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Some augmentation quotients of integral group rings

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Abstract. Let G be a group and H be a subgroup of G. A complete description of $\Delta(G)\Delta^n(H)/\Delta^{n+1}(H)$ is given, and as a consequence the structures of $\Delta(G)/\Delta(H)$ and $\Delta^2(G)/\Delta^2(H)$ are determined. Also, the structure of $\Delta^n(G)/\Delta^n(H)$ for all $n \geq 1$ is determined when G is a free group.

Keywords. Integral group ring; augmentation ideal; augmentation quotient.

1. Introduction

Let G be a group, ZG the integral group ring of G, and $\Delta(G)$ its augmentation ideal. Let $\Delta^n(G)$ denote the n-th associative power of $\Delta(G)$, with $\Delta^0(G) = ZG$. The additive abelian group $Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G)$ is known as the n-th augmentation quotient of the group G. It is well-known that $Q_1(G)$ is isomorphic to G/G'. For an abelian group G, Passi [14] showed that $Q_2(G)$ is isomorphic to $\mathrm{Sp}^2(G)$, the second symmetric power of G. Sandling [16] in his Ph.D. thesis showed that if G is any finite group, then the canonical homomorphism ϕ: $G'/\gamma_3(G) \to \Delta^2(G)/\Delta^3(G)$ given by $g\gamma_3(G) \to (g-1) + \Delta^3(G)$ is a split monomorphism, that is $Q_2(G) \cong G'/\gamma_3(G) \oplus M$ for some abelian group M. Getting a clue from it, Losey [13] proved that $Q_2(G) \cong G'/\gamma_3(G) \oplus \mathrm{Sp}^2(G/G')$ for a finitely generated group G. Extending this result of Losey, Hales and Passi [4] showed that the exact sequence

$$0 \to G'/\gamma_3(G) \to \Delta^2(G)/\Delta^3(G) \to \mathrm{Sp}^2(G/G') \to 0$$

splits for several rather large classes of groups G. For an arbitrary $n \geq 1$, Passi [14] proved that $Q_n(G) \cong \mathrm{Sp}^n(G)$ for a free abelian group G and for a free group F, Sandling and Tahara [17] showed that

$$Q_n(F) \cong \sum \otimes_{i=1}^{n} \mathrm{Sp}^{a_i}(\gamma_i(F)/\gamma_{i+1}(F)),$$

where summation runs over all non-negative integers $a_1, a_2, \ldots, a_n$ such that $\sum_{i=1}^{n} i a_i = n$.

Khambadkone [12] extended Losey's result in another direction. She proved that if H is a finitely generated normal subgroup of a group G such that G is the semidirect product of H by a subgroup K, then

$$\frac{\Delta(G)\Delta(H)}{\Delta^2(G)\Delta(H)} \cong (K/K' \otimes H/H') \oplus \frac{\Delta^2(H)}{\Delta^3(H) + \Delta([H,K])\Delta(H)}.$$

Karan and Vermani [8] generalized these results of Khambadkone and proved that if H is any subgroup of G, then

$$\frac{\Delta(G)\Delta(H)}{\Delta^2(G)\Delta(H)} \cong (G/HG' \otimes H/H') \oplus \frac{\Delta^2(G)\Delta(H)+\Delta^2(H)}{\Delta^2(G)\Delta(H)}$$

if the exact sequence $1 \to HG'/G' \to G/G' \to G/HG' \to 1$ splits. Let G be the semidirect product of a normal subgroup H by a subgroup K. If G is finite and both H and K are nilpotent, Khambadkone [10, 11] gave a description of $\Delta^2(G)\Delta(K)/\Delta^3(G)\Delta(K)$ and $\Delta^2(G)\Delta(H)/\Delta^3(G)\Delta(H)$. Karan and Vermani [9] gave descriptions of the same quotients when G is not necessarily finite, and neither of H and K is assumed to be nilpotent.

