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On n-weak amenability of Rees semigroup algebras

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Abstract. Let S be a Rees matrix semigroup. We show that $l^1(S)$ is $(2k+1)$-weakly amenable for $k \in \mathbb{Z}^+$.

Keywords. n-Weak amenability; bounded derivation; dual space; semigroup; Banach algebra.

1. Introduction

In [4], Dales, Ghahramani and Gronbaek introduced the concept of n-weak amenability for Banach algebras for $n \in \mathbb{N}$. They determined some relations between m- and n-weak amenability for general Banach algebras and for Banach algebras in various classes, and proved that, for each $n \in \mathbb{N}$, $(n+2)$-weak amenability always implies n-weak amenability. Let A be a weakly amenable Banach algebra. Then it is also proved in [4] that in the case where A is an ideal in its second dual $(A'', \Box)$, A is necessarily $(2m-1)$-weakly amenable for each $m \in \mathbb{N}$. The authors of [4] asked the following questions: (i) Is a weakly amenable Banach algebra necessarily 3-weakly amenable? (ii) Is a 2-weakly amenable Banach algebra necessarily 4-weakly amenable? A counter-example resolving question (i) was given by Zhang in [11], but it seems that question (ii) is still open.

It is also shown in Corollary 5.4 of [4] that for certain Banach space E the Banach algebra $\mathcal{N}(E)$ of nuclear operators on E is n-weakly amenable if and only if n is odd.

A class of Banach algebras that was not considered in [4] is the Banach algebras on semigroups. In this work, we shall consider this class of Banach algebras. We examine the n-weak amenability of some semigroup algebras, and give an easier example of a Banach algebra which is n-weakly amenable if n is odd.

Let $L^1(G)$ be the group algebra of a locally compact group G (§3.3 of [3]). Then Johnson has proved that $L^1(G)$ is amenable if and only if G is amenable ([8], Theorem 5.6.42 of [3]) and that $L^1(G)$ is always weakly amenable ([9], Theorem 5.6.48 of [3]). It is proved in Theorem 4.1 of [4] that each group algebra is n-weakly amenable whenever n is odd, and it is conjectured that $L^1(G)$ is n-weakly amenable for each $n \in \mathbb{N}$; this is true whenever G is amenable, and it is true when G is a free group [10].

2. Preliminaries

First, we recall some standard notions; for further details, see [3].

Let A be an algebra. The character space of A is denoted by Φ_A. Let X be an A-bimodule. A *derivation* from A to X is a linear map D: $A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A).$$

For example, δ_x: $a \to a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations*.

Let X be a Banach space. Then the spaces $X^{(n)}$ for $n \in \mathbb{Z}^+$ are the iterated duals of X, where we take $X^{(0)} = X$. Let $\lambda \in X'$. We denote by $\lambda^{(2n)} \in X^{(2n+1)}$ the $2n$-th dual of λ for $n \in \mathbb{Z}^+$, where $\lambda^{(0)} = \lambda$. Clearly $\lambda^{(2n)}|X = \lambda$, where we regard X as a closed subspace of $X^{(2n)}$.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k \|a\| \|x\|, \quad \|x \cdot a\| \leq k \|a\| \|x\| \quad (a \in A, x \in X).$$

By renorming X, we can suppose that $k = 1$. For example, A itself is a Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x, a \cdot \lambda\rangle = \langle x \cdot a, \lambda\rangle, \quad \langle x, \lambda \cdot a\rangle = \langle a \cdot x, \lambda\rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X. Successively, the duals $X^{(n)}$ are Banach A-bimodules; in particular $A^{(n)}$ is a Banach A-bimodule for each $n \in \mathbb{N}$. We take $X^{(0)} = X$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X).$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X and *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$. Further, as in [4], A is *n-weakly amenable* for $n \in \mathbb{N}$ if $\mathcal{H}^1(A, A^{(n)}) = \{0\}$, and A is *permanently weakly amenable* if it is n-weakly amenable for each $n \in \mathbb{N}$. For instance, each C^*-algebra is permanently weakly amenable (Theorem 2.1 of [4]). As we stated, each group algebra is n-weakly amenable whenever n is odd.

