Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 4, November 2008, pp. 557–567.

On P-coherent endomorphism rings

LI-XIN MAO

Institute of Mathematics, Nanjing Institute of Technology, Nanjing 211167, People's Republic of China
E-mail: maolx2@hotmail.com

MS received 22 April 2007; revised 19 May 2008

Abstract. A ring is called right P-coherent if every principal right ideal is finitely presented. Let M_R be a right R-module. We study the P-coherence of the endomorphism ring S of M_R. It is shown that S is a right P-coherent ring if and only if every endomorphism of M_R has a pseudokernel in add M_R; S is a left P-coherent ring if and only if every endomorphism of M_R has a pseudocokernel in add M_R. Some applications are given.

Keywords. P-coherent ring; *PP* ring; torsionfree module; divisible module; precover; preenvelope; pseudokernel; pseudocokernel.

1. Introduction

As a generalization of coherent rings, the concept of P-coherent rings was introduced in [11]. From [11], R is called a *right P-coherent ring* if every principal right ideal of R is finitely presented, or equivalently, if the right annihilator of a in R is a finitely generated right ideal for any $a \in R$. The examples of right P-coherent rings include right coherent rings and domains. Recall that R is a *right PP ring* (resp. *PF ring*) if every principal right ideal of R is projective (resp. flat). It is obvious that R is a right *PP* ring if and only if R is a right P-coherent and *PF* ring. Another interesting fact is that R is a right coherent ring if and only if every $n \times n$ matrix ring $M_n(R)$ is a right P-coherent ring for every $n \geq 1$ (see Proposition 2.4 of [11]). P-coherent rings are also closely related to two kinds of classical modules, i.e., torsionfree modules and divisible modules. Following [10], a left R-module N is called *torsionfree* if $\mathrm{Tor}_1^R(R/aR, N) = 0$ for all $a \in R$. A right R-module Q is said to be *divisible* if $\mathrm{Ext}_R^1(R/aR, Q) = 0$ for all $a \in R$. It has been shown that R is a right P-coherent ring if and only if any direct product of torsionfree left R-modules is torsionfree if and only if any direct limit of divisible right R-modules is divisible (see Theorem 2.7 of [11]).

Let M_R be a right R-module and S the endomorphism ring of M_R. Many authors have studied the coherence of S (see, for example, [2, 3, 8, 9]). In this paper, we will consider the P-coherence of the endomorphism ring S. It is shown that S is a right P-coherent ring if and only if every cyclically M-copresented right R-module has an add M_R-precover if and only if every endomorphism of M_R has a pseudokernel in add M_R; S is a left P-coherent ring if and only if every cyclically M-presented right R-module has an add M_R-preenvelope if and only if every endomorphism of M_R has a pseudocokernel in add M_R. We also consider when the endomorphism ring S is *PP*, and when every injective right S-module is torsionfree.

Next we recall some notions and definitions needed in the later sections.

Let M_R be a right R-module. We call a right R-module L *cyclically M-presented* if there exists an exact sequence $M \to M \to L \to 0$, and call a right R-module N *cyclically M-copresented* if there exists an exact sequence $0 \to N \to M \to M$. A right R-module is called *cyclically presented* (resp. *cyclically copresented*) if it is cyclically R_R-presented (resp. cyclically R_R-copresented).

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [5] or [6], we say that a homomorphism $\phi: C \to M$ is a *$\mathcal{C}$-precover of M* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(C', \phi): \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for every $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi: C \to M$ is said to be a *$\mathcal{C}$-cover of M* if every endomorphism $g: C \to C$ such that $\phi g = \phi$ is an isomorphism. A $\mathcal{C}$-cover $\phi: C \to M$ is said to *have the unique mapping property* [4] if for any homomorphism $f: C' \to M$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g: C' \to C$ such that $\phi g = f$. Dually we have the definitions of a $\mathcal{C}$-preenvelope and a $\mathcal{C}$-envelope (with the unique mapping property). $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

Following the terminology of Auslander, we say that, in add M_R, a morphism $f: A \to B$ is a *pseudokernel* (resp. *kernel*) of $g: B \to C$ when $gf = 0$ and if $gh = 0$, then h factors through f (resp. h factors through f in a unique way). Dually we have the definition of a pseudocokernel (cokernel) of a morphism in add M_R.