For an arbitrary group G and a subgroup H of G, Karan and Vermani [7] gave a complete description of the quotient $\Delta(G)\Delta^{n-1}(H)/\Delta(G)\Delta^n(H)$ for all $n \geq 1$. In §2 of this paper, we give a complete description of the quotient $\Delta(G)\Delta^n(H)/\Delta^{n+1}(H)$ for all $n \geq 1$. As a consequence, we determine the structures of $\Delta(G)/\Delta(H)$ and $\Delta^2(G)/\Delta^2(H)$. We also give a complete description of the quotient $\Delta(G)\Delta^{n-1}(H)/\Delta(G)\Delta^n(K)$, where K is a subgroup of G contained in H, thereby generalizing the result of Karan and Vermani [7]. In §3, we determine the structure of the quotient $(\Delta(G)\Delta(H) + \Delta(H)\Delta(G))/(\Delta(G)\Delta^2(H)+\Delta(H)\Delta(G)\Delta(H)+\Delta^2(H)\Delta(G))$ when H is normal in G and $H/[H, G]$ is a direct sum of cyclic groups or divisible or torsion free and completely decomposable (i.e. a direct sum of rank one groups).

If R is a normal subgroup of a free group F, then Gruenberg (Lemma III.5 of [1]) showed that $\Delta^n(F)\Delta^m(R)/\Delta^{n+1}(F)\Delta^m(R)$ is a free abelian group for all $n, m \geq 1$. For arbitrary R, Karan and Kumar [6] determined a basis of this quotient and also described it completely. They also determined a basis of $\Delta^n(F)\Delta^m(R)/\Delta^{n-1}(F)\Delta^{m+1}(R)$ for all $m \geq 0$ and $n \geq 2$, and as a result, they described the structures of the quotients $\Delta^n(F)\Delta^m(R)/\Delta^{n-1}(F)\Delta^{m+1}(R)$ and $\Delta^n(F)\Delta^m(R)/\Delta^n(F)\Delta^{m+1}(R)$ for all $m \geq 1$ and $n \geq 2$. In §4, we give a complete description of the quotient $\Delta^n(F)/\Delta^n(R)$ for all $n \geq 1$ when R is any subgroup of F. We also show that if R is normal in F, then $\Delta^p(R)\Delta^n(F)\Delta^q(R)/\Delta^p(R)\Delta^{n+1}(F)\Delta^q(R)$ is a free abelian for all $p, q, n \geq 1$, and as a consequence we prove that the group $\gamma_3(R)/\gamma_4(R)[R \cap F', R \cap F', R]$ is a free abelian.

2. The quotient $\Delta(G)\Delta^n(H)/\Delta^{n+1}(H)$

Let T be a left transversal of H in G containing 1. Then every element g of G can be uniquely written as $th,\ t \in T,\ h \in H$. Let $\Delta(T)$ denote the Z-submodule of ZG generated by $\{t-1 \mid 1 \neq t \in T\}$. We begin with the following simple observation.

Lemma 2.1. $\Delta(T) \cong \Delta(G)/ZG\Delta(H)$.

Proof. Every element of $\Delta(T)$ is a finite sum of the form $\sum m_i(t_i-1),\ m_i \in Z,\ 1 \neq t_i \in T$. The map ϕ: $\Delta(T) \to \Delta(G)/ZG\Delta(H)$ defined as

$$\phi\left(\sum m_i(t_i-1)\right) = \sum \overline{m_i(t_i-1)}$$

can be easily seen to be an isomorphism. □

Theorem 2.2. $\Delta(G)\Delta(H)/\Delta^2(H) \cong (\Delta(G)/ZG\Delta(H)) \otimes \Delta(H)$ *and hence is a free abelian.*

Proof. Since $\Delta(G)/ZG\Delta(H)$ is a free Z-module on $\{t-1 \mid 1 \neq t \in T\}$, the map

$$\psi\colon (\Delta(G)/ZG\Delta(H)) \times \Delta(H) \to \Delta(G)\Delta(H)/\Delta^2(H)$$

defined as $\psi(\overline{(t-1)}, a) = \overline{(t-1)a}$, $t \in T, a \in \Delta(H)$, induces a homomorphism

$$\psi\colon (\Delta(G)/ZG\Delta(H)) \otimes \Delta(H) \to \Delta(G)\Delta(H)/\Delta^2(H).$$