Arens in [1] defined two products, $\Box$ and $\Diamond$, on the bidual A'' of Banach algebra A; A'' is a Banach algebra with respect to each of these products, and each algebra contains A as a closed subalgebra. The products are called the *first* and *second Arens products* on A'', respectively. For the general theory of Arens products, see [3, 5]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a, \lambda \cdot \Phi\rangle = \langle \Phi, a \cdot \lambda\rangle, \quad \langle a, \Phi \cdot \lambda\rangle = \langle \Phi, \lambda \cdot a\rangle \quad (a \in A, \lambda \in A'),$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \Box \Psi, \lambda\rangle = \langle \Phi, \Psi \cdot \lambda\rangle, \quad \langle \Phi \Diamond \Psi, \lambda\rangle = \langle \Psi, \lambda \cdot \Phi\rangle \quad (\lambda \in A').$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for certain nets (a_α) and (b_β) in A. Then $\Phi \,\Box\, \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta$ and $\Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta$, where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''.

We define the product $\Box$ on $A^{(2n)}$ for $n \in \mathbb{N}$ inductively. Indeed, assume that $\Box$ is defined on $A^{(2n)}$, and set $B = (A^{(2n)}, \Box)$. Then $(A^{(2n+2)}, \Box) = (B'', \Box)$. Let $\varphi \in \Phi_A$. Then it is clear that $\varphi^{(2n)}$ is a character on $(A^{(2n)}, \Box)$.

Let S be a non-empty set. Then

$$\ell^1(S) = \left\{ f \in \mathbb{C}^S \colon \sum_{s \in S} |f(s)| < \infty \right\},$$

with the norm $\|\cdot\|_1$ given by $\|f\|_1 = \sum_{s\in S} |f(s)|$ for $f \in \ell^1(S)$. We write δ_s for the characteristic function of $\{s\}$ when $s \in S$.

Now suppose that S is a semigroup. For $f, g \in \ell^1(S)$, we set

$$(f \star g)(t) = \left\{ \sum f(r)g(s) \colon r, s \in S, rs = t \right\} \quad (t \in S)$$

so that $f \star g \in \ell^1(S)$. It is standard that $(\ell^1(S), \star)$ is a Banach algebra, called the *semigroup algebra on S*. For further discussion of this algebra, see [3, 5], for example. In particular, with $A = \ell^1(S)$, we identify A' with $C(\beta S)$, where βS is the Stone–Čech compactification of S, and $(A'', \Box)$ with $(M(\beta S), \Box)$, where $M(\beta S)$ is the space of regular Borel measures on βS of S; in this way, $(\beta S, \Box)$ is a compact, right topological semigroup that is a subsemigroup of $(M(\beta S), \Box)$ after the identification of $u \in \beta S$ with $\delta_u \in M(\beta S)$.

There is at least one character on the Banach algebra $\ell^1(S)$: this is the *augmentation character*

$$\varphi_S \colon f \mapsto \sum_{s \in S} f(s), \quad l^1(S) \to \mathbb{C}.$$

Let S be a semigroup, and let $o \in S$ be such that $so = os = o$; $(s \in S)$. Then o is a *zero* for the semigroup S. Suppose that $o \notin S$; set $S^o = S \cup \{o\}$, and define $so = os = o$ $(s \in S)$ and $o^2 = o$. Then S^o is a semigroup containing S as a subsemigroup; we say that S is formed by *adjoining a zero to S*.

We recall that S is a *right zero semigroup* if the product in S is such that

$$st = t \quad (s, t \in S).$$

In this case, $f \star g = \varphi_S(f) g$ $(f, g \in \ell^1(S))$.

