{aligned}
-(\Delta f)(u) &= \sum_{i=2}^{n} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} f(\gamma_i(t))\\
&= 4\sum_{i=2}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \{\langle A(\gamma_i(t)), \gamma_i(t)\rangle\, \langle A(\gamma_i'(t)), \gamma_i(t)\rangle\}\\
&= 4\sum_{i=2}^{n} \{2\,\langle Au, e_i\rangle^2 + \langle Au, u\rangle\,[\langle Ae_i, e_i\rangle - \langle Au, u\rangle]\}\\
&= 4\sum_{i=2}^{n} \{2\,\langle Au, e_i\rangle^2 - \langle Au, u\rangle^2 + \langle Au, u\rangle\,\langle Ae_i, e_i\rangle\}\\
&= 4\{2\| Au \|^2 - (n+2)\,\langle Au, u\rangle^2 + \mathrm{Tr}(A)\,\langle Au, u\rangle\}.
\end{aligned}$$

By divergence theorem $\int_{S^{n-1}} \Delta f(u) = 0$. Therefore

$$\int_{S^{n-1}} \{2\| Au \|^2 - (n+2)\,\langle Au, u\rangle^2 + \mathrm{Tr}(A)\,\langle Au, u\rangle\} = 0.$$

This completes the proof. □

2.1 *Proof of the results*

Proof of Theorem 1. We use Lemma 1 to prove Theorem 1.

Let M be a hypersurface of $\mathbb{M}(\kappa)$. For every point $m \in M$, let $A\colon T_mM \to T_mM$ be the Weingarten map defined by $Av := -\nabla_v N$ where N is a unit normal to M at the point m.

Let us recall that $K_M(X, Y) = K_{\mathbb{M}(\kappa)}(X, Y) + \langle AX, X\rangle \langle AY, Y\rangle - \langle AX, Y\rangle^2$ for every pair of orthonormal vectors $X, Y \in T_pM$ where K_M and $K_{\mathbb{M}(\kappa)}$ denotes the sectional curvature of M and $\mathbb{M}(\kappa)$ respectively. Let $u \in U_mM$ and $\{e_2, e_3, \ldots, e_n\}$ be a set of orthonormal vectors such that $\{u, e_2, e_3, \ldots, e_n\}$ is an orthonormal basis of T_mM. Then

$$\begin{aligned}\mathrm{Ric}_M(u, u) &= \sum_{i=1}^{n-1} K_M(u, e_i)\\ &= \sum_{i=1}^{n-1}\{K_{\mathbb{M}(\kappa)}(u, e_i) + \langle Au, u\rangle \langle Ae_i, e_i\rangle - \langle Au, e_i\rangle^2\}\\ &= (n-1)\kappa + \langle Au, u\rangle \sum_{i=1}^{n-1} \langle Ae_i, e_i\rangle - \sum_{i=1}^{n-1} \langle Au, e_i\rangle^2\\ &= (n-1)\kappa + \mathrm{Tr}(A)\,\langle Au, u\rangle - \| Au \|^2.\end{aligned}$$

Therefore

$$\mathrm{Tr}(A)\,\langle Au, u\rangle = \mathrm{Ric}_M(u, u) - (n-1)\kappa + \| Au \|^2.$$

We substitute this in eq. (1) and use Cauchy–Schwartz inequality $\langle Au, u\rangle^2 \leq \| Au \|^2$ for every $u \in U_mM$ to get

$$\begin{aligned}0 &= \int_{U_mM} \{\mathrm{Ric}_M(u, u) - (n-1)[\kappa + \langle Au, u\rangle^2] + 3(\| Au \|^2 - \langle Au, u\rangle^2)\}\\ &\geq \int_{U_mM} \{\mathrm{Ric}_M(u, u) - (n-1)[\kappa + \langle Au, u\rangle^2]\}. \qquad (2)\end{aligned}$$

Since $\mathrm{Ric}_M(u, u) - (n-1)[\kappa + \langle Au, u\rangle^2] \geq 0$ for every $u \in U_mM$, it follows that equality must hold in Cauchy–Schwartz inequality. That is $\langle Au, u\rangle^2 = \| Au \|^2$ for every $u \in U_mM$ and for every point $m \in M$. Therefore the Weingarten map $A\colon T_mM \to T_mM$ is diagonal for every point $m \in M$. Since M is connected, M is totally umbilic. □

Proof of Corollary 1. It is well-known that totally umbilic complete, connected hypersurfaces in the simply connected space forms are precisely the ones we have listed. For a proof of these results we refer to [1]. □

Proof of Theorem 2. From eq. (2) of Theorem 1 it follows that

$$\int_{U_mM} \mathrm{Ric}_M(u, u) \leq (n-1) \int_{U_mM} [\kappa + \langle Au, u\rangle^2]. \qquad (3)$$

It is known that (see [2] for instance), for every point m in M,

$$\int_{U_mM} \mathrm{Ric}_M(u, u) = \frac{\mathrm{vol}(S^{n-1})}{n}\mathrm{Scal}_m.$$

Therefore

$$\begin{aligned}\mathrm{Scal}_m &= \frac{n}{\mathrm{vol}(S^{n-1})}\int_{U_m M}\mathrm{Ric}_M(u,u)\\ &\le \frac{n(n-1)}{\mathrm{vol}(S^{n-1})}\int_{U_m M}[\kappa + \langle Au,u\rangle^2].\end{aligned}$$

Moreover equality holds iff M is totally umbilic. □

Proof of Corollary 2. When M is a surface in $\mathbb{R}^3$, the inequality

$$\mathrm{Scal}_m \le \frac{n(n-1)}{\mathrm{vol}(S^{n-1})}\int_{U_m M}[\kappa + \langle Au,u\rangle^2]$$

becomes

$$K(m) \le \frac{1}{2\pi}\int_{U_m M}[\kappa + \langle Au,u\rangle^2],$$

where K denotes the sectional curvature of the surface M.

We integrate this inequality over M and use Gauss–Bonnet theorem (see p. 358 of [3]) to get

$$\begin{aligned}2\pi\chi(M) &= \int_M K(m)\,\mathrm{d}m\\ &\le \int_M \frac{1}{2\pi}\int_{U_m M}[\kappa + \langle Au,u\rangle^2]\mathrm{d}v\mathrm{d}m\\ &= \frac{1}{2\pi}\int_{UM}[\kappa + \langle Au,u\rangle^2]\mathrm{d}\mu.\end{aligned}$$

Hence

$$\chi(M) \le \frac{1}{4\pi^2}\int_{UM}[\kappa + \langle Au,u\rangle^2]\mathrm{d}\mu.$$

Further the equality holds iff M is a totally umbilic surface in $\mathbb{M}(\kappa)$. Since the only connected compact umbilic surfaces in $\mathbb{M}(\kappa)$ are spheres the result follows. □

References

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