Conversely, the map

$$\phi\colon G \to (\Delta(G)/ZG\Delta(H)) \otimes \Delta(H)$$

defined as $\phi(g) = \phi(th) = \overline{(t-1)} \otimes (h-1)$ extends, by linearity, to a homomorphism

$$\phi\colon ZG \to (\Delta(G)/ZG\Delta(H)) \otimes \Delta(H).$$

For $g = th \in G$ and $h_1 \in H$,

$$\begin{aligned}\phi((g-1)(h_1-1)) = \phi((th-1)(h_1-1)) &= \phi(thh_1 - th - h_1 + 1)\\ &= \overline{(t-1)} \otimes (hh_1 - 1) - \overline{(t-1)} \otimes (h-1)\\ &= \overline{(t-1)} \otimes h(h_1-1).\end{aligned}$$

Thus, restriction of ϕ to $\Delta(G)\Delta(H)$ induces a homomorphism

$$\phi\colon \Delta(G)\Delta(H) \to (\Delta(G)/ZG\Delta(H)) \otimes \Delta(H).$$

Obviously ϕ vanishes on $\Delta^2(H)$ and hence induces a homomorphism

$$\phi\colon \Delta(G)\Delta(H)/\Delta^2(H) \to (\Delta(G)/ZG\Delta(H)) \otimes \Delta(H).$$

That $\phi\psi$ is the identity map on $(\Delta(G)/ZG\Delta(H)) \otimes \Delta(H)$ is clear. Conversely, if $g = th \in G$ and $h_1 \in H$, then

$$\begin{aligned}(\psi\phi)(\overline{(g-1)(h_1-1)}) &= \psi(\overline{(t-1)} \otimes h(h_1-1)) = \overline{(t-1)h(h_1-1)}\\ &= \overline{(g-h)(h_1-1)} = \overline{((g-1)-(h-1))(h_1-1)}\\ &= \overline{(g-1)(h_1-1)}.\end{aligned}$$

Thus $\psi\phi$ is the identity map on $\Delta(G)\Delta(H)/\Delta^2(H)$. Hence both ϕ and ψ are isomorphisms and this completes the proof. □

Remark. Modulo $\Delta^2(H)$, $\Delta(G)\Delta(H)$ is freely generated by distinct elements of the form $(t-1)(h-1), t \in T, h \in H$.

Using similar arguments, we can in fact prove the following.

Theorem 2.3. $\Delta(G)\Delta^n(H)/\Delta^{n+1}(H) \cong (\Delta(G)/ZG\Delta(H)) \otimes \Delta^n(H)$ *and hence is free abelian for all* $n \geq 1$.

COROLLARY 2.4

If H is a normal subgroup of G, then

$$\Delta(G)\Delta^n(H)/\Delta^{n+1}(H) \cong \Delta(G/H) \otimes \Delta^n(H)$$

and hence is free abelian for all $n \geq 1$.

COROLLARY 2.5

For $m \geq 1$ and $n \geq m+1$,

$$\frac{\Delta(G)\Delta^m(H)}{\Delta^n(H)} \cong \frac{\Delta^{m+1}(H)}{\Delta^n(H)} \oplus \left(\frac{\Delta(G)}{ZG\Delta(H)} \otimes \Delta^m(H)\right).$$

Proof. Follows from the following split exact sequence:

$$0 \to \frac{\Delta^{m+1}(H)}{\Delta^n(H)} \to \frac{\Delta(G)\Delta^m(H)}{\Delta^n(H)} \to \frac{\Delta(G)\Delta^m(H)}{\Delta^{m+1}(H)} \to 0.$$

□

COROLLARY 2.6

$\Delta(G)/\Delta(H) \cong (\Delta(G)/ZG\Delta(H)) \otimes ZH$ *and hence is free abelian.*