Throughout this paper, all rings are associative with identity and all modules are unitary. M_R (${}_RM$) denotes a right (left) R-module. For a module M_R, we denote by $S = \mathrm{End}(M_R)$ the endomorphism ring of M_R and by add M_R the category consisting of all modules isomorphic to direct summands of finite direct sums of copies of M_R. M^I stands for the direct product of copies of M indexed by a set I.

2. When is the endomorphism ring of a right module right P-coherent?

We start with the following theorem.

Theorem 2.1. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent*:

(1) *S is a right P-coherent ring.*
(2) *Every cyclically M-copresented right R-module has an* add *M_R-precover.*
(3) *Every endomorphism of M_R has a pseudokernel in* add *M_R.*

Proof.

(1) $\Rightarrow$ (2). Let N be a cyclically M-copresented right R-module. Then there is a right R-module exact sequence $0 \to N \to M \overset{\alpha}{\to} M$, which gives rise to the right S-module exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to S \overset{\alpha_*}{\to} S.$$

So $\mathrm{Hom}_R(M, N)$ is a finitely generated right S-module since $\mathrm{im}(\alpha_*)$ is finitely presented by (1). Thus N has an add M_R-precover of N by Lemma 3(2) of [3].

(2) $\Rightarrow$ (1). Let $\phi \in S$. Then there is a right R-module exact sequence $0 \to K \to M \overset{\phi}{\to} M$. Since K has an add M_R-precover by (2), $\mathrm{Hom}_R(M, K)$ is a finitely generated right S-module by Lemma 3(2) of [3]. On the other hand, the sequence $0 \to K \to M \overset{\phi}{\to} M$

induces the exactness of the right S-module sequence

$$0 \to \operatorname{Hom}_R(M, K) \to \operatorname{Hom}_R(M, M) \xrightarrow{\phi_*} \operatorname{Hom}_R(M, M).$$

Thus $\phi S = \phi_*(S)$ is finitely presented. It follows that S is a right P-coherent ring.

(3) $\Rightarrow$ (2). Let N be a cyclically M-copresented right R-module. Then there is a right R-module exact sequence $0 \to N \xrightarrow{i} M \xrightarrow{\alpha} M$ (we may regard i as the inclusion). By (3), α has a pseudokernel β: $A \to M$ with $A \in \operatorname{add} M_R$. Since $\alpha\beta = 0$, we have $\operatorname{im}(\beta) \subseteq \ker(\alpha) = N$. So there exists β': $A \to N$ such that $i\beta' = \beta$. It is easy to verify that β' is an add M_R-precover of N since β is a pseudokernel in add M_R.

(2) $\Rightarrow$ (3). Let $\psi \in S$. Then there is a right R-module exact sequence $0 \to K \xrightarrow{i} M \xrightarrow{\psi} M$. So the cyclically M-copresented right R-module K has an add M_R-precover γ: $G \to K$ by (2). It is easy to check that $i\gamma$: $G \to M$ is a pseudokernel of ψ in add M_R. ■

If we take $M_R = R_R$ in Theorem 2.1, we obtain the following corollary.

COROLLARY 2.2

The following conditions are equivalent for a ring R:

(1) *R is a right P-coherent ring.*
(2) *Every cyclically copresented right R-module has a finitely generated projective precover.*
(3) *Every endomorphism of R_R has a pseudokernel in the category of finitely generated projective right R-modules.*

Next we consider some conditions stronger than those in Theorem 2.1.

PROPOSITION 2.3

Let M_R be a right R-module with $S = \operatorname{End}(M_R)$. Then the following conditions are equivalent:

(1) *Every endomorphism of M_R has a kernel in* add M_R.
(2) *Every cyclically M-copresented right R-module has an* add M_R*-cover with the unique mapping property.*
(3) *For any cyclically M-copresented right R-module X, there exists an* add M_R*-precover α: $N \to X$ such that* $\operatorname{Hom}_R(M, \ker(\alpha)) = 0$.

Proof. It is straightforward by definitions. ■

Now, we characterize when S is a right *PP* ring and $_SM$ is torsionfree. Before that, we need the following lemma.