3. Munn algebras

Let A be a unital algebra, $m, n \in \mathbb{N}$ and $P = (p_{r,s})$ be a matrix in $\mathcal{M}_{n,m}(A)$. Then $\mathcal{M}_{m,n}(A)$ is an algebra for the product

$$a \circ b = aPb \quad (a, b \in \mathcal{M}_{m,n}(A))$$

(in the sense of matrix products). This is the *Munn algebra over A with sandwich matrix P*, and it is denoted by

$$\mathcal{A} = \mathcal{M}(A, P, m, n).$$

Now suppose that A is a unital Banach algebra and that each non-zero element in P has norm 1. Then $\mathcal{M}(A, P, m, n)$ is also a Banach algebra for the norm given by

$$\|(a_{ij})\| = \sum\{\|a_{ij}\|: i \in \mathbb{N}_m,\ j \in \mathbb{N}_n\} \quad ((a_{ij}) \in \mathcal{M}_{m,n}(A)). \tag{3.1}$$

These Banach algebras are special cases of those defined by Esslamzadeh in Definition 3.1 of [6].

We may make the following assumptions if necessary: each non-zero element of P is invertible, and P has no zero rows or columns.

Next let E be a Banach A-bimodule, and define $\mathcal{E} = \mathcal{M}_{m,n}(E)$. We shall regard $\mathcal{E}$ as a Banach $\mathcal{A}$-bimodule in the following way:

$$a \cdot x = aPx, \quad x \cdot a = xPa \quad (a \in \mathcal{A}, x \in \mathcal{E}),$$

again in the sense of matrix products. We now make the following conjecture.

Conjecture. Suppose that each continuous derivation from A to E is inner. Then each continuous derivation from $\mathcal{A}$ to $\mathcal{E}$ is inner.

The question is; of what use would this be? Well, it would give a result about Rees semigroup algebras (see below). Also it has some interest in its own right.

Consider the special case in which $\mathcal{A}$ has an identity, so that, by Proposition 2.16 of [5], $m = n$ and P is invertible in $\mathcal{M}_{m,n}(A)$). Then the argument of Theorem 2.7(iii) of [5] gives the result.

Let A be a Banach algebra, $n, m \in \mathbb{N}$, and set $\mathcal{A} = \mathcal{M}_n(A)$. $A^{(m)}$ the m-th dual of A is a Banach A-bimodule. As in [5], we shall identify $\mathcal{A}^{(m)}$ the m-th dual of $\mathcal{A}$ with $\mathcal{M}_n(A^m)$, using the duality

$$\langle a, \Lambda\rangle = \sum_{i,j=1}^{n} a_{ij} \cdot \lambda_{ij} \quad (a = (a_{ij}) \in \mathcal{A}^{(m-1)}, \quad \Lambda = (\lambda_{ij}) \in \mathcal{A}^{(m)}).$$

We note that

$$(a \cdot \Lambda)_{ij} = \sum_{k=1}^{n} a_{jk} \cdot \lambda_{ik} \quad \text{and} \quad (\Lambda \cdot a)_{ij} = \sum_{k=1}^{n} \lambda_{kj} \cdot a_{ki}, \tag{3.2}$$

for $(a = (a_{ij}) \in \mathcal{A}, \Lambda = (\lambda_{ij}) \in \mathcal{A}^{(m)})$.

Let $D: A \to A^{(m)}$ be a continuous derivation, and define $\mathcal{D}: \mathcal{A} \to \mathcal{A}^{(m)}$ by setting $\mathcal{D}(a)_{ij} = (D(a_{ji}))$, where we note the transposition of i and j. Clearly $\mathcal{D}$ is a continuous linear map. We claim that for $m \in \mathbb{N}$, m odd, $\mathcal{D}$ is a derivation, by showing that

$$\langle c, \mathcal{D}(ab)\rangle = \langle ca, \mathcal{D}b\rangle + \langle bc, \mathcal{D}a\rangle \quad (a, b \in \mathcal{A}, c \in \mathcal{A}^{(m-1)}). \tag{3.3}$$

By definition, we have

$$\begin{aligned}\langle c, \mathcal{D}(ab)\rangle &= \sum_{i,j=1}^{n} \langle c_{ij}, D((ab)_{ji})\rangle \\ &= \sum_{i,j,k=1}^{n} \langle c_{ij}, a_{jk} \cdot D(b_{ki}) + D(a_{jk}) \cdot b_{ki}\rangle\end{aligned}$$

$$= \sum_{i,j,k=1}^{n} (\langle c_{ij}a_{jk}, D(b_{ki})\rangle + \langle b_{ki}c_{ij}, D(a_{jk})\rangle)$$

$$\langle ca, \mathcal{D}b\rangle = \sum_{i,j,k=1}^{n} \langle c_{ik}a_{kj}, D(b_{ji})\rangle = \sum_{i,j,k=1}^{n} \langle c_{ij}a_{jk}, D(b_{ki})\rangle.$$