Proof. It follows from the split exact sequence

$$0 \to \Delta(G)\Delta(H)/\Delta^2(H) \to \Delta(G)/\Delta(H) \to \Delta(G)/ZG\Delta(H) \to 0$$

and Theorem 2.2, that

$$\begin{aligned}\Delta(G)/\Delta(H) &\cong \Delta(G)/ZG\Delta(H) \oplus (\Delta(G)/ZG\Delta(H) \otimes \Delta(H))\\ &\cong \Delta(G)/ZG\Delta(H) \otimes (Z \oplus \Delta(H))\\ &\cong \Delta(G)/ZG\Delta(H) \otimes ZH.\end{aligned}$$

□

Theorem 2.7. *$\Delta^2(G)/\Delta^2(H)$ is isomorphic to*

$$((\Delta^2(G)+\Delta(H))/ZG\Delta(H)) \oplus H \cap G'/H' \oplus (\Delta(G)/ZG\Delta(H) \otimes \Delta(H)).$$

Proof. The kernel of the projection map

$$p\colon \Delta^2(G)/\Delta^2(H) \to (\Delta^2(G) + \Delta(H))/ZG\Delta(H)$$

is $(\Delta(G)\Delta(H) + \Delta(H \cap G'))/\Delta^2(H)$ and thus we have the exact sequence

$$0 \to \frac{\Delta(G)\Delta(H) + \Delta(H \cap G')}{\Delta^2(H)} \to \frac{\Delta^2(G)}{\Delta^2(H)} \to \frac{\Delta^2(G) + \Delta(H)}{ZG\Delta(H)} \to 0.$$

The sequence splits because the last factor is a subgroup of the free abelian group $\Delta(G)/ZG\Delta(H)$. Thus

$$\frac{\Delta^2(G)}{\Delta^2(H)} \cong \frac{\Delta^2(G) + \Delta(H)}{ZG\Delta(H)} \oplus \frac{\Delta(G)\Delta(H) + \Delta(H \cap G')}{\Delta^2(H)}.$$

To describe $(\Delta(G)\Delta(H)+\Delta(H\cap G'))/\Delta^2(H)$, we consider the exact sequence of abelian groups

$$0 \to \frac{H\cap G'}{H'} \to \frac{\Delta(G)\Delta(H)+\Delta(H\cap G')}{\Delta^2(H)} \to \frac{\Delta(G)\Delta(H)+\Delta(H\cap G')}{\Delta^2(H)+\Delta(H\cap G')} \to 0.$$

The last factor is obviously isomorphic to $\Delta(G)\Delta(H)/\Delta^2(H)$, and thus the result follows by Theorem 2.2. □

Let H be an arbitrary subgroup of a group G. Karan and Vermani [7] proved that

$$\Delta(G)\Delta^{n-1}(H)/\Delta(G)\Delta^n(H) \cong Q_n(H) \oplus (\Delta(G)/ZG\Delta(H) \otimes Q_{n-1}(H))$$

for all $n \geq 1$. We generalize this result to the following:

Theorem 2.8. *Let H and K be subgroups of a group G such that $K \subseteq H$, then*

$$\frac{\Delta(G)\Delta^{n-1}(H)}{\Delta(G)\Delta^n(K)} \cong \frac{\Delta^n(H)}{\Delta(H)\Delta^n(K)} \oplus \left(\frac{\Delta(G)}{ZG\Delta(H)} \otimes \frac{\Delta^{n-1}(H)}{\Delta(H)\Delta^n(K)+\Delta^n(K)}\right)$$

for all $n \geq 1$.

We need the following simple observations.