Lemma 2.4. *Let M_R be a right R-module with $S = \operatorname{End}(M_R)$. Then the following conditions are equivalent*:

(1) *$_SM$ is torsionfree.*
(2) *Every cyclically M-copresented right R-module is generated by M_R.*

Proof.

(1) $\Rightarrow$ (2). Let N be a cyclically M-copresented right R-module. Then there is a right R-module exact sequence $0 \to N \overset{\lambda}{\to} M \overset{\gamma}{\to} M$ (we may regard λ as the inclusion). Let $x \in N$. Then $\gamma x = \gamma(x) = 0$. Since ${}_SM$ is torsionfree, by Proposition 4.1 of [13], we have $x = \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_m y_m$ with $\gamma_i \in S$, $y_i \in M$ and $\gamma\gamma_i = 0$, $i = 1, 2, \ldots, m$. Thus $\mathrm{im}(\gamma_i) \subseteq N$, so we can define $\gamma_i'\colon M \to N$ by $\gamma_i'(y) = \gamma_i(y)$. Then $x = \gamma_1'(y_1) + \gamma_2'(y_2) + \cdots + \gamma_m'(y_m)$. Therefore $N = \sum\{\mathrm{im}(f)\colon f \in \mathrm{Hom}_R(M, N)\}$. So, by Corollary 8.13(1) of [1], N is generated by M_R.

(2) $\Rightarrow$ (1). Let $\alpha \in S$. Then there is a right R-module exact sequence $0 \to K \overset{\lambda}{\to} M \overset{\alpha}{\to} M$. We will show that the sequence $0 \to \alpha S \otimes_S M \to S \otimes_S M \cong M_R$ is exact. Let $\alpha \otimes u \in \alpha S \otimes_S M$ such that $\alpha u = \alpha(u) = 0$. Then $u \in K$. Since K is generated by M_R, we have $u = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)$ with $f_i \in \mathrm{Hom}(M, K)$ and $x_i \in M$. So

$$\begin{aligned} \alpha \otimes u &= \alpha \otimes (f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)) \\ &= \alpha \otimes (\lambda f_1(x_1)) + \alpha \otimes (\lambda f_2(x_2)) + \cdots + \alpha \otimes (\lambda f_n(x_n)) \\ &= \alpha(\lambda f_1) \otimes x_1 + \alpha(\lambda f_2) \otimes x_2 + \cdots + \alpha(\lambda f_n) \otimes x_n = 0 \text{ in } \alpha S \otimes_S M. \end{aligned}$$

Thus ${}_SM$ is torsionfree. ■

It is clear that R is a right *PP* ring if and only if every cyclically copresented right R-module is a direct summand of R. In general, we have the following result.

Theorem 2.5. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent*:

(1) *S is a right PP ring and ${}_SM$ is torsionfree.*
(2) *Every cyclically M-copresented right R-module is a direct summand of M_R.*

Proof.

(1) $\Rightarrow$ (2). Let N be a cyclically M-copresented right R-module. Then there is a right R-module exact sequence $0 \to N \overset{i}{\to} M \overset{\gamma}{\to} M$ (where i is the inclusion), which gives rise to the right S-module exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \overset{i_*}{\to} S \overset{\gamma_*}{\to} S.$$

Since ${}_SM$ is torsionfree, we obtain the right R-module exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \otimes_S M \overset{i_* \otimes 1}{\to} S \otimes_S M \overset{\gamma_* \otimes 1}{\to} S \otimes_S M.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M, N) \otimes_S M & \longrightarrow & S \otimes_S M & \longrightarrow & S \otimes_S M \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \xrightarrow{\;i\;} & M & \xrightarrow{\;\gamma\;} & M. \end{array}$$

By the Five lemma, σ is an isomorphism. Since S is a right *PP* ring, i_* is a split monomorphism, and so $i_* \otimes 1$ is also a split monomorphism. Thus i is split by the above diagram. Hence N is a direct summand of M_R.