Similarly,

$$\langle bc, \mathcal{D}a\rangle = \sum_{i,j,k=1}^{n} \langle b_{ki}c_{ij}, D(a_{jk})\rangle.$$

Thus $\mathcal{D}$ is a derivation for $m \in \mathbb{N}$, m odd.

PROPOSITION 3.1

Let A be a unital Banach algebra. Then the Banach algebra $\mathcal{A} = \mathcal{M}_n(A)$ is $(2k+1)$-weakly amenable if and only if A is $(2k+1)$-weakly amenable.

Proof. Suppose $\mathcal{A}$ is $(2k+1)$-weakly amenable. Let $D: A \to A^{(2k+1)}$ be a continuous derivation. Define $\mathcal{D}: \mathcal{A} \to \mathcal{A}^{(2k+1)}$ by setting $\mathcal{D}(a)_{ij} = (D(a_{ji}))$, where we note the transposition of i and j. Then $\mathcal{D}$ is a continuous derivation where $a = (a_{ij}) \in \mathcal{A}$. Since $\mathcal{A}$ is $(2k+1)$-weakly amenable, there exists $\Lambda = (\lambda_{ij}) \in \mathcal{A}^{(2k+1)}$ such that

$$\mathcal{D}(a) = a \cdot \Lambda - \Lambda \cdot a \quad (a \in \mathcal{A}).$$

Take $a \in A$ and identify a with the matrix that has a in the $(1,1)$-th position and 0 elsewhere. Then $\lambda_{1,1} \in A^{((2k+1)}$ and

$$D(a) = \mathcal{D}(a)_{1,1} = (a \cdot \lambda - \lambda \cdot a)_{1,1} = a \cdot \lambda_{1,1} - \lambda_{1,1} \cdot a \quad (a \in A)$$

and so $D: A \to A^{(2k+1)}$ is inner. Hence A is $(2k+1)$-weakly amenable.

For the converse, we identify $\mathcal{A}$ with $\mathcal{M}_n \otimes A$ where $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$. Now suppose A is $(2k+1)$-weakly amenable, and let $\mathcal{D}: \mathcal{A} \to \mathcal{A}^{(2k+1)}$ be a continuous derivation. $\mathcal{M}_n$ is regarded as a subalgebra of $\mathcal{A}$ and since $\mathcal{M}_n$ is amenable, there exists an element $\Lambda = (\lambda_{ij}) \in \mathcal{A}^{(2k+1)} = \mathcal{M}_n(A^{(2k+1)})$ with $\mathcal{D}|\mathcal{M}_n = \mathbf{d}_\Lambda|\mathcal{M}_n$. By replacing $\mathcal{D}$ by $\mathcal{D} - \mathbf{d}_\Lambda$, we may suppose that $\mathcal{D}|\mathcal{M}_n = 0$. Let $a \in A$ and for $r, s \in \mathbb{N}_n$, consider the elements $(a)_{rs} = E_{rs} \otimes a \in \mathcal{A}$ so

$$\mathcal{D}((a)_{rs}) = (d_{ij}^{(r,s)} : i, j \in \mathbb{N}) \in \mathcal{M}_n(A^{(2k+1)}),$$

with $d_{11}^{(1,1)} = d(a)$. We have

$$\mathcal{D}((a)_{rs}) = \mathcal{D}(E_{r1}(a)_{11}E_{1s}) = E_{r1} \cdot \mathcal{D}((a)_{11}) \cdot E_{1s}$$

because $\mathcal{D}(E_{r1}) = \mathcal{D}(E_1 s) = 0$, and so by using (3.2), $d_{ij}^{(r,s)} = 0 \quad (i, j \in \mathbb{N})$ except where $(i, j) = (s, r)$ and in this case $d_{sr}^{(r,s)} = d(a)$. We may regard the map $d: a \to d(a)$ as a map from A into $A^{(2k+1)}$ and clearly d is a continuous derivation. Since A is $(2k+1)$-weakly amenable, there exists $\lambda \in A^{(2k+1)}$ such that