Lemma 2.9.

$$\frac{\Delta(G)\Delta^{n-1}(H)}{\Delta(G)\Delta^n(K)+\Delta^n(H)} \cong \frac{\Delta(G)\Delta^{n-1}(H)+\Delta(H)}{\Delta(G)\Delta^n(K)+\Delta(H)}.$$

Lemma 2.10. *Modulo $\Delta(H)$*

$$\Delta(G)\Delta^{n-1}(H) \equiv \textstyle\sum_{1\neq t\in T}(t-1)\Delta^{n-1}(H), \text{ and}$$

$$\Delta(G)\Delta^n(K) \equiv \textstyle\sum_{1\neq t\in T}(t-1)(\Delta(H)\Delta^n(K)+\Delta^n(K)).$$

Lemma 2.11.

$$\Delta(T)\otimes\frac{\Delta^{n-1}(H)}{\Delta(H)\Delta^n(K)+\Delta^n(K)} \cong \frac{\sum_{1\neq t\in T}(t-1)\Delta^{n-1}(H)}{\sum_{1\neq t\in T}(t-1)(\Delta(H)\Delta^n(K)+\Delta^n(K))}.$$

Proof. The multiplication map

$$\psi\colon \Delta(T)\times\frac{\Delta^{n-1}(H)}{\Delta(H)\Delta^n(K)+\Delta^n(K)} \to \frac{\sum_{1\neq t\in T}(t-1)\Delta^{n-1}(H)}{\sum_{1\neq t\in T}(t-1)(\Delta(H)\Delta^n(K)+\Delta^n(K))}$$

induces a homomorphism

$$\psi\colon \Delta(T)\otimes\frac{\Delta^{n-1}(H)}{\Delta(H)\Delta^n(K)+\Delta^n(K)} \to \frac{\sum_{1\neq t\in T}(t-1)\Delta^{n-1}(H)}{\sum_{1\neq t\in T}(t-1)(\Delta(H)\Delta^n(K)+\Delta^n(K))}$$

given by $\psi(a\otimes\bar{b}) = \overline{ab}$. The map ψ can be easily seen to be an isomorphism. □

Proof of Theorem 2.8. Consider the exact sequence

$$0 \to \frac{\Delta^n(H)}{\Delta(H)\Delta^n(K)} \xrightarrow{i} \frac{\Delta(G)\Delta^{n-1}(H)}{\Delta(G)\Delta^n(K)} \xrightarrow{p} \frac{\Delta(G)\Delta^{n-1}(H)}{\Delta(G)\Delta^n(K)+\Delta^n(H)} \to 0,$$

where i is the inclusion map and p is the natural projection. The map $\lambda\colon G \to \Delta(H)/\Delta(H)\Delta^n(K)$, given by $\lambda(g) = \lambda(th) = (h-1) + \Delta(H)\Delta^n(K)$, can be extended to a homomorphism $\lambda\colon ZG \to \Delta(H)/\Delta(H)\Delta^n(K)$ of abelian groups. Clearly λ defines a homomorphism

$$\lambda\colon \Delta(G)\Delta^{n-1}(H) \to \Delta^n(H)/\Delta(H)\Delta^n(K),$$

which vanishes on $\Delta(G)\Delta^n(K)$. Now it is clear that λ is a splitting homomorphism for the above exact sequence and therefore the sequence splits. Since $\Delta(T)$ is isomorphic to $\Delta(G)/ZG\Delta(H)$, therefore, using Lemmas 2.9 to 2.11, the last factor of the above exact sequence is isomorphic to $\Delta(G)/ZG\Delta(H) \otimes \Delta^{n-1}(H)/(\Delta(H)\Delta^n(K) + \Delta^n(K))$. This completes the proof of the theorem. □

On the similar lines, we can prove that the exact sequence

$$0 \to \frac{\Delta^{n-1}(H)}{ZH\Delta^n(K)} \xrightarrow{i} \frac{ZG\Delta^{n-1}(H)}{ZG\Delta^n(K)} \xrightarrow{p} \frac{ZG\Delta^{n-1}(H)}{ZG\Delta^n(K)+\Delta^{n-1}(H)} \to 0$$

splits. The last factor in this sequence is isomorphic to

$$\frac{\Delta(G)\Delta^{n-1}(H)}{\Delta(G)\Delta^n(K)+\Delta^n(H)}$$

and hence we have the following.