(2) $\Rightarrow$ (1). Let $\alpha \in S$. Then there is a right R-module exact sequence

$$0 \to K \to M \to \alpha(M) \to 0.$$

Since K is a direct summand of M_R by (2), the short exact sequence is split, and so we get the split right S-module exact sequence

$$0 \to \mathrm{Hom}_R(M, K) \to \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, \alpha(M)) \to 0.$$

Thus $\alpha S = \alpha_*(S) \cong \mathrm{Hom}_R(M, \alpha(M))$ is a projective right S-module, and hence S is a right *PP* ring. On the other hand, it is clear that ${}_SM$ is torsionfree by Lemma 2.4. ■

3. When is the endomorphism ring of a right module left *P*-coherent?

Theorem 3.1. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent*:

(1) *S is a left P-coherent ring.*
(2) *Every cyclically M-presented right R-module has an* add M_R*-preenvelope.*
(3) *Every endomorphism of M_R has a pseudocokernel in* add M_R.

Proof. The proof is dual to that of Theorem 2.1 by using Lemma 3(1) of [3].

If we take $M_R = R_R$ in Theorem 3.1, we obtain the following.

COROLLARY 3.2

The following conditions are equivalent for a ring R:

(1) *R is a left P-coherent ring.*
(2) *Every cyclically presented right R-module has a finitely generated projective preenvelope.*
(3) *Every endomorphism of R_R has a pseudocokernel in the category of finitely generated projective right R-modules.*

The following result is dual to Proposition 2.3.

PROPOSITION 3.3

Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:

(1) *Every endomorphism of M_R has a cokernel in* add M_R.
(2) *Every cyclically M-presented right R-module has an* add M_R*-envelope with the unique mapping property.*
(3) *For any cyclically M-presented right R-module X, there exists an* add M_R*-preenvelope $\alpha: X \to N$ such that* $\mathrm{Hom}_R(\mathrm{coker}(\alpha), M) = 0$.

Recall that a right R-module M is *quasi-injective* if for any submodule N of M, the sequence $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(N, M) \to 0$ is exact, and M is *quasi-projective* if the sequence $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, L) \to 0$ is exact for any quotient L of M.

Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. By Theorem 3.1, S is a left P-coherent ring if and only if every cyclically M-presented right R-module has an add M_R-preenvelope. Moreover, if M_R is quasi-injective, we can give special descriptions of add M_R-preenvelopes as follows.

Theorem 3.4. *Let M_R be a quasi-injective right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent*:

(1) *S is a left P-coherent ring.*
(2) *For any cyclically M-presented right R-module X, there exist a nonnegative integer n and $\alpha: X \to M^n$ such that $\mathrm{Hom}_R(\ker(\alpha), M) = 0$.*

Proof.

(1) ⇒ (2). Let X be a cyclically M-presented right R-module. Since M_R is quasi-injective, M_R is X-injective by Proposition 16.13(1) of [1]. In addition, by (1) and Theorem 3.1, X has an add M_R-preenvelope $\alpha: X \to M^n$, where n is a nonnegative integer. So we get an exact sequence

$$0 \to \mathrm{Hom}_R(\mathrm{im}(\alpha), M) \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(\ker(\alpha), M) \to 0.$$

But $\mathrm{Hom}_R(\mathrm{im}(\alpha), M) \to \mathrm{Hom}_R(X, M) \to 0$ is exact. Thus $\mathrm{Hom}_R(\ker(\alpha), M) = 0$.

(2) ⇒ (1). For any cyclically M-presented right R-module X, there exist a nonnegative integer n and a right R-homomorphism $\alpha: X \to M^n$ such that $\mathrm{Hom}_R(\ker(\alpha), M) = 0$ by (2). So the exact sequence $0 \to \ker(\alpha) \to X \to \mathrm{im}(\alpha) \to 0$ induces the exactness of the sequence

$$0 \to \mathrm{Hom}_R(\mathrm{im}(\alpha), M) \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(\ker(\alpha), M) = 0.$$

But M is M^n-injective by Proposition 16.13(2) of [1], and so we get the exact sequence

$$\mathrm{Hom}_R(M^n, M) \to \mathrm{Hom}_R(\mathrm{im}(\alpha), M) \to 0.$$

Hence we have the exact sequence

$$\mathrm{Hom}_R(M^n, M) \to \mathrm{Hom}_R(X, M) \to 0.$$

Thus $\alpha: X \to M^n$ is an add M_R-preenvelope of X. It follows that S is a left P-coherent ring by Theorem 3.1. ■

In order to characterize when every cyclically M-presented right R-module has a monic add M_R-preenvelope, we need the following lemma.