$$d(a) = a \cdot \lambda - \lambda \cdot a \quad (a \in A).$$

Take $\Lambda \in \mathcal{A}^{(2k+1)} = \mathcal{M}_n(A^{(2k+1)})$ to be the matrix that has λ in each diagonal position and 0 elsewhere. Then using eq. (3.2),

$$\mathcal{D}((a_{ij})) = (a_{ij}) \cdot \Lambda - \Lambda \cdot (a_{ij}) \quad ((a_{ij}) \in \mathcal{A}, \Lambda \in \mathcal{M}_n(A^{(2k+1)}))$$

and so $\mathcal{D}$ is inner. Thus $\mathcal{A}$ is $(2k+1)$-weakly amenable. □

4. *n*-Weak amenability of Rees semigroup algebras

Let S be a semigroup. It is not known in general when the semigroup algebra $\ell^1(S)$ is weakly amenable; partial results and a conjecture are given in [2]. Thus we cannot determine when $\ell^1(S)$ is n-weakly amenable. Here we give some special cases; we describe Rees semigroups, and show that, for each such semigroup S, $\ell^1(S)$ is $(2k+1)$-weakly amenable for each $k \in \mathbb{Z}^+$.

Rees semigroups are described in §3.2 of [7] and Chapter 3 of [5]. Indeed, let G be a group, and $m, n \in \mathbb{N}$; the zero adjoined to G is o. A *Rees semigroup* has the form $S = \mathcal{M}(G, P, m, n)$; here $P = (a_{ij}) \in \mathbb{M}_{n,m}(G)$, the collection of $n \times m$ matrices with components in G. For $x \in G$, $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, let $(x)_{ij}$ be the element of $\mathbb{M}_{m,n}(G^o)$ with x in the (i, j)-th place and o elsewhere. As a set, S consists of the collection of all these matrices $(x)_{ij}$. Multiplication in S is given by the formula

$$(x)_{ij}(y)_{k\ell} = (xa_{jk}y)_{i\ell} \quad (x, y \in G, i, k \in \mathbb{N}_m, j, \ell \in \mathbb{N}_n);$$

it is shown in Lemma 3.2.2 of [7] that S is a semigroup.

Similarly, we have the semigroup $\mathcal{M}^o(G, P, m, n)$, where the elements of this semigroup are those of $\mathcal{M}(G, P, m, n)$, together with the element o, identified with the matrix that has o in each place (so that o is the zero of $\mathcal{M}^o(G, P, m, n)$), and the components of P are now allowed to belong to G^o. The matrix P is called the *sandwich matrix* in each case. The semigroup $\mathcal{M}^o(G, P, m, n)$ is a *Rees matrix semigroup with a zero over G*.

We write $\mathcal{M}^o(G, P, n)$ for $\mathcal{M}^o(G, P, n, n)$ in the case where $m = n$.

The above sandwich matrix P is *regular* if every row and column contains at least one entry in G; the semigroup $\mathcal{M}^o(G, P, m, n)$ is regular as a semigroup if and only if the sandwich matrix is regular.

Let $S = \mathcal{M}^o(G, P, m, n)$. For $x \in G$, $(x)_{ij}$ is identified with the element of $\mathbb{M}_{m,n}(\ell^1(G))$ which has δ_x in the (i, j)-th position and 0 elsewhere, and o is identified with δ_o. Thus an element of $\ell^1(S)$ is identified with an element of $\mathbb{M}_{m,n}(\ell^1(G)) \cup \mathbb{C}\delta_o$. The sandwich matrix $P \in \mathbb{M}_{n,m}(G^o)$ is identified with a matrix $P \in \mathbb{M}_{n,m}(\ell^1(G))$ as follows: if the first matrix P has $a \in G$ in the (i, j)-th position, then the new matrix P has the point mass δ_a in the (i, j)-th position; if the first matrix P has the element o in the (i, j)-th position, then the new matrix P has the element $0 \in \ell^1(G)$ in the (i, j)-th position. Thus, as in [5], we can write

$$\ell^1(S) = \mathcal{M}^o(\ell^1(G), P, m, n) = \mathcal{M}(\ell^1(G), P, m, n) \oplus \mathbb{C}\delta_o;$$

the multiplication is given explicitly in pp. 61, 62 of [5]. With this identification, we have the next result.