Theorem 2.12.

$$\frac{ZG\Delta^{n-1}(H)}{ZG\Delta^n(K)} \cong \frac{\Delta^{n-1}(H)}{\Delta(H)\Delta^n(K)+\Delta^n(K)} \oplus \left(\frac{\Delta(G)}{ZG\Delta(H)} \otimes \frac{\Delta^{n-1}(H)}{\Delta(H)\Delta^n(K)+\Delta^n(K)}\right)$$

for all $n \geq 1$.

Taking $K = H$ in the theorem, we get the following.

COROLLARY 2.13

$$\frac{ZG\Delta^{n-1}(H)}{ZG\Delta^n(H)} \cong Q_{n-1}(H) \oplus \left(\frac{\Delta(G)}{ZG\Delta(H)} \otimes Q_{n-1}(H)\right)$$

for all $n \geq 1$.

We remark here that when H is a normal subgroup of G, then the free abelian group $\Delta(G)/ZG\Delta(H)$ can be replaced by $\Delta(G/H)$ in the above results.

3. A splitting theorem

Let G be a group and H be a normal subgroup of G. In this section, our object is to describe the quotient

$$X = \frac{\Delta(G)\Delta(H)+\Delta(H)\Delta(G)}{\Delta(G)\Delta^2(H)+\Delta(H)\Delta(G)\Delta(H)+\Delta^2(H)\Delta(G)}.$$

We shall use the following notations to avoid prolonged expressions:

$$M = \Delta^3(H) + \Delta([H, G])\Delta(H) + \Delta(H)\Delta([H, G]),$$
$$N = \Delta(G)\Delta^2(H) + \Delta^2(H) + \Delta([H, G]),$$
$$W = (\Delta^2(H) + \Delta([H, G]))/M,$$
$$Y = (\Delta(G)\Delta(H) + \Delta([H, G]))/N,$$
$$\bar{H} = H/[H, G].$$

The group X is the middle term of the exact sequence

$$0 \to W \xrightarrow{i} X \xrightarrow{p} Y \to 0, \tag{1}$$

where i and p are inclusion and projection maps respectively. Define a map λ: $G \to \Delta(H)/M$, by $\lambda(th) = (h-1) + M, t \in T, h \in H$. The map λ can be extended to a homomorphism λ: $ZG \to \Delta(H)/M$ of abelian groups. Since

$$\Delta(H)\Delta(G) + \Delta(G)\Delta(H) = \Delta(G)\Delta(H) + \Delta([H, G]),$$

therefore λ defines a homomorphism

$$\lambda\colon \Delta(H)\Delta(G) + \Delta(G)\Delta(H) \to (\Delta^2(H) + \Delta([H, G]))/M.$$

Also $\Delta(G)\Delta^2(H) + \Delta(H)\Delta(G)\Delta(H) + \Delta^2(H)\Delta(G) \subseteq \Delta(G)\Delta^2(H) + M$, and thus λ defines a homomorphism λ: $X \to W$. Clearly λ is a splitting homomorphism for the exact sequence (1) and hence $X \cong W \oplus Y$. We first determine the abelian group

$$Y = \frac{\Delta(G)\Delta(H) + \Delta([H, G])}{\Delta(G)\Delta^2(H) + \Delta^2(H) + \Delta([H, G])}.$$

Since $\Delta(G)\Delta(H) \cap (\Delta(G)\Delta^2(H) + \Delta^2(H) + \Delta([H, G])) = \Delta(G)\Delta^2(H) + \Delta^2(H) + \Delta^2([H, G]) + \Delta(H') = \Delta(G)\Delta^2(H) + \Delta^2(H)$, it follows that Y is isomorphic to $\Delta(G)\Delta(H)/(\Delta(G)\Delta^2(H) + \Delta^2(H))$, which is isomorphic to

$$\Delta(G/H) \otimes \Delta(H)/\Delta^2(H) \cong \Delta(G/H) \otimes H/H'$$

(Theorem 2.4 of [7]). We now determine the structure of the abelian group W.