Lemma 3.5. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent*:

(1) *${}_SM$ is divisible.*
(2) *Every cyclically M-presented right R-module is cogenerated by M_R.*

Proof.

(1) $\Rightarrow$ (2). Let L be a cyclically M-presented right R-module. Then there is a right R-module exact sequence $M \overset{\varphi}{\to} M \overset{\pi}{\to} L \to 0$. If L is not cogenerated by M_R, then by Corollary 8.13(2) of [1], there exists $x \in M$ such that $0 \neq \bar{x} \in L$ and $f(\bar{x}) = 0$ for all $f \in \mathrm{Hom}_R(L, M)$. Define ψ: $S\varphi \to M$ by $\psi(s\varphi) = sx$. We claim that ψ is well-defined. In fact, if $s\varphi = 0$, then $\mathrm{im}(\varphi) \subseteq \ker(s)$, and so there exists α: $L \to M$ such that $\alpha\pi = s$. Hence $sx = \alpha\pi(x) = \alpha(\bar{x}) = 0$. Thus by (1), there exists a left S-homomorphism g: $S \to M$ such that $x = \psi(\varphi) = g(\varphi) = \varphi g(1) \in \varphi M$, and so $\bar{x} = 0$, a contradiction. Therefore L is cogenerated by M_R.

(2) $\Rightarrow$ (1). Let $a \in S$ and f: $Sa \to {}_SM$ be any left S-homomorphism. We will prove that $f(a) \in aM$. Consider the exact sequence $M \overset{a}{\to} M \overset{\pi}{\to} M/aM \to 0$. If $f(a) \notin aM$, then $\overline{f(a)} \neq 0$. Since M/aM is cogenerated by M_R, there exists g: $M/aM \to M$ such that $(g\pi)(f(a)) = g(\overline{f(a)}) \neq 0$. But $g\pi \in S$, so $(g\pi)f(a) = f((g\pi)a) = 0$, a contradiction. Thus $f(a) \in aM$, and hence there is $b \in M$ such that $f(a) = ab$. Define g: $S \to {}_SM$ by $g(s) = sb$. It is clear that g extends f, and so ${}_SM$ is divisible. ■

PROPOSITION 3.6

Let M_R be a quasi-injective right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:

(1) *S is a left P-coherent ring and ${}_SM$ is divisible.*
(2) *Every cyclically M-presented right R-module has a monic* add *M_R-preenvelope.*
(3) *Every cyclically M-presented right R-module embeds in L with $L \in$* add *M_R.*

Proof.

(1) $\Rightarrow$ (2). Let X be a cyclically M-presented right R-module. Since S is a left P-coherent ring, X has an add M_R-preenvelope f: $X \to N$ by Theorem 3.1. In addition, since ${}_SM$ is divisible, there exists a monic R-homomorphism λ: $X \to M^I$ by Lemma 3.5. Let π_i: $M^I \to M$ be the i-th projection. There exist g_i: $N \to M$ such that $g_i f = \pi_i\lambda$. So there exists h: $N \to M^I$ such that $g_i = \pi_i h$. Thus $\pi_i hf = \pi_i\lambda$ and hence $hf = \lambda$. Therefore f is a monic add M_R-preenvelope.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) follows from Theorem 3.4 and Lemma 3.5. ■

Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. In Proposition 2.11(ii) of [8], under the hypothesis that ${}_SM$ is *FP*-injective, it is shown that S is a left *PP* ring if and only if every finitely M-generated submodule of M is a direct summand of M. The following result may be viewed as an improvement of Proposition 2.11(ii) of [8].

PROPOSITION 3.7

Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. The following conditions are equivalent:

(1) *S is a left PP ring and ${}_SM$ is divisible.*
(2) *If N is a cyclically M-presented right R-module*, *which has a presentation $M \to M \overset{\pi}{\to} N \to 0$, then $M \overset{\pi}{\to} N$ is split.*

Proof.