Theorem 4.1. *$l^1(S)$ is $(2k+1)$-weakly amenable for each Rees matrix semigroup $S = \mathcal{M}^0(G, P, n)$ $n, k \in \mathbb{N}$.*

Proof. By [4], $l^1(G)$ is $(2k+1)$-weakly amenable. Take $A = l^1(G)$, then $\mathcal{A} = l^1(S)$, and so the result follows from Proposition 3.1. □

Let S be a semigroup. We recall that S is regular if, for each $s \in S$, there exists $t \in S$ with $sts = s$. An element $p \in S$ is an idempotent if $p^2 = p$; the set of idempotents of S is denoted by $E(S)$. Let S be a semigroup with a zero 0. Then an idempotent p is primitive if $p \neq 0$ or $q = 0$ whenever $q \in E(S)$ with $q \leq p$, where $\leq$ is partially ordered on $E(S)$ defined as $p \leq q$ if $p = pq = qp$ for every $p, q \in E(S)$. S is 0-simple if $S_{[2]} \neq \{0\}$ and the only ideals in S are $\{0\}$ and S, and S is completely 0-simple if it is 0-simple and contains a primitive idempotent.

COROLLARY 4.2

Let S be an infinite, completely 0*-simple semigroup with finitely many idempotents. Then $l^1(S)$ is $(2k+1)$-weakly amenable for $k \in \mathbb{N}$.*

Proof. By Theorem 3.13 of [5], S is isomorphic as a semigroup to a regular Rees matrix semigroup with a zero $\mathcal{M}^0(G, P, n)$, thus by Theorem 4.1, $l^1(S)$ is $(2k+1)$-weakly amenable.

We recall from Definition 2.8.65 of [3] that a closed ideal I of a Banach algebra A has the trace extension property if, for each $\lambda \in I'$ with $a \cdot \lambda = \lambda \cdot a$ $(a \in A)$, there is a continuous trace τ on A such that $\tau \mid I = \lambda$. Also, a linear functional τ on A is a trace if $\tau(ab) = \tau(ba)$ $(a, b \in A)$. With this, we have the following result. □

PROPOSITION 4.3

Let A be a Banach algebra with a closed ideal I. Suppose that A is n-weakly amenable for $n \in N$, n odd and I has the trace extension property. Then A/I is n-weakly amenable.

Proof. We use the fact that $A \subset A^{(n)}$ for n even and $A' \subset A^{(n)}$ for n odd in the proof.

Let $\pi\colon A \to A/I$ be the quotient map and $\pi_n\colon (A/I)^{(n)} \to A^{(n)}$ be the n-th adjoint of π for $n \in \mathbb{N}$, n odd. Let $D\colon A/I \to (A/I)^{(n)}$ be a continuous derivation, and let $\tilde{D} = \pi_n \circ D \circ \pi$. Then $\tilde{D}\colon A \to A^{(n)}$ is a continuous derivation, and so there exists $\lambda \in A^{(n)}$ with $\tilde{D}a = a \cdot \lambda - \lambda \cdot a$ $(a \in A)$ since A is n-weakly amenable. Since I has the trace extension property, there exists $\tau \in A' \subset A^{(n)}$ $(n$ odd) and $\tau \mid I = \lambda \mid I$. Then $\lambda - \tau \in (A/I)^{(n)}$ and

$$D(a+I) = a \cdot (\lambda - \tau) - (\lambda - \tau) \cdot a \quad (a \in A).$$

Thus D is inner. Hence A/I is n-weakly amenable. □

PROPOSITION 4.4

Let A be a Banach algebra with a closed ideal I. Suppose I and A/I are $(2k+1)$-weakly amenable. Then A is $(2k+1)$-weakly amenable, $k \in \mathbb{Z}^+$.