Lemma 3.1. *If $\bar{H}$ is a direct sum of cyclic groups or divisible or torsion free and completely decomposable* (*i.e. a direct sum of rank one groups*), *then*

$$W \cong \mathrm{Sp}^2(\bar{H}) \oplus [H, G]/[H, G, H].$$

Proof. The last factor of the exact sequence

$$0 \to \frac{\Delta([H, G])}{M \cap \Delta([H, G])} \to \frac{\Delta^2(H) + \Delta([H, G])}{M} \to \frac{\Delta^2(H) + \Delta([H, G])}{M + \Delta([H, G])} \to 0$$

is isomorphic to the abelian group $\Delta^2(\bar{H})/\Delta^3(\bar{H})$ and therefore to $\mathrm{Sp}^2(\bar{H})$. Under the given conditions, the exact sequence

$$0 \to \langle a \otimes b - b \otimes a \rangle \to \bar{H} \otimes \bar{H} \to \mathrm{Sp}^2(\bar{H}) \to 0$$

splits [4], where $a, b \in \bar{H}$. Now the multiplication map

$$f\colon \bar{H} \times \bar{H} \to \frac{\Delta^2(H) + \Delta([H, G])}{M}$$

induces a homomorphism

$$f\colon \bar{H} \otimes \bar{H} \to \frac{\Delta^2(H) + \Delta([H, G])}{M}$$

which gives rise to the following commutative diagram

$$\begin{array}{ccc} \bar{H} \otimes \bar{H} & \longrightarrow & \mathrm{Sp}^2(\bar{H}) \\ \downarrow & & \| \\ (\Delta^2(H) + \Delta([H, G]))/M & \longrightarrow & \mathrm{Sp}^2(\bar{H}). \end{array}$$

Now it is clear that the given exact sequence splits. To determine the first factor, let $z \in M \cap \Delta([H, G])$. Then $z = \sum m_i(x_i - 1)$, where $x_i \in [H, G], m_i \in Z$. Take $x = \prod x_i^{m_i}$, then $x - 1 \equiv z(\mathrm{mod}(\Delta^2([H, G])))$ and thus $x \in H \cap (1 + M) = [H, G, H]$ (Theorem 5.9 of [15]). Therefore $M \cap \Delta([H, G]) = \Delta^2([H, G]) + \Delta([H, G, H])$ and hence

$$\frac{\Delta([H, G])}{M \cap \Delta([H, G])} = \frac{\Delta([H, G])}{\Delta^2([H, G]) + \Delta([H, G, H])} \cong [H, G]/[H, G, H]$$

(Corollary 2.9 of [8]). This completes the proof. □

Hence we have the following.

Theorem 3.2. *If $\bar{H}$ is same as in the above lemma, then*

$$X \cong \mathrm{Sp}^2(\bar{H}) \oplus [H, G]/[H, G, H] \oplus (\Delta(G/H) \otimes H/H').$$

4. The quotient $\Delta^n(F)/\Delta^n(R)$

Theorem 4.1. *Let R be a subgroup of a free group F, then*

$$\Delta^n(F)/\Delta^n(R) \cong \oplus \textstyle\sum_{m=1}^{n} (Q_{n-m}(F) \otimes \Delta(F)/ZF\Delta(R) \otimes Q_{m-1}(R))$$

and hence free abelian for all $n \geq 1$.

Proof. Consider the exact sequence

$$0 \to \frac{\Delta^{n-1}(F)\Delta(R)}{\Delta^n(R)} \to \frac{\Delta^n(F)}{\Delta^n(R)} \to \frac{\Delta^n(F)}{\Delta^{n-1}(F)\Delta(R)} \to 0.$$

By Theorem 3.2 of [6], the last factor of the above sequence is isomorphic to the free abelian group $Q_{n-1}(F) \otimes \Delta(F)/ZF\Delta(R)$. The sequence therefore splits and hence

$$\frac{\Delta^n(F)}{\Delta^n(R)} \cong \frac{\Delta^{n-1}(F)\Delta(R)}{\Delta^n(R)} \oplus \left(Q_{n-1}(F) \otimes \frac{\Delta(F)}{ZF\Delta(R)}\right).$$