(1) $\Rightarrow$ (2). Let N be a cyclically M-presented right R-module, which has a presentation $M \overset{\alpha}{\to} M \overset{\pi}{\to} N \to 0$. Then we have the left S-module exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \overset{\pi^*}{\to} S \overset{\alpha^*}{\to} S.$$

Since S is a left *PP* ring, π^* is a split monomorphism, and so π^{**} is a split epimorphism. In addition, since ${}_SM$ is divisible, we have the right R-module exact sequence

$$\mathrm{Hom}_S(S, M) \overset{\alpha^{**}}{\to} \mathrm{Hom}_S(S, M) \overset{\pi^{**}}{\to} \mathrm{Hom}_S(\mathrm{Hom}_R(N, M), M) \to 0.$$

By the Five lemma, $N \cong \mathrm{Hom}_S(\mathrm{Hom}_R(N, M), M)$. Thus π is split.

(2) $\Rightarrow$ (1). Let $\alpha \in S$. Then there is a right R-module exact sequence $M \overset{\alpha}{\to} M \overset{\pi}{\to} N \to 0$. Since π is split, we get the split left S-module exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(\alpha(M), M) \to 0.$$

Thus $S\alpha = \alpha^*(S) \cong \mathrm{Hom}_R(\alpha(M), M)$ is a projective left S-module, and hence S is a left *PP* ring. On the other hand, ${}_SM$ is divisible by Lemma 3.5. ■

Letting $M_R = R_R$ in Proposition 3.7, we obtain a characterization of von Neumann regular rings, which is due to Xue (see Theorem 3 of [14]).

COROLLARY 3.8

R is a von Neumann regular ring if and only if R is a left PP ring and ${}_RR$ is divisible.

The following lemma will be useful for dealing with other ring properties.

Lemma 3.9. *The following conditions are equivalent for a right R-module N*:

(1) *N is a torsionfree right R-module.*
(2) *For any cyclically presented right R-module A, any homomorphism $f: A \to N$ factors through a finitely generated projective right R-module.*

Proof.

(1) $\Rightarrow$ (2). There is an exact sequence $0 \to K \overset{i}{\to} P \overset{\alpha}{\to} N \to 0$ with P projective and i the inclusion. Let $a \in R$ and $f: R/aR \to N$ be any homomorphism. Then there exist g and h such that the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & aR & \xrightarrow{\lambda} & R & \xrightarrow{\pi} & R/aR & \longrightarrow & 0 \\
 & & \Big\downarrow h & & \Big\downarrow g & & \Big\downarrow f & & \\
0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{\alpha} & N & \longrightarrow & 0.
\end{array}$$

So $g(a) = g\lambda(a) = ih(a) = h(a) \in K$. Note that $g(a) \otimes \bar{1} = g(1)a \otimes \bar{1} = g(1) \otimes \bar{a} = 0$ in $P \otimes (R/Ra)$, thus $g(a) \otimes \bar{1} = 0$ in $K \otimes (R/Ra)$ since the sequence $0 \to K \otimes (R/Ra) \to P \otimes (R/Ra)$ is exact by (1). Hence there exists $k \in K$ such that $g(a) = ka$ by Lemma 3.64 of [12]. Define $\beta: R \to K$ by $\beta(r) = kr$, then β is well-defined and $\beta\lambda = h$. Therefore there exists $\gamma\colon R/aR \to P$ such that $\alpha\gamma = f$ by Lemma 8.4 of [7]. Since R/aR is finitely generated, f factors through a finitely generated projective right R-module.

(2) $\Rightarrow$ (1). There exists an exact sequence $0 \to K \xrightarrow{i} P \xrightarrow{\alpha} N \to 0$, where P is projective and i is the inclusion. Let $a \in R$. We will show that the sequence $0 \to K \otimes (R/Ra) \to P \otimes (R/Ra)$ is exact. Let $k \otimes \bar{1} \in K \otimes (R/Ra)$ such that $k \otimes \bar{1} = 0$ in $P \otimes (R/Ra)$. Then there exists $p \in P$ such that $k = pa$ by Lemma 3.64 of [12]. Define $g\colon R \to P$ by $g(r) = pr$, and define $f\colon R/aR \to N$ by $f(\bar{r}) = \alpha(p)r$. Then f is well-defined, and $f\pi = \alpha g$. Therefore there exists $h\colon aR \to K$ such that the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & aR & \xrightarrow{\lambda} & R & \xrightarrow{\pi} & R/aR & \longrightarrow & 0 \\ & & {\scriptstyle h}\,\vdots & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{\alpha} & N & \longrightarrow & 0. \end{array}$$