Proof. Let $i\colon I \to A$ be the natural embedding, $i_{(2k+1)}\colon A^{(2k+1)} \to I^{(2k+1)}$ be the $(2k+1)$-th adjoint of i and $\pi\colon A \to A/I$ be the quotient map. Let $D\colon A \to A^{(2k+1)}$ be a continuous derivation. Then $i_{(2k+1)} \circ D \circ i\colon I \to I^{(2k+1)}$ is a continuous derivation, and since I is $(2k+1)$-weakly amenable, there exists $\lambda \in I^{(2k+1)}$ with $(i_{(2k+1)} \circ D)(a) = \delta_\lambda(a)$ $(a \in I)$; extending λ to be an element of $A^{(2k+1)}$. By replacing D by $D - \delta_\lambda$, we may suppose that $(i_{(2k+1)} \circ D)|I = 0$. For $a, b \in I$ and $c \in A^{(2k)}$ (where $A^{(0)} = A$ and $A \subset A^{(2k)}$), we have

$$\langle c, D(ab)\rangle = \langle ca, (i_{(2k+1)} \circ D)(b)\rangle + \langle bc, (i_{(2k+1)} \circ D)(a)\rangle = 0$$

and so, $D|I^2 = 0$.

Since I is $(2k+1)$-weakly amenable, using the idea of Proposition 2.8.63(i) of [3], we have that I is essential, that is $\overline{I^2} = I$. Then $D \mid I = 0$.

We set $F = \overline{IA^{(2k)} + A^{(2k)}I}$. Then F is a closed A-submodule of $A^{(2k)}$, and $A^{(2k)}/F$ is clearly a Banach A/I-bimodule.

For each $a \in A$ and $b \in I$, we have $a \cdot D(b) = D(ab) = 0$ since $D|I = 0$, and so $D(a) \cdot b = 0$. Take $c \in A^{(2k)}$. Then

$$\langle b \cdot c, D(a)\rangle = \langle c, D(a) \cdot b\rangle = 0$$

and so $D(a)|I \cdot A^{(2k)} = 0$. Similarly, $D(a)|A^{(2k)} \cdot I = 0$, and so $D(a)|F = 0$. Thus $D(A) \subset (A/I)^{(n)}$, and that map $D_I \colon a + I \to D(a), \quad A/I \to (A/I)^{(n)}$, is a continuous derivation. By hypothesis, A/I is $(2k+1)$-weakly amenable, and so there exists $\lambda_1 \in (A/I)^{(n)}$ with $D_I = \delta_{\lambda_1}$. It follows that A is $(2k+1)$-weakly amenable. □

Let S be a semigroup. A principal series of ideals for S is a chain

$$S = I_1 \supset I_2 \supset \cdots \supset I_{m-1} \supset I_m = K(S)$$

where $I_1, I_2, \ldots, I_m$ are ideals in S and there is no ideals of S strictly between I_j and I_{j+1} for each $j \in \mathbb{N}_{m-1}$ and $K(S)$ is the minimum ideal of S.

Let S be a regular semigroup with finitely many idempotents. Then by Theorem 3.12 and Theorem 3.13 of [5], $K(S)$ exists and S has a principal series. In this case each quotient I_j/I_{j+1} is a Rees matrix semigroup of the form $\mathcal{M}^0(G, P, n)$, where $n \in \mathbb{N}$, and the sandwich matrix P is invertible in $\mathcal{M}_n(l^1(G))$. With this idea, we have the following theorem.

Theorem 4.5. *Let S be a regular semigroup with finitely many idempotents. Then* $l^1(S)$ *is* $(2k+1)$*-weakly amenable for* $k \in \mathbb{Z}^+$.

Proof. By Theorem 3.12 and Theorem 3.13 of [5], S has the principal series and each quotient I_j/I_{j+1} is a Rees matrix semigroup of the form $T = \mathcal{M}^0(G, P, n)$. Thus by Theorem 4.1, $l^1(T)$ is $(2k+1)$-weakly amenable for each quotient $T = I_j/I_{j+1}$ as above. Hence $l^1(S)$ is $(2k+1)$-weakly amenable by Proposition 4.4. □

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