To determine the structure of the first summand, consider the exact sequence

$$0 \to \frac{\Delta^{n-2}(F)\Delta^2(R)}{\Delta^n(R)} \to \frac{\Delta^{n-1}(F)\Delta(R)}{\Delta^n(R)} \to \frac{\Delta^{n-1}(F)\Delta(R)}{\Delta^{n-2}(F)\Delta^2(R)} \to 0.$$

Repetition of the process gives the desired result. □

Theorem 4.2. *If R is a normal subgroup of F, then the group*

$$\Delta^p(R)\Delta^n(F)\Delta^q(R)/\Delta^p(R)\Delta^{n+1}(F)\Delta^q(R)$$

is a free abelian group for all $p, q, n \geq 1$ with basis consisting of distinct elements of the form

$$(y_{11}-1)(y_{12}-1)\cdots(y_{1p}-1)(x_1-1)(x_2-1)\cdots(x_n-1)(y_{21}-1)(y_{22}-1)\cdots(y_{2q}-1),$$

where $x_i \in X$, and $y_i \in Y$.

Proof. We prove that $\Delta(R)\Delta^n(F)\Delta(R)/\Delta(R)\Delta^{n+1}(F)\Delta(R)$ is a free abelian group with basis consisting of distinct elements of the form

$$(y_1 - 1)(x_1 - 1)(x_2 - 1)\cdots(x_n - 1)(y_2 - 1).$$

The result then follows by induction on p and q respectively. It is clear that elements of the above type generate $\Delta(R)\Delta^n(F)\Delta(R)/\Delta(R)\Delta^{n+1}(F)\Delta(R)$. For freeness, suppose that modulo $\Delta(R)\Delta^{n+1}(F)\Delta(R)$,

$$\sum_i m_i(y_{1i} - 1)(x_{1i} - 1)(x_{2i} - 1)\cdots(x_{ni} - 1)(y_{2i} - 1) \equiv 0. \tag{2}$$

Collecting coefficients of distinct y_{1i} and y_{2i}, and using the fact that $ZF\Delta(R)$ is a free right as well as left ZF-module on $\{y - 1 \mid y \in Y\}$ (Proposition I.1.12 of [3]), we can show that modulo $\Delta^{n+1}(F)$,

$$\sum_{y_{1i}=y_{1k},\, y_{2i}=y_{2j}} m_i(x_{1i} - 1)(x_{2i} - 1)\cdots(x_{ni} - 1) \equiv 0. \tag{3}$$

But modulo $\Delta^{n+1}(F)$, $\Delta^n(F)$ is a free abelian group with basis consisting of distinct elements of the form $(x_1 - 1)(x_2 - 1)\cdots(x_n - 1)$ (Remark I.1.11 of [3]). Thus each m_i in (5) and hence each m_i in (4) is zero. □

It follows from Hurley and Sehgal [5] that $\gamma_3(R)/\gamma_4(R)\gamma_3(R \cap F')$ is a free abelian group. We end up with the following.

COROLLARY 4.3

$\gamma_3(R)/\gamma_4(R)[R \cap F', R \cap F', R]$ *is a free abelian group.*

Proof. Define a map

$$\phi\colon \gamma_3(R) \to \Delta(R)\Delta(F)\Delta(R)/\Delta(R)\Delta^2(F)\Delta(R)$$

by $\phi(r) = (r - 1) + \Delta(R)\Delta^2(F)\Delta(R)$, $r \in \gamma_3(R)$. Then ϕ is a group homomorphism and by Theorem B of [2]

$$r \in \operatorname{Ker}\phi \Leftrightarrow r - 1 \in \Delta(R)\Delta^2(F)\Delta(R) \Leftrightarrow r \in \gamma_4(R)[R \cap F', R \cap F', R].$$

Thus ϕ induces a monomorphism

$$\phi\colon \gamma_3(R)/\gamma_4(R)[R \cap F', R \cap F', R] \to \Delta(R)\Delta(F)\Delta(R)/\Delta(R)\Delta^2(F)\Delta(R).$$

The result now follows from the above theorem. □

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