By (2), f factors through a finitely generated projective right R-module, i.e., there exist a finitely generated projective right R-module F, homomorphisms $\beta\colon R/aR \to F$ and $\gamma\colon F \to N$ such that $f = \gamma\beta$. In addition, there exists $\delta\colon F \to P$ such that $\gamma = \alpha\delta$ since F is projective. Note that $\alpha(\delta\beta) = \gamma\beta = f$. Thus there exists $\varphi\colon R \to K$ such that $h = \varphi\lambda$ by Lemma 8.4 of [7]. So $k = pa = g(a) = h(a) = \varphi(a) = \varphi(1)a$. Consequently, $k \otimes \bar{1} = \varphi(1)a \otimes \bar{1} = \varphi(1) \otimes \bar{a} = 0$ in $K \otimes (R/Ra)$, as desired. ■

COROLLARY 3.10

The following conditions are equivalent for a ring R:

(1) *Every injective right R-module is torsionfree.*
(2) *Every cyclically presented right R-module embeds in a finitely generated projective right R-module.*
Moreover, if R_R is injective, then the above conditions are equivalent to
(3) *R is a left P-coherent ring and ${}_RR$ is divisible.*
(4) *Every cyclically presented right R-module has a monic finitely generated projective envelope.*

Proof.

(1) $\Leftrightarrow$ (2) is easy by Lemma 3.9 since every module embeds in its injective envelope.

(2) $\Leftrightarrow$ (3) $\Leftarrow$ (4) hold by Proposition 3.6.

(2) $\Rightarrow$ (4). Let N be a cyclically presented right R-module. We will prove that the injective envelope $i\colon N \to E(N)$ is a finitely generated projective envelope of N. First we claim that $E(N)$ is a finitely generated projective. In fact, by (2), there exists a monomorphism $g\colon N \to P$ with P finitely generated projective. Thus there exists $f\colon P \to E(N)$ such that $i = fg$. On the other hand, we have P is injective since R_R is injective, and so there exists $h\colon E(N) \to P$ such that $g = hi$. Therefore $fhi = i$, and hence fh is an isomorphism. Thus $E(N)$ is isomorphic to a direct summand of P, and hence $E(N)$ is a finitely generated projective. It is easy to see that i is a finitely generated projective preenvelope by the above proof, and hence i is a finitely generated projective envelope since i is an injective envelope. ■

Remark 3.11. It would be interesting to compare the results of Proposition 4.2 of [11] and Corollary 3.10. By Proposition 4.2 of [11] for a left P-coherent ring R, every injective right R-module is torsionfree if and only if ${}_RR$ is divisible if and only if every left R-module has an epic divisible cover. On the other hand, for a right self-injective ring R, Corollary 3.10 shows that every injective right R-module is torsionfree if and only if R is a left P-coherent ring and ${}_RR$ is divisible if and only if every cyclically presented right R-module has a monic finitely generated projective envelope.

In the end of the paper, we consider a more general setting than Corollary 3.10, i.e., characterize when every injective right S-module is torsionfree.

Theorem 3.12. *Let M_R be a quasi-projective right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent*:

(1) *Every injective right S-module is torsionfree.*
(2) *For any cyclically M-presented right R-module X, there exist a nonnegative integer n and $\alpha: X \to M^n$ such that* $\mathrm{Hom}_R(M, \ker(\alpha)) = 0$.

Proof.

(1) $\Rightarrow$ (2). The right R-module exact sequence $M \xrightarrow{\gamma} M \to X \to 0$ gives rise to the right S-module exact sequence $S \xrightarrow{\gamma_*} S \to \mathrm{Hom}_R(M, X) \to 0$ since M_R is quasi-projective. So $\mathrm{Hom}_R(M, X)$ is a cyclically presented right S-module. By (1) and Corollary 3.10, there is a monic S-homomorphism $\varphi: \mathrm{Hom}_R(M, X) \to S^n$. By the Five lemma, it is easy to see that the canonical map $\beta: \mathrm{Hom}_R(M, X) \otimes_S M \to X$ is an isomorphism. Put

$$\alpha = \psi(\varphi \otimes 1)\beta^{-1}: X \xrightarrow{\beta^{-1}} \mathrm{Hom}_R(M, X) \otimes_S M \xrightarrow{\varphi\otimes 1} S^n \otimes_S M \xrightarrow{\psi} M^n.$$

Then the exact sequence $0 \to \ker(\alpha) \to X \xrightarrow{\alpha} M^n$ induces the exact sequence

$$0 \to \mathrm{Hom}_R(M, \ker(\alpha)) \to \mathrm{Hom}_R(M, X) \xrightarrow{\alpha_*} S^n.$$

It is easy to verify that $\alpha_* = \varphi$, and so α_* is monic. Hence $\mathrm{Hom}_R(M, \ker(\alpha)) = 0$.

(2) $\Rightarrow$ (1). Let $S \xrightarrow{\varphi} S \to L \to 0$ be an exact sequence of right S-modules. By tensoring with ${}_SM$, we get an induced right R-module exact sequence $M \to M \to L \otimes_S M \to 0$. By (2), there exists a right R-homomorphism $\alpha: L \otimes_S M \to M^n$ such that $\mathrm{Hom}_R(M, \ker(\alpha)) = 0$. So we get the right S-module exact sequence $0 \to \mathrm{Hom}_R(M, L \otimes_S M) \to S^n$. On the other hand, since M_R is quasi-projective, we have $L \cong \mathrm{Hom}_R(M, L \otimes_S M)$ by the Five lemma. Thus L embeds in S^n. So every injective right S-module is torsionfree by Corollary 3.10. ∎

Acknowledgements

This research was partially supported by SRFDP (No. 20050284015), NSFC (No. 10771096), the Natural Science Foundation of Jiangsu Province of China (No. BK2008365), Science Research Fund of Nanjing Institute of Technology (No. KXJ07061), Jiangsu 333 Project, and Jiangsu Qinglan Project. The author would like to thank Professor Nanqing Ding for his constant encouragement and the referee for the valuable comments and suggestions in shaping the paper into its present form.

References

[1] Anderson F W and Fuller K R, Rings and Categories of Modules (New York: Springer-Verlag) (1974)

[2] Angeleri-Hügel L, On Some Precovers and Preenvelopes (München: Habilitationsschrift) (2000)

[3] Angeleri-Hügel L, Endocoherent modules, *Pacific J. Math.* **212** (2003) 1–11

[4] Ding N Q, On envelopes with the unique mapping property, *Comm. Algebra* **24** (1996) 1459–1470

[5] Enochs E E, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981) 189–209

[6] Enochs E E and Jenda O M G, Relative Homological Algebra (Berlin-New York: Walter de Gruyter) (2000)

[7] Fuchs L and Salce L, Modules over Non-noetherian Domains, Math. Surveys and Monographs, vol. 84 (Providence: Amer. Math. Soc.) (2001)

[8] Garcia J L, Martinez Hernandez J and Gomez Sanchez P L, Endomorphism rings and the category of finitely generated projective modules, *Comm. Algebra* **28** (2000) 3837–3852

[9] Gomez Pardo J L and Martinez Hernandez J, Coherence of endomorphism rings, *Arch. Math.* **48** (1987) 40–52

[10] Hattori A, A foundation of torsion theory for modules over general rings, *Nagoya Math. J.* **17** (1960) 147–158

[11] Mao L X and Ding N Q, On divisible and torsionfree modules, *Comm. Algebra* **36** (2008) 708–731

[12] Rotman J J, An Introduction to Homological Algebra (New York: Academic Press) (1979)

[13] Shamsuddin A, n-injective and n-flat modules, *Comm. Algebra* **29** (2001) 2039–2050

[14] Xue W M, On *PP* rings, *Kobe J. Math.* **7** (1990) 